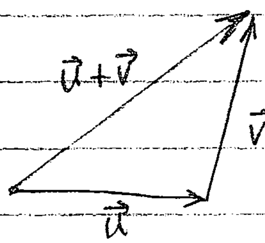
Section 12.2: Vectors

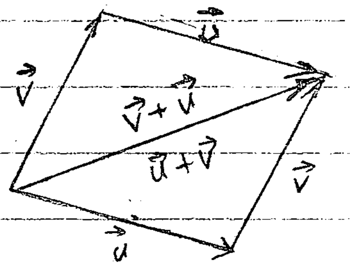
p. 798 **Def. of Vector Addition.** If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal of of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$

The Triangle Law :



The Parallelogram Law : $\vec{u}+\vec{v} = \vec{v}+\vec{u}$.

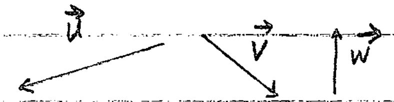
p. 799



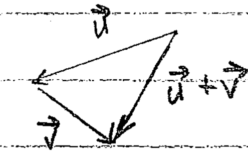
p. 799 **Def. of Scalar Multiplication**, If c is a scalar and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is the vector whose length is $|c|$ times the length of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $c>0$ and is opposite to $\vec{v}$ if $c<0$. If $c=0$ or $\vec{v}=\vec{0}$, then $c\vec{v}=\vec{0}$

* Two nonzero vectors are parallel if they are scalar multiples of each other
* We write $-\vec{v}=(-1)\vec{v}$ and call it the negative of $\vec{v}$.
* The difference $\vec{u}-\vec{v}$ is the vector $\vec{u}-\vec{v} = \vec{u}+(-\vec{v})$.

12.2.5 | Given



(a)



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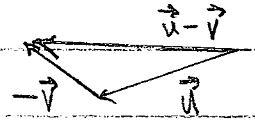

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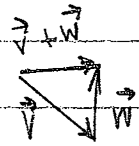

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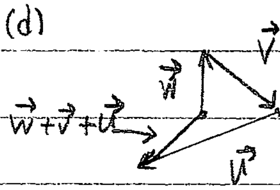


Properties of Vectors (p. 802).

$\vec{a}+\vec{b} = \vec{b}+\vec{a}$;

$(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$;

$\vec{a}+\vec{0} = \vec{a}$;

$\vec{a}+(-\vec{a}) = \vec{0}$;

$\alpha(\vec{a}+\vec{b}) = \alpha\vec{a}+\alpha\vec{b}$;

$(\alpha+\beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$;

$(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$;

$1\vec{a} = \vec{a}$

where α, β are scalars.

p. 801 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\vec{AB}$ is

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

In other words,

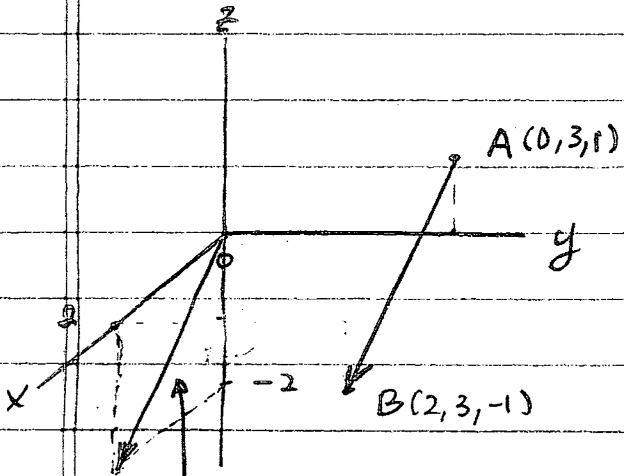
$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$\boxed{12.2.11}$ Given $A(0,3,1)$, $B(2,3,-1)$

So

$\vec{AB} = \langle 2-0, 3-3, -1-1 \rangle$

$= \langle 2, 0, -2 \rangle$



the equivalent representation of $\vec{AB}$ starting at the origin

p. 801. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and c is a scalar, then

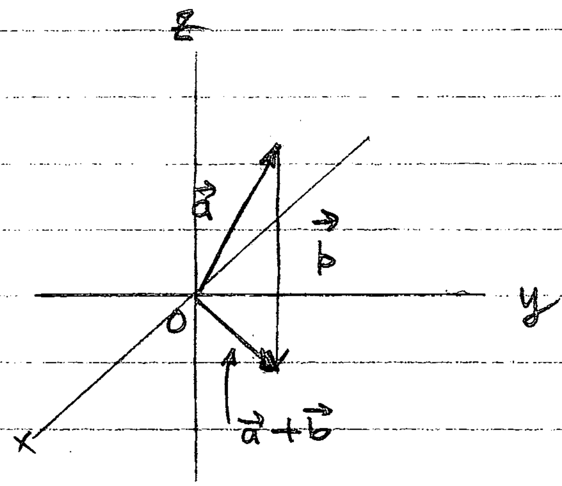
$\vec{a} \pm \vec{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$;

$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$

$\boxed{12.2.15}$ Given $\vec{a} = \langle 0, 1, 2 \rangle$, $\vec{b} = \langle 0, 0, -3 \rangle$

Then $\vec{a} + \vec{b} = \langle 0+0, 1+0, 2-3 \rangle$

$= \langle 0, 1, -1 \rangle$



p. 801. The length or magnitude of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is given by

$$\|\vec{a}\| = |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

p. 802. Definition of standard basis vectors

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

12.2.21 Given

$\vec{a} = \vec{i} - 2\vec{j} + \vec{k} = \langle 1, -2, 1 \rangle$

$\vec{b} = \vec{j} + 2\vec{k} = \langle 0, 1, 2 \rangle$.

Then

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$;

$\vec{a} + \vec{b} = \langle 1+0, -2+1, 1+2 \rangle$

$= \langle 1, -1, 3 \rangle$;

$\vec{a} - \vec{b} = \langle 1-0, -2-1, 1-2 \rangle$

$= \langle 1, -3, -1 \rangle$;

$2\vec{a} = 2\langle 1, -2, 1 \rangle$

$= \langle 2, -4, 2 \rangle$;

$3\vec{a} + 4\vec{b}$

$= 3\langle 1, -2, 1 \rangle + 4\langle 0, 1, 2 \rangle$

$= \langle 3, -6, 3 \rangle + \langle 0, 4, 8 \rangle$

$= \langle 3, -2, 11 \rangle$

p. 803. A unit vector is a vector $\vec{v}$ such that $|\vec{v}| = 1$.

In general, if $\vec{w} \neq \vec{0}$, the the unit vector that has the same direction as $\vec{w}$ is

$$\frac{\vec{w}}{|\vec{w}|} = \frac{1}{|\vec{w}|}\vec{w},$$

and the unit that has the opposite direction is

$$-\frac{\vec{w}}{|\vec{w}|} = -\frac{1}{|\vec{w}|}\vec{w}.$$

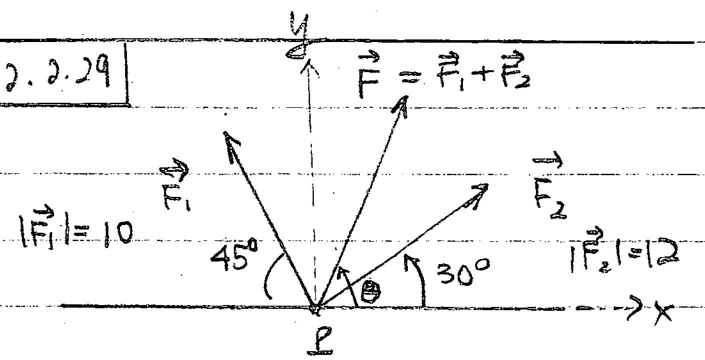
12.2.25 The length of the vector $8\vec{i} - \vec{j} + 4\vec{k} = \langle 8, -1, 4 \rangle$ is

$$|\langle 8, -1, 4 \rangle| = \sqrt{8^2 + (-1)^2 + 4^2}$$
$$= 9$$

So the unit vector that has the same direction as $\langle 8, -1, 4 \rangle$ is

$$\frac{\langle 8, -1, 4 \rangle}{|\langle 8, -1, 4 \rangle|} = \left\langle \frac{8}{9}, -\frac{1}{9}, \frac{4}{9} \right\rangle.$$

12.2.29



Two forces $\vec{F}_1$ and $\vec{F}_2$ with magnitudes 10 lb and 12 lb acts on an object at a point P as shown. Find the resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2$ acting at P, its magnitude, and its direction.

Relative to P (take P as the origin),

$\vec{F}_1 = \langle -|\vec{F}_1| \cos 45°, |\vec{F}_1| \sin 45° \rangle$

$= |\vec{F}_1| \langle -\cos 45°, \sin 45° \rangle$

$= 10 \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

$= \langle -5\sqrt{2}, 5\sqrt{2} \rangle$,

$\vec{F}_2 = \langle |\vec{F}_2| \cos 30°, |\vec{F}_2| \sin 30° \rangle$

$= |\vec{F}_2| \langle \cos 30°, \sin 30° \rangle$

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$$\vec{F_2} = 12\langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle = \langle 6\sqrt{3}, 6 \rangle.$$

So $\vec{F} = \vec{F_1} + \vec{F_2}$

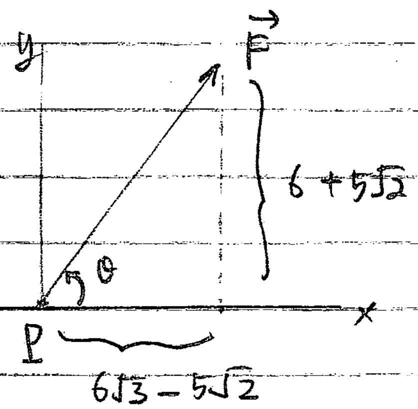
$$= \langle -5\sqrt{2} + 6\sqrt{3}, 5\sqrt{2} + 6 \rangle.$$

Hence

$$|\vec{F}| = \left[(-5\sqrt{2}+6\sqrt{3})^2 + (5\sqrt{2}+6)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{244 + 60\sqrt{2}(1-\sqrt{3})}$$

$$\simeq 13.4864 \text{ lb}$$



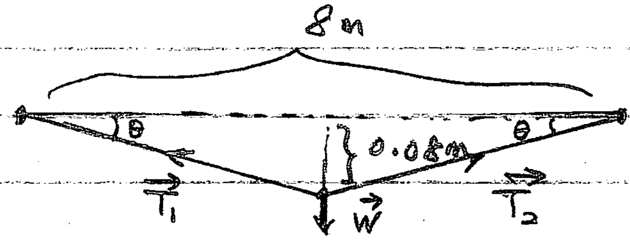
$$\tan\theta = \frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{6+5\sqrt{2}}{6\sqrt{3}-5\sqrt{2}}\right)$$

$$\simeq 75.7434^\circ$$

See also example 7 on p. 804 →

$\boxed{12.2.33}$ A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.



Let $\vec{T_1}$ and $\vec{T_2}$ be the tensions. Note that $\vec{T_1}$ and $\vec{T_2}$ have equal vertical components and opposite horizontal components. So we can write

$$\vec{T_1} = \langle -a, b \rangle,$$
$$\vec{T_2} = \langle a, b \rangle, \quad a > 0, b > 0.$$

Alternatively, let $\vec{W}$ be the weight of the shirt due to gravity:

$$\vec{W} = \langle 0, -0.8\,g \rangle$$

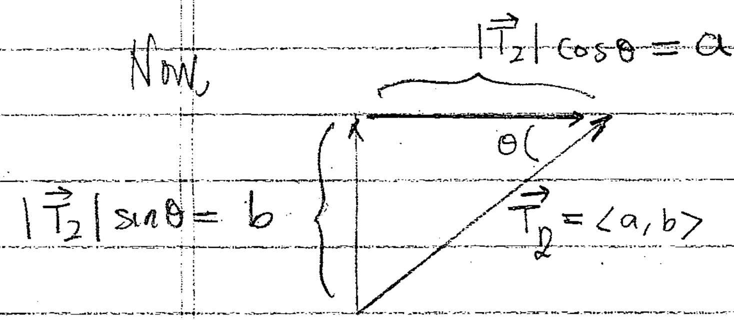
acceleration due to gravity ($\simeq -9.8\,m/s^2$)

Then by Newton's second law:

$$\vec{T_1} + \vec{T_2} + \vec{W} = \vec{0}, \qquad (*)$$

from which we see that the horizontal components of $\vec{T}_1$ and $\vec{T}_2$ must be the negative of each other since $\vec{w}$ has no horizontal components.

Now,



However, we also have

$$\tan\theta = \frac{0.08}{4} = 0.02.$$

Hence $\frac{b}{a} = \tan\theta = 0.02$

or $a = 50b$

On the other hand, considering the vertical component of (*), we get

$$2b + 0.8g = 0$$

or $b = -0.4g$

$\simeq 3.92$ N

and thus $a \simeq 50(3.92) = 196$ N.

Thus the tensions are

$\vec{T}_1 \simeq \langle -196, 3.92\rangle$ N

$\vec{T}_2 \simeq \langle 196, 3.92\rangle$ N.

$\boxed{12.2.39}$ Given $\vec{r} = \langle x, y, z\rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$.

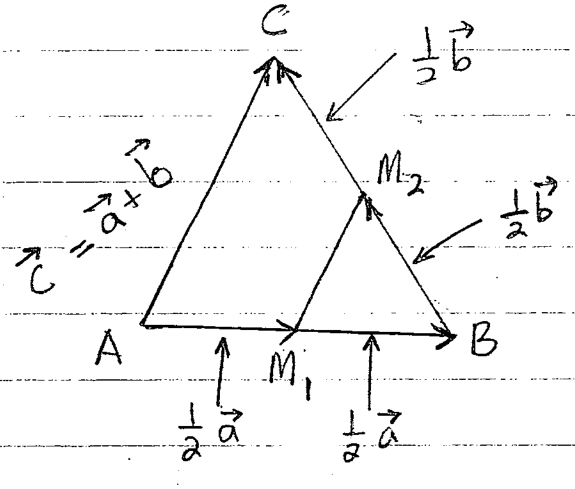
Then

$$|\vec{r} - \vec{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Therefore the set of all points (x, y, z) such that $|\vec{r} - \vec{r}_0| = 1$ is the sphere with radius 1 and center (x_0, y_0, z_0).

$\boxed{12.3.43}$ Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

$\longrightarrow$ next page.



Let $\vec{a} = \vec{AB}$, $\vec{b} = \vec{BC}$, $\vec{c} = \vec{AC}$

Let M_1, M_2 be the midpoints of AB and BC respectively.

Then

$$\vec{c} = \vec{a} + \vec{b} \;;$$
$$\vec{AM_1} = \vec{M_1B} = \frac{1}{2}\vec{a} \;;$$
$$\vec{BM_2} = \vec{M_2C} = \frac{1}{2}\vec{b}.$$

Note that

$$\begin{aligned}\vec{M_1M_2} &= \vec{M_1B} + \vec{BM_2} \\ &= \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} \\ &= \frac{1}{2}(\vec{a} + \vec{b}) \\ &= \frac{1}{2}\vec{c}\end{aligned}$$

Hence M_1M_2 is parallel to AC.

Furthermore, the length of M_1M_2 is

$$\begin{aligned}|\vec{M_1M_2}| &= |\tfrac{1}{2}\vec{c}| \\ &= \frac{1}{2}|\vec{c}| \\ &= \frac{1}{2} \times \text{the length of } AC.\end{aligned}$$