Common Student Mistakes in Midterm 1 (Math 162A - Autumn 08)

- *Mistake:* For problem 2(b), some students conclude right away from the limits $\lim_{n\to\infty} \ln(2n^2+13) = \infty$ and $\lim_{n\to\infty} \ln(5n^2-13) = \infty$ that

  – $\lim_{n\to\infty}[\ln(2n^2+13) - \ln(5n^2-13)] = 0$,
  – $\lim_{n\to\infty}[\ln(2n^2+13) - \ln(5n^2-13)] = \pm\infty$, or
  – $\lim_{n\to\infty}[\ln(2n^2+13) - \ln(5n^2-13)]$ does not exist.

  Comment 1: Given two sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$, the limit $\lim_{n\to\infty}(a_n - b_n)$ cannot be determined before the nth term of the sequence $\{a_n - b_n\}$ is further simplified. This is so since any of the following may occur:

  – $\lim_{n\to\infty}(a_n - b_n) = 0$ (example: $a_n = b_n = n$);
  – $\lim_{n\to\infty}(a_n - b_n) = L$, $0 < |L| < \infty$ (example: $a_n = \ln(2n)$, $b_n = \ln(n)$);
  – $\lim_{n\to\infty}(a_n - b_n) = \infty$ (example: $a_n = n^2$, $b_n = n$);
  – $\lim_{n\to\infty}(a_n - b_n) = -\infty$ (example: $a_n = n$, $b_n = n^2$).

  The expression "$\infty - \infty$" has no definite meaning in itself.

  Comment 2: Given two sequences $\{a_n\}$ and $\{b_n\}$, it is safe to use the rule

  $$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

  when both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are finite. **However**, it is safe to use the same rule when either $\lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} b_n$ is infinite only under certain circumstances.

- *Mistake:* For problem 3, some students use the alternating series test to conclude that the series are absolutely convergent.

  Comment: No! The alternating series test can **never** test the absolute convergence of a series.

- *Mistake:* For problem 3(a), many students are still verifying the hypotheses in the integral test for the sequence $\{\frac{1}{n\ln n}\}$ of the terms of the series, and not for the corresponding function $f(x) = \frac{1}{x\ln x}$.

  Comment: Verifying the hypotheses for the sequence is insufficient! See the comment following the solution of problem 2, quiz #1.

- *Mistake:* For problem 3(c), some students conclude the divergence of the series from the fact that the series fails the hypothesis of the alternating series test, i.e., $\lim_{n\to\infty}(-4)^n/(2^n - 3^n)$ does not exist.

  Comment: No! The alternating series test test only the convergence of an alternating series. The test that allows us to conclude the divergence of the series from the nonexistence of the limit $\lim_{n\to\infty}(-4)^n/(2^n - 3^n)$ is the test for divergence.

- *Mistake:* For problem 3(c), many students shows that

$$\lim_{n\to\infty}\left|\frac{(-4)^{n+1}}{2^{n+1}-3^{n+1}}\cdot\frac{2^n-3^n}{(-4)^n}\right|=4$$

or

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-4)^n}{2^n-3^n}\right|}=4$$

in the attempt of applying the ratio test or the root test.

Comment: These limits are both equal to $\frac{4}{3}$. To see this,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-4)^{n+1}}{2^{n+1}-3^{n+1}}\cdot\frac{2^n-3^n}{(-4)^n}\right| &= \lim_{n\to\infty}\frac{4^{n+1}}{4^n}\cdot\frac{3^n-2^n}{3^{n+1}-2^{n+1}}\\ &= \lim_{n\to\infty}4\cdot\frac{1-(2/3)^n}{3-2(2/3)^n}\\ &= \frac{4}{3}\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-4)^n}{2^n-3^n}\right|} &= \lim_{n\to\infty}\frac{4}{\sqrt[n]{3^n-2^n}}\\ &= \lim_{n\to\infty}\frac{4}{\sqrt[n]{3^n(1-(2/3)^n}}\\ &= \lim_{n\to\infty}\frac{4}{3}\cdot\frac{1}{\sqrt[n]{1-(2/3)^n}}\\ &= \frac{4}{3}\end{aligned}$$

since it can be shown by taking the logarithm that $\lim_{n\to\infty}\sqrt[n]{1-(2/3)^n}=1$.

- *Mistake:* For problem 4, quite a number of students conclude from the "fact" that

$$\sum_{n=1}^{\infty}\frac{2}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n+2},$$

together with the divergence of the two series on the right-hand side, the series $\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$ is divergent.

Comment: Addition and substraction of two divergent series may **not** make sense!!! See the comment following the solution of problem 1, quiz #1.

- *Mistake:* For problem 5, many students thought that

$$\lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}=1$$

in the attempt of applying the ratio test.

Comment: In fact,

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n} &= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}(n+1)\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n(n+1)\\ &= \infty.\end{aligned}$$

This is so since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. This limit is given in the solution of problem 11.7.35 of the homework.

- *Mistake:* For problem 5, some students forget the absolute value sign in the attempt of applying the ratio test or the root test.