

Homework Set Hmwk1 due 10/13/08 at 6:30 AM

This set covers sections 10.1-10.4 of the text.

You may need to give 4 or 5 significant digits for some (floating point) numerical answers in order to have them accepted by the computer.

1.(1 pt) Find a Cartesian equation relating x and y corresponding to the parametric equations

$$x = 2 \sin(3t) \quad y = 9 \cos(3t)$$

Write your answer in the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y such that the coefficient of y^2 is 4.

Answer: _____ = 0

Find the equation of the tangent line to the curve at the point corresponding to $t = \pi/9$.

Answer: $y =$ _____

2.(1 pt) Find a Cartesian equation relating x and y corresponding to the parametric equations

$$x = e^{3t} \quad y = e^{-6t}$$

Write your answer in the form

$$y = f(x)$$

Answer: $y =$ _____

3.(1 pt) To find the surface area of the surface generated by revolving the curve defined by the parametric equations

$$x = 3t^3 + 3t, \quad y = t, \quad 1 \leq t \leq 7$$

around the x -axis you'd have to compute

$$\int_a^b f(t) dt$$

where $a =$ _____,

$b =$ _____,

and $f(t) =$ _____

4.(1 pt) Consider the parametric curve given by the equations

$$x(t) = t^2 + 30t + 31$$

$$y(t) = t^2 + 30t - 40$$

How many units of distance are covered by the point $P(t) = (x(t), y(t))$ between $t=0$, and $t=9$?

5.(1 pt) Find the area of the region inside: $r = 9 \sin \theta$ but outside: $r = 2$

6.(1 pt) Find the area of the region outside $r = 10 + 10 \sin \theta$, but inside $r = 30 \sin \theta$.

Problem 1. Using the identity $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{9}\right)^2 = \sin^2 3t + \cos^2 3t = 1$$

or, equivalently,

$$\frac{x^2}{4} + \frac{y^2}{81} - 1 = 0.$$

To turn the coefficient of y^2 to 4, we multiply the last equation by $4/81$:

$$\frac{x^2}{81} + 4y^2 - 4 = 0.$$

Hence

$$P(x, y) = \frac{x^2}{81} + 4y^2 - 4.$$

Next, to compute the tangent line to the curve at the point corresponding to $t = \pi/9$, we compute

- $x = 2 \sin 3 \cdot \frac{\pi}{9} = \sqrt{3}$;
- $y = 9 \cos 3 \cdot \frac{\pi}{9} = \frac{9}{2}$;
- $\frac{dx}{dt} = 6 \cos 3t \stackrel{t=\pi/9}{=} 3$;
- $\frac{dy}{dt} = -27 \sin 3t \stackrel{t=\pi/9}{=} -\frac{27\sqrt{3}}{2}$;
- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \stackrel{t=\pi/9}{=} -\frac{9\sqrt{3}}{2}$.

Hence an equation for the tangent line is

$$y - \frac{9}{2} = -\frac{9\sqrt{3}}{2}(x - \sqrt{3})$$

or, equivalently,

$$y = \frac{9}{2} - \frac{9\sqrt{3}}{2}(x - \sqrt{3}).$$

Problem 2. From $x = e^{3t}$ and $y = e^{-6t}$, we obtain

$$\ln x = 3t \text{ and } \ln y = -6t.$$

So

$$t = \frac{\ln x}{3} = -\frac{\ln y}{6}.$$

Thus

$$\ln y = -6 \cdot \frac{\ln x}{3} = -2 \ln x = \ln x^{-2}.$$

In consequence,

$$y = \frac{1}{x^2}.$$

Problem 3. Use formula (7) on page 665 of the text. So $a = 1$, $b = 7$, and

$$\begin{aligned} f(t) &= 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= 2\pi t \sqrt{(9t^2 + 3)^2 + 1}. \end{aligned}$$

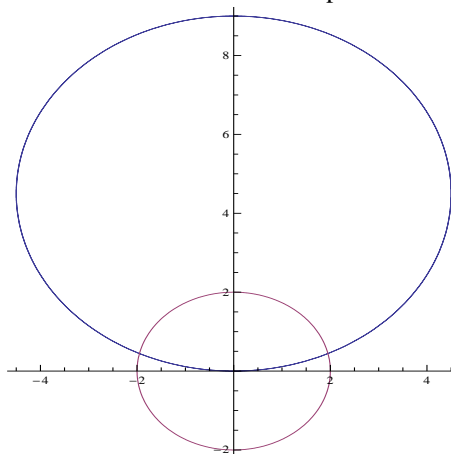
Problem 4. Note that $x - 31 = t^2 + 30t = y + 40$ or, equivalently, $y = x - 71$, so that the parametric curve lies in straight line with unit slope. In fact, since

$$\frac{dx}{dt} = \frac{dy}{dt} = 2t + 30 > 0$$

for $0 \leq t \leq 9$, so that both x and y are strictly increasing with respect to t , each point on the curve is traversed precisely once as t increases from 0 to 9. Hence the distance traveled by the point $P(t) = (x(t), y(t))$ as t varies from 0 to 9 is given by the length of the curve, i.e.,

$$\begin{aligned} L &= \int_0^9 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^9 \sqrt{2(2t + 30)^2} dt \\ &= \int_0^9 \sqrt{2}(2t + 30) dt \\ &= \sqrt{2} [t^2 + 30t]_0^9 \\ &= 351\sqrt{2}. \end{aligned}$$

Problem 5. The curves are depicted as follows:



They intersect when $9 \sin \theta = 2$ or

$$\theta = \sin^{-1} \frac{2}{9} \text{ or } \pi - \sin^{-1} \frac{2}{9}.$$

Hence the area of the region inside the curve $r = 9 \sin \theta$ and outside the curve $r = 2$ is given by

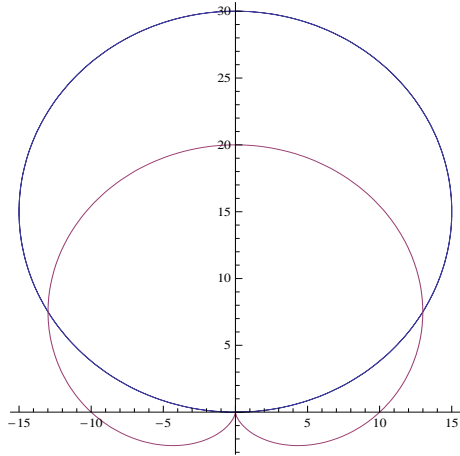
$$\begin{aligned}
A &= \int_{\sin^{-1} 2/9}^{\pi - \sin^{-1} 2/9} \left[\frac{1}{2} (9 \sin \theta)^2 - \frac{1}{2} (2)^2 \right] d\theta \\
&= \frac{1}{2} \int_{\sin^{-1} 2/9}^{\pi - \sin^{-1} 2/9} (81 \sin^2 \theta - 4) d\theta \\
&= \frac{1}{2} \int_{\sin^{-1} 2/9}^{\pi - \sin^{-1} 2/9} \left(81 \cdot \frac{1 - \cos 2\theta}{2} - 4 \right) d\theta \\
&= \frac{1}{4} \int_{\sin^{-1} 2/9}^{\pi - \sin^{-1} 2/9} (73 - \cos 2\theta) d\theta.
\end{aligned}$$

By symmetry, this can also be calculated as

$$\begin{aligned}
A &= 2 \cdot \frac{1}{4} \int_{\sin^{-1} 2/9}^{\pi/2} (73 - \cos 2\theta) d\theta \\
&= \frac{1}{2} \left[73\theta - \frac{\sin 2\theta}{2} \right]_{\sin^{-1} 2/9}^{\pi/2} \\
&= \frac{1}{2} [73\theta - \cos \theta \sin \theta]_{\sin^{-1} 2/9}^{\pi/2} \\
&= \frac{1}{2} \left[73 \left(\frac{\pi}{2} - \sin^{-1} \frac{2}{9} \right) - \left(\cos \frac{\pi}{2} \sin \frac{\pi}{2} - \cos \sin^{-1} \frac{2}{9} \sin \sin^{-1} \frac{2}{9} \right) \right] \\
&= \frac{1}{2} \left[73 \left(\frac{\pi}{2} - \sin^{-1} \frac{2}{9} \right) + \frac{\sqrt{77}}{9} \cdot \frac{2}{9} \right] \\
&= \frac{73\pi}{4} - \frac{73}{2} \sin^{-1} \frac{2}{9} + \frac{\sqrt{77}}{81}.
\end{aligned}$$

Problem 6.* The curves are depicted as follows: $\sin \theta = 1/2$, i.e., when

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}.$$



So the area of the region inside the curve $r = 30 \sin \theta$ and outside the curve $r = 10 + 10 \sin \theta$ is given by

$$\begin{aligned}
A &= \int_{\pi/6}^{5\pi/6} \left[\frac{1}{2} (30 \sin \theta)^2 - \frac{1}{2} (10 + 10 \sin \theta)^2 \right] d\theta \\
&= \int_{\pi/6}^{5\pi/6} (400 \sin^2 \theta - 100 \sin \theta - 50) d\theta \\
&= \int_{\pi/6}^{5\pi/6} \left(400 \cdot \frac{1 - \cos 2\theta}{2} - 100 \sin \theta - 50 \right) d\theta \\
&= \int_{\pi/6}^{5\pi/6} (150 - 200 \cos 2\theta - 100 \sin \theta) d\theta.
\end{aligned}$$

They intersect when $30 \sin \theta = 10 + 10 \sin \theta$ or By symmetry, this can also be calculated as

*Compare this problem to problem 10.4.27 in the text.

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} (150 - 200 \cos 2\theta - 100 \sin \theta) d\theta \\ &= [150\theta - 100 \sin 2\theta + 100 \cos \theta]_{\pi/6}^{\pi/2} \\ &= \left[150 \left(\frac{\pi}{2} - \frac{\pi}{6} \right) - 100 \left(\sin 2 \cdot \frac{\pi}{2} - \sin 2 \cdot \frac{\pi}{6} \right) + 100 \left(\cos \frac{\pi}{2} - \cos \frac{\pi}{6} \right) \right] \\ &= 50\pi. \end{aligned}$$