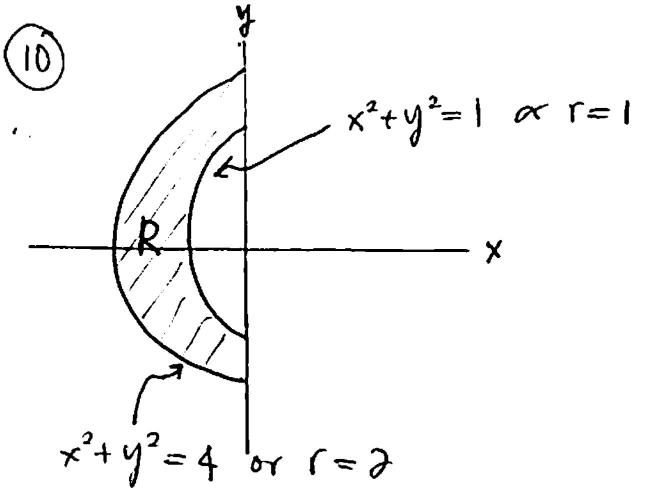
Solutions to some problems from section 15.4

(10)



$x = r\cos\theta$
$y = r\sin\theta$

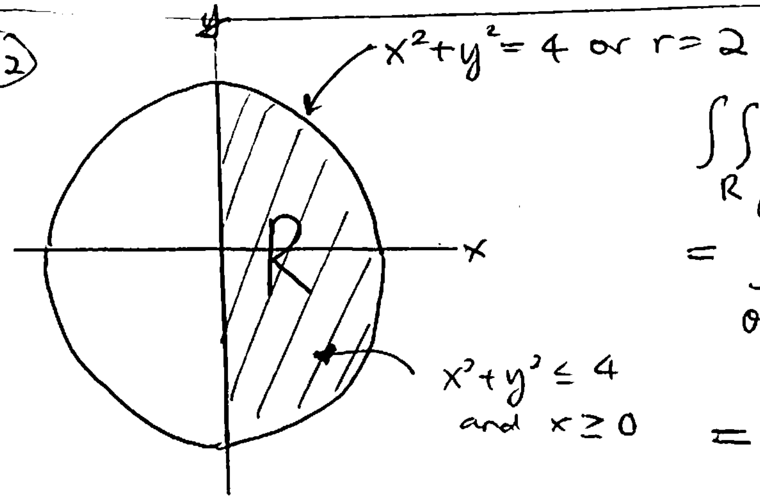
$$\iint_R (x+y)\,dA$$

$$= \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \int_{r=1}^{r=2} \underbrace{r(\cos\theta+\sin\theta)}_{x+y}\,\underbrace{r\,dr\,d\theta}_{\text{“}dA\text{”}}$$

$$= \int_{r=1}^{r=2} r^2\,dr \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} (\cos\theta+\sin\theta)\,d\theta$$

$$= -\frac{14}{3}$$

(12)



$x^2+y^2=r^2$

$$\iint_R \sqrt{4-x^2-y^2}\,dA$$

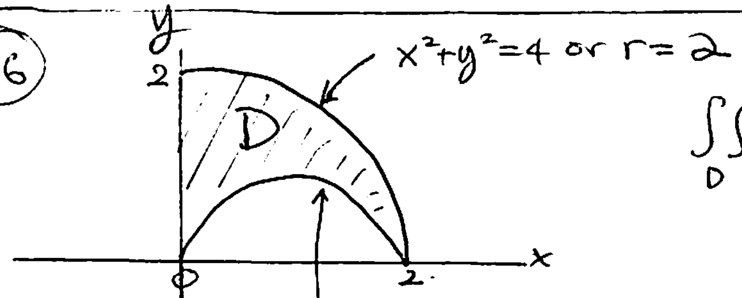
$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2} \underbrace{\sqrt{4-r^2}}_{\sqrt{4-x^2-y^2}}\ \underbrace{r\,dr\,d\theta}_{\text{“}dA\text{”}}$$

$$= \underbrace{\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} d\theta}\ \int_{r=0}^{r=2} r\sqrt{4-r^2}\,dr$$

$$= \Big[\theta\Big]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left[-\frac{1}{2}\cdot\frac{2}{3}\left(4-r^2\right)^{\frac{3}{2}}\right]_{r=0}^{2}$$

$$= \frac{8}{3}\pi$$

(16)



$x^2+y^2 = 2x$
$x^2-2x+y^2 = 0$
$(x-1)^2+y^2 = 1$

$x^2+y^2=2x$
or $r^2 = 2r\cos\theta$
or $r = 2\cos\theta$

$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4,\\ x\ge 0,\ y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1,\\ y\ge 0}} x\,dA$$

$$= \iint_{\substack{0\le r\le 2,\\ 0\le\theta\le\frac{\pi}{2}}} x\,dA - \int_{\substack{0\le r\le 2\cos\theta,\\ 0\le\theta\le\frac{\pi}{2}}} x\,dA$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2} \underbrace{r\cos\theta}_{x}\ \underbrace{r\,dr\,d\theta}_{\text{“}dA\text{”}} - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r\cos\theta\ r\,dr\,d\theta$$

(16) Continued from last page

$$\iint_D x\,dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_{r=0}^{2} \cos\theta\,d\theta - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_{r=0}^{r=2\cos\theta} \cos\theta\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{8}{3}\cos\theta\,d\theta - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{8}{3}\cos^4\theta\,d\theta$$

$$= \left[\frac{8}{3}\sin\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} - \frac{8}{3}\int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\right) d\theta$$

$$= \frac{8}{3} - \frac{8}{3}\left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \frac{8}{3} - \frac{\pi}{2}$$

<u>Exercise</u>: Use the identity $\cos 2\gamma = 2\cos^2\gamma - 1$ to show $\cos^4\theta = \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$.

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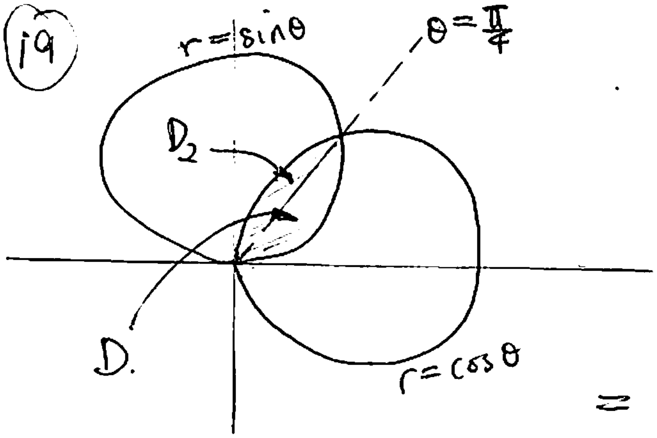
(19)



$\sin\theta = \cos\theta$

$\tan\theta = 1$

$\theta = \pm\frac{\pi}{4}, \pm\frac{5\pi}{4}, \ldots$

From the picture, $\theta = +\frac{\pi}{4}$

Area of D (where $D = D_1 \cup D_2$)

$$= \iint_{D_1} 1\,dA + \iint_{D_2} 1\,dA$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=\sin\theta} 1\cdot r\,dr\,d\theta + \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\cos\theta} 1\cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{\sin^2\theta}{2}\,d\theta + \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{\cos^2\theta}{2}\,d\theta$$

$$= \frac{\pi-2}{16} + \frac{\pi}{8} - \frac{\pi+2}{16}$$

$$= \frac{\pi-2}{8}$$

or, from symmetry

Area of D

$$= 2\iint_{D_1} 1\,dA$$

$$= 2\int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=\sin\theta} 1\cdot r\,dr\,d\theta$$

$$= 2\left(\frac{\pi-2}{16}\right)$$

$$= \frac{\pi-2}{8}$$

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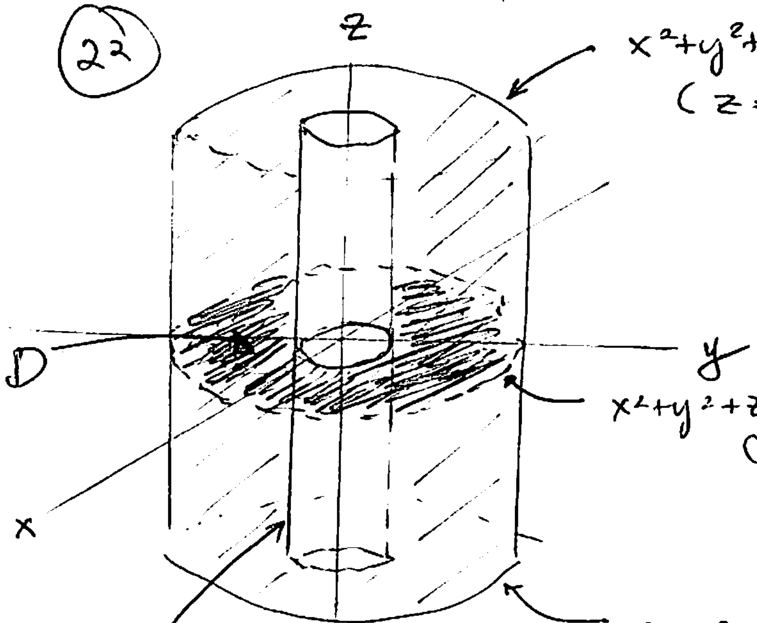

$x^2+y^2+z^2=16,\ z\geq 0$
$(z=\sqrt{16-x^2-y^2})$

$x^2+y^2+z^2=16$ intersects with the xy-plane $(z=0)$
$(x^2+y^2=16)$
$(r=4)$

$x^2+y^2+z^2=16,\ z\leq 0$
$(z=-\sqrt{16-x^2-y^2})$

$x^2+y^2=4\ (r=2)$

By symmetry, the volume is given by

$$2\iint_D \sqrt{16-x^2-y^2}\,dA = 2\int_{\theta=0}^{2\pi}\int_{r=2}^{r=4}\sqrt{16-r^2}\,r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi} d\theta \int_{r=2}^{r=4} r\sqrt{16-r^2}\,dr$$

$$= 32\sqrt{3}\,\pi$$

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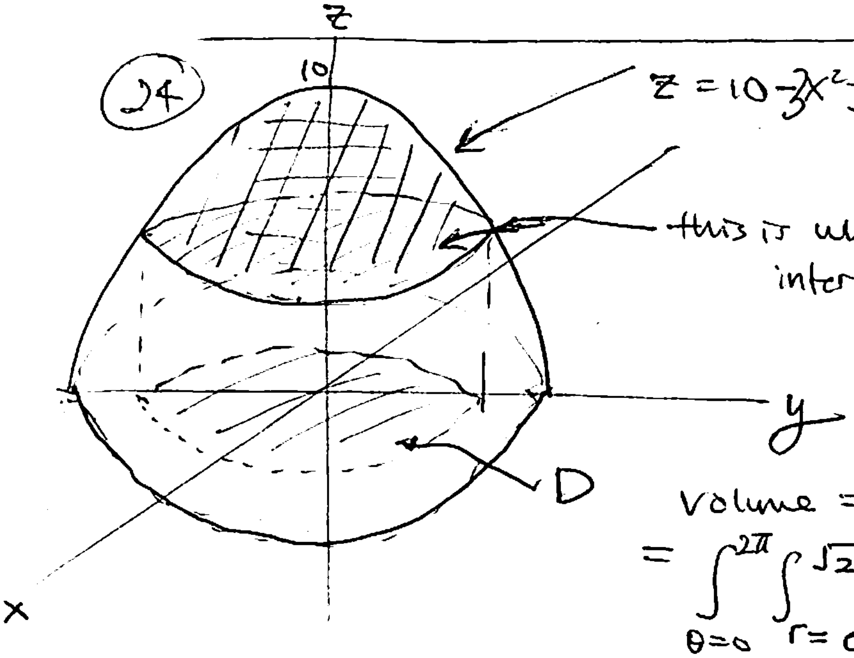


$z=10-3x^2-3y^2$ paraboloid

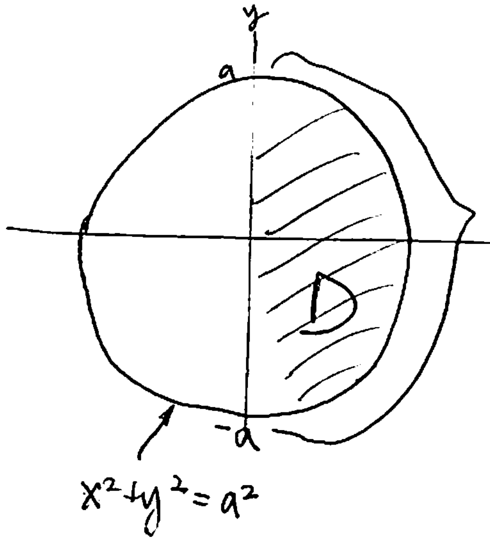
this is where the paraboloid intersects the plane $z=4$
$(4=10-3x^2-3y^2$, so
$x^2+y^2=2)$

$$\text{volume} = \iint_D [10-3x^2-3y^2] - 4\,dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} (6-3r^2)\,r\,dr\,d\theta$$

$$= 6\pi$$

(30) $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} (x^2+y^2)^{3/2}\,dx\,dy = \iint_D (x^2+y^2)^{3/2}\,dA$



$x^2+y^2=a^2$
or $x=\sqrt{a^2-y^2}$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{a} (r^2)^{3/2}\, r\,dr\,d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{a} r^4\,dr\,d\theta$$

$$= \frac{\pi a^5}{5} \quad (\text{assuming } a>0)$$

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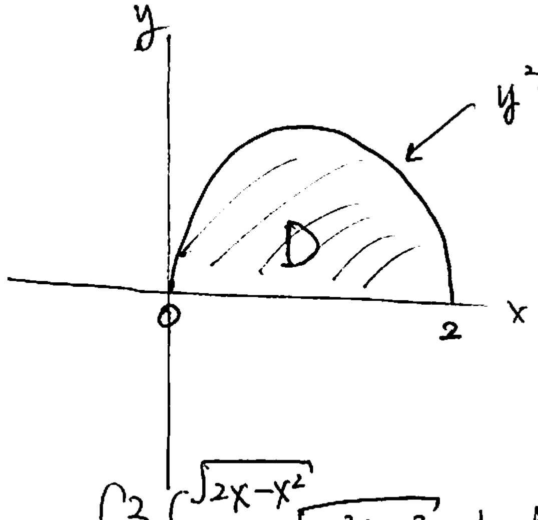
(32) $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = ?$

$$\begin{cases} y^2 = 2x - x^2 \\ y^2 + x^2 - 2x = 0 \\ y^2 + (x-1)^2 = 1 \end{cases}$$



$$y^2 = 2x - x^2$$
$$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$$
$$r^2(\cos^2\theta+\sin^2\theta) = 2r\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

$$= \iint_D \sqrt{x^2+y^2}\,dA = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2\cos\theta} r\cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\,d\theta$$

$$= \frac{8}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{\theta=0}^{\frac{\pi}{2}} = \frac{16}{9}$$