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Sec 12.4: The Cross Product

p.814 If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

where for any scalars $\alpha, \beta, \gamma, \delta$,

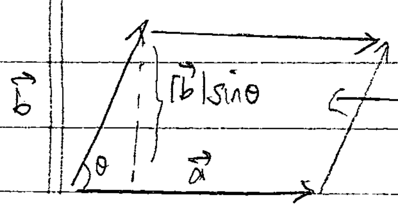
$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma$$

p816 1. $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$.

2. $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$

p.817 3. If $\vec{a} \ne \vec{0}$, $\vec{b} \ne \vec{0}$, then they are parallel if and only if $\vec{a}\times\vec{b} = \vec{0}$.

4. $|\vec{a}\times\vec{b}|$ = the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



area $= |\vec{a}|\,(|\vec{b}|\sin\theta)$

$= |\vec{a}\times\vec{b}|$

12.4.1 $\vec{a} = \langle 1, 2, 0 \rangle$, $\vec{b} = \langle 0, 3, 1 \rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$$

$$= \vec{i}(2\times1 - 0\times3) - \vec{j}(1\times1 - 0\times0) + \vec{k}(1\times3 - 2\times0)$$

$$= 2\vec{i} - \vec{j} + 3\vec{k}$$

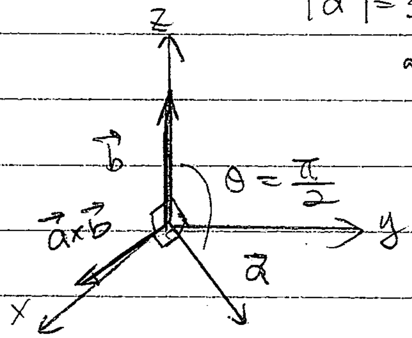
12.4.7 $\vec{a} = \langle t, t^2, t^3 \rangle$, $\vec{b} = \langle 1, 2t, 3t^2 \rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} - \vec{j}\begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} + \vec{k}\begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}$$

$$= \langle t^4, -2t^3, t^2 \rangle$$

12.4.12 Given $\vec{a}$ lies in the xy-plane, $|\vec{a}| = 3$, $|\vec{b}| = 2$ as shown



(a) so $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$= (3)(2)\sin\frac{\pi}{2}$

$= 6$

p. 816

$\boxed{\text{The right-hand rule:}}$ If $\vec{a}$, $\vec{b}$ are represented by directed line segments with the same initial point, then $\vec{a}\times\vec{b}$ points in a direction perpendicular to the plane through $\vec{a}$ and $\vec{b}$. So if the fingers of one's right hand curl in the direction through the angle θ between $\vec{a}$ and $\vec{b}$, from $\vec{a}$ to $\vec{b}$, then the thumb points in the direction of $\vec{a}\times\vec{b}$.

(b) The only component of $\vec{a}\times\vec{b}$ that is zero is the $\vec{k}$-component

$\boxed{12.4.15}$ Let $\vec{a} = \langle 1,-1,1\rangle$, $\vec{b} = \langle 0,4,4\rangle$

Then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 0 & 4 & 0 \end{vmatrix} = \langle -4, 0, 4\rangle$$

is a vector orthogonal to $\vec{a}$ and $\vec{b}$.

A unit vector that is orthogonal to $\vec{a}$ and $\vec{b}$ is given by

$$\vec{u} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{\langle -4,0,4\rangle}{\sqrt{4^2+0^2+4^2}}$$

$$= \frac{\langle -1,0,1\rangle}{\sqrt{2}}$$

and another is its negative $-\vec{u} = \frac{\langle 1,0,-1\rangle}{\sqrt{2}}$

$\boxed{12.4.19}$ Prove $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$

Proof.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

$$= -\langle b_2a_3 - b_3a_2,\ b_3a_1 - b_1a_3,\ b_1a_2 - b_2a_1\rangle$$

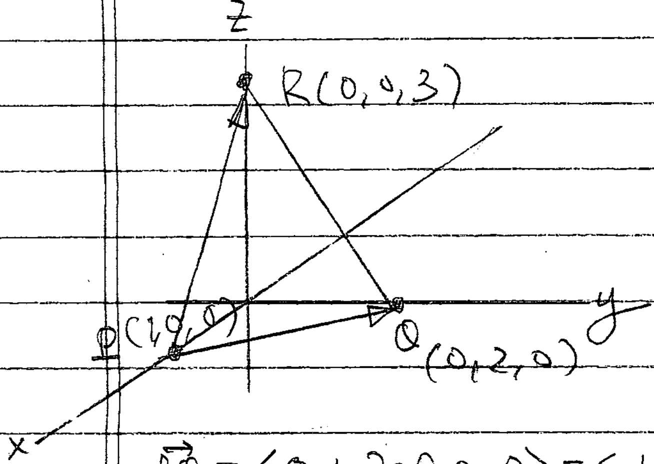
$$= -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\vec{b}\times\vec{a}$$

*p. 818. Properties of Cross Product.

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors and α a scalar.

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
2. $(\alpha\vec{a})\times\vec{b} = \alpha(\vec{a}\times\vec{b}) = \vec{a}\times(\alpha\vec{b})$
3. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$
4. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$
5. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$
6. $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$\boxed{12.4.25}$ (a) Find a vector orthogonal to the plane through the points $P(1,0,0)$, $Q(0,2,0)$, and $R(0,0,3)$



$\vec{PQ} = \langle 0-1, 2-0, 0-0\rangle = \langle -1, 2, 0\rangle$

$\vec{PR} = \langle 0-1, 0-0, 3-0\rangle = \langle -1, 0, 3\rangle$

$\vec{PQ} \times \vec{PR}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, -2\rangle$$

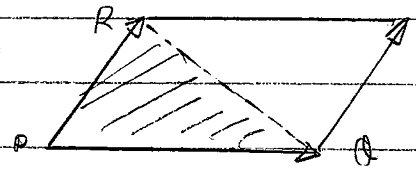
is orthogonal to $\vec{PQ}$ and $\vec{PR}$, while $\vec{PQ}$ and $\vec{PR}$ are lying in the plane through the points P, Q, and R. Hence $\vec{PQ} \times \vec{PR} = \langle 6, 3, -2\rangle$ is orthogonal to the plane through P, Q, and R.

(b) The area of the parallelogram determined by $\vec{PQ}$ and $\vec{PR}$ is

$$|\vec{PQ} \times \vec{PR}| = \sqrt{6^2 + 3^2 + 2^2}$$
$$= 7 \text{ unit}^2$$

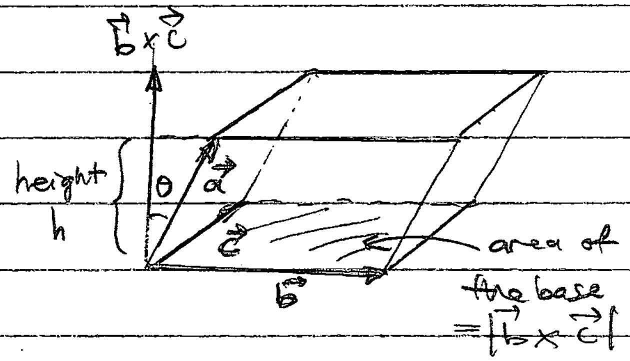
So the area of the $\triangle PQR$ is

$$\frac{|\vec{PQ} \times \vec{PR}|}{2} = \frac{7}{2} \text{ unit}^2$$



p. 819. The volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is

$$V = |\vec{a} \cdot b \times \vec{c}|$$



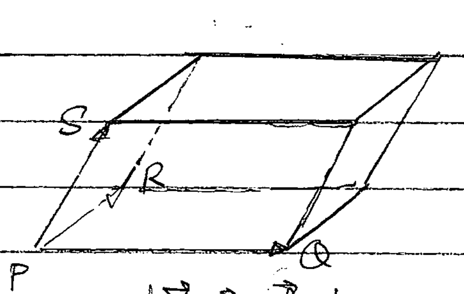
$$V = \text{area of the base} \times \text{height}$$
$$= |\vec{b} \times \vec{c}|\,|\vec{a}|\,|\cos\theta|$$
$$= |\vec{a} \cdot \vec{b} \times \vec{c}|$$

$\boxed{12.4.31}$ Given $P(2,0,-1)$, $Q(4,1,0)$, $R(3,-1,1)$, $S(2,-2,2)$. Then

$\vec{PQ} = \langle 2, 1, 1\rangle$

$\vec{PR} = \langle 1, -1, 2\rangle$

$\vec{PS} = \langle 0, -2, 3\rangle$



$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -3, -3\rangle$$

So the volume of the parallelepiped with adjacent edges PQ, PQ, and PS is

$$|\vec{PS} \cdot \vec{PQ} \times \vec{PR}| = |(0)(3) + (-2)(-3) + (3)(-3)| =$$

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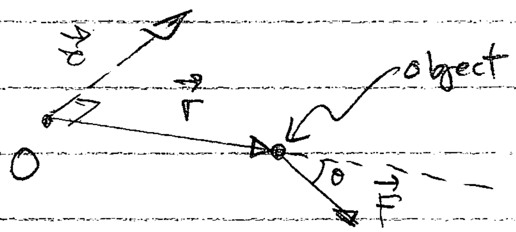
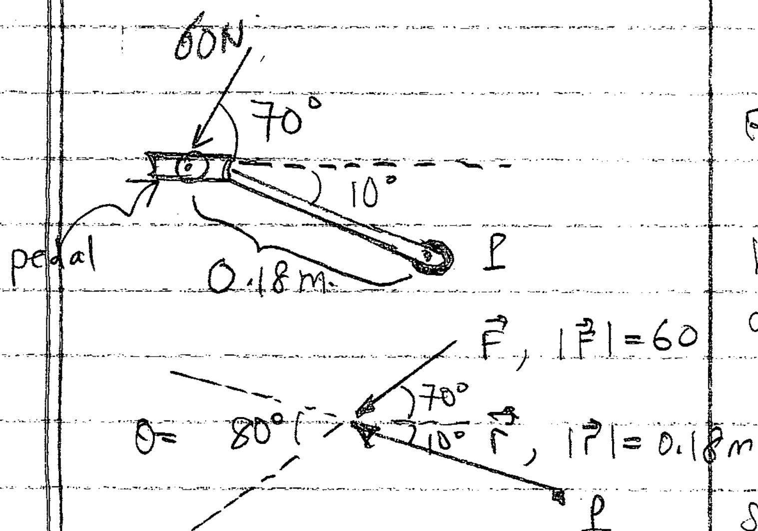
p. 819 Suppose a force $\vec{F}$ is acting on an object at a point with position vector $\vec{r}$. Then the torque relative to the origin is given by

$$\vec{\tau} = \vec{r} \times \vec{F}.$$



12.4.35 A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm = 0.18 m long. Find the magnitude of the torque $\vec{\tau}$ about P.



Note that the angle between the position vector $\vec{r}$ (relative to P) and the force is $\theta = 80°$. So

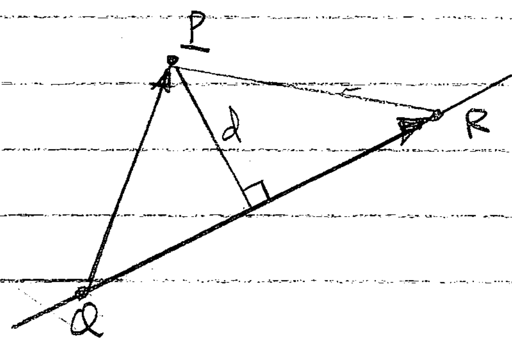
$$|\vec{\tau}| = |\vec{r} \times \vec{F}|$$
$$= |\vec{r}||\vec{F}| \sin\theta$$
$$= (0.18)(60) \sin 80°$$
$$\simeq 10.6 J.$$

12.4.39 (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from P to L is

$$d = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|},$$

where $\vec{a} = \vec{QR}$ and $\vec{b} = \vec{QP}$.

Proof.



Recall that the area of ΔPQR is given by $\frac{1}{2}|\vec{QR} \times \vec{QP}|$. On the other hand, it is clear that the same area is also given by $\frac{1}{2}|\vec{QR}|d$. So

$$\frac{1}{2}|\vec{QR} \times \vec{QP}| = \frac{1}{2}|\vec{QR}|d,$$

so that $d = \dfrac{|\vec{QR} \times \vec{QP}|}{|\vec{QR}|}$. #

(b) Find the distance from P(1,1,1) to the line through Q(0,6,8) and R(-1,4,7)

$\vec{QR} = \langle -1, -2, -1 \rangle$

$\vec{QP} = \langle 1, -5, -7 \rangle$

$$\vec{QR} \times \vec{QP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -1 \\ 1 & -5 & -7 \end{vmatrix}$$

$$= \langle 9, -8, 7 \rangle$$

So $|\vec{QR} \times \vec{QP}|$

$$= \sqrt{9^2 + 8^2 + 7^2}$$

$$= \sqrt{194}$$

$$|\vec{QR}| = \sqrt{1^2 + 2^2 + 1^2}$$

$$= \sqrt{6}$$

Therefore $d = \dfrac{|\vec{QR} \times \vec{QP}|}{|\vec{QR}|}$

$$= \sqrt{\frac{97}{3}} \text{ unit.}$$

$\boxed{12.4.41}$ Prove that

$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

Proof. Use theorem 8 on p. 878

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b} \quad \text{by property 3}$$

$$= \underbrace{\vec{a} \times \vec{a} - \vec{b} \times \vec{a}}_{\text{by properties 2 and 4}} + \underbrace{\vec{a} \times \vec{b} - \vec{b} \times \vec{b}}_{\text{by properties 2 and 4}}$$

$$= \vec{0} + \underbrace{\vec{a} \times \vec{b}}_{\text{by problem 19}} + \vec{a} \times \vec{b} - \vec{0}$$

Note For any vector $\vec{a}$, $\vec{a} \times \vec{a} = \vec{0}$

$$= 2(\vec{a} \times \vec{b})$$