Math 162A (Autumn 08) - Quiz # 1

Problem 1. Show that $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ is divergent.

Solution 1. Use the limit comparison test. First some estimates:

$$\begin{aligned} 3n+1 &\approx 3n, \\ n(n+1) &\approx n^2, \\ \frac{3n+1}{n(n+1)} &\approx \frac{3n}{n^2} = \frac{3}{n}. \end{aligned}$$

Now,

- $\frac{3}{n} > 0$, $\frac{3n+1}{n(n+1)} > 0$;
- $\lim_{n\to\infty} \left[\frac{3n+1}{n(n+1)}\right]/[\frac{3}{n}] = 1 > 0$;
- $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ diverges. □

Solution 2. Use the comparison test. Note that

- $\frac{3n+1}{n(n+1)} > \frac{3n}{n(n+1)} = \frac{3}{n+1} > 0$;
- $\sum_{n=1}^{\infty} \frac{3}{n+1}$ diverges.

So $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ diverges. □

Solution 3. Use the comparison test. Note that

- $\frac{3n+1}{n(n+1)} = \frac{1}{n} + \frac{2}{n+1} > \frac{1}{n} > 0$;
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ diverges. □

Solution 4. Use the fact that *if the nth partial sum of a series $\sum a_n$ is the sum of the nth partial sums of two divergent series $\sum b_n$ and $\sum c_n$ of positive terms, then $\sum a_n$ is also divergent.* To this end, note that

- $\frac{3n+1}{n(n+1)} = \frac{1}{n} + \frac{2}{n+1}$;
- $\frac{1}{n} > 0$ and $\frac{2}{n+1} > 0$;
- $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1} \frac{2}{n+1}$ diverge.

So $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ diverges. □

Be warned! It is incorrect to write $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{2}{n+1}$ and conclude that $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ is divergent from the divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{2}{n+1}$. This is so since equalities and sums become ill-defined when divergent series are involved.

Solution 5. Use the integral test. Set $f(x) = \frac{3x+1}{x(x+1)} = \frac{1}{x} + \frac{2}{x+1}$ for $x \geq 1$. Note that

- f is continuous;
- $f(x) > 0$;
- f is clearly decreasing.

Hence $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$ is convergent if and only if so is $\int_1^{\infty} f(x)\,dx$. Now

$$\begin{aligned} &\int_1^{\infty} f(x)\,dx \\ &= \int_1^{\infty} \left(\frac{1}{x} + \frac{2}{x+1}\right) dx \\ &= [\ln x + 2\ln(x+1)]_1^{\infty} \\ &= \{\ln[x(x+1)^2]\}_1^{\infty} \\ &= \lim_{x\to\infty} \ln[x(x+1)] - \ln[1(1+1)] \\ &= \infty, \end{aligned}$$

i.e., $\int_1^{\infty} f(x)\,dx$ diverges. Hence so does $\sum_{n=1}^{\infty} \frac{3n+1}{n(n+1)}$. □

Problem 2. Show that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent.

Solution. Use the integral test. Set $f(x) = \frac{\ln x}{x^2}$ for $x \geq 1$. Note that

- f is continuous because $\ln x$ and $\frac{1}{x^2}$ are continuous;
- $f(x) \geq 0$ since $\ln x \geq 0$ and $\frac{1}{x^2} > 0$;
- $f'(x) = \frac{1-2\ln x}{x^3} < 0$ eventually, i.e., for $x > e^{1/2}$, and this implies that $f(x)$ is eventually decreasing.

Hence $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent if and only so is

$\int_1^\infty f(x)\,dx$. Now

$$\begin{aligned}
&\int_1^\infty f(x)\,dx \\
&= \int_1^\infty \frac{\ln x}{x^2}\,dx \\
&= \left[-\frac{\ln x}{x}\right]_1^\infty - \int_1^\infty -\frac{1}{x^2}\,dx \\
&= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^\infty \\
&= \lim_{x\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right) \\
&= 1
\end{aligned}$$

since $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$ by l'Hôpital's rule. In particular, $\int_1^\infty f(x)\,dx$ is convergent and thus so is $\sum_{n=1}^\infty \frac{\ln n}{n^2}$. $\square$

Be warned! It is insufficient to verify that $f(n)$, where n is an integer variable, satisfies the hypotheses in the integral test. This is so since even if $f(n) > 0$ for all positive integers n, it may still happen that $f(x) < 0$ for some non-integer value of x. It may also happen that $f(n)$ is decreasing, i.e., $f(n+1) < f(n)$ for all positive integers n, but $f(x)$, where x is a real variable, is not. Also, the integrals $\int_1^\infty f(n)\,dn$ and $\int_1^\infty f(x)\,dx$ are totally different if one is meant by the first integral an integral over all positive integers, while the second integral an integral over the interval $[1,\infty)$.