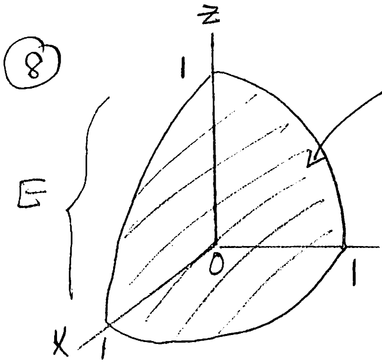
Section 15.8: Triple Integrals in Cylindrical and Spherical Coordinates

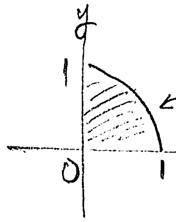
(8)



$z = 1 - x^2 - y^2$ or $z = 1 - r^2$

$$\iiint_E (x^3 + xy^2)\, dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, \overbrace{r\, dz\, dr\, d\theta}^{dV}$$



when $z = 0$, $x^2 + y^2 = 1$ or $r = 1$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^4 \cos\theta\, \underbrace{(\cos^2\theta + \sin^2\theta)}_{1}\, dz\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^4 \cos\theta\, dz\, dr\, d\theta$$

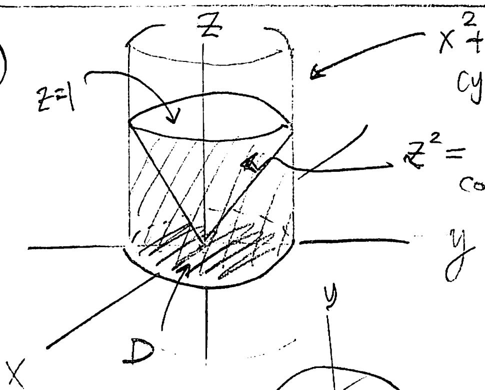
$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^4(1-r^2)\cos\theta\, dr\, d\theta$$

$$= \frac{2}{35}$$

$z = z$	$x = r\cos\theta$
$r^2 = x^2 + y^2$	$y = r\sin\theta$

---

(11)



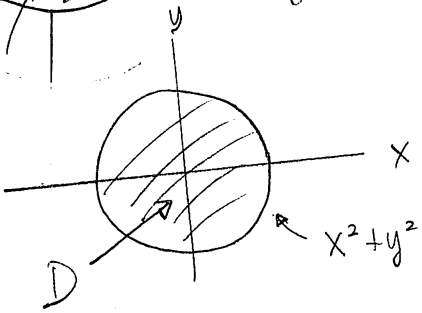
$x^2 + y^2 = 1$ cylinder

$z^2 = 4x^2 + 4y^2$, $z \geq 0$ cone

intersect when $z^2 = 4$ or $z = 2$

$$\iiint_E x^2\, dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4r^2}} r^2\cos^2\theta\, \overbrace{r\, dz\, dr\, d\theta}^{dV}$$

$$= \frac{2\pi}{5}$$

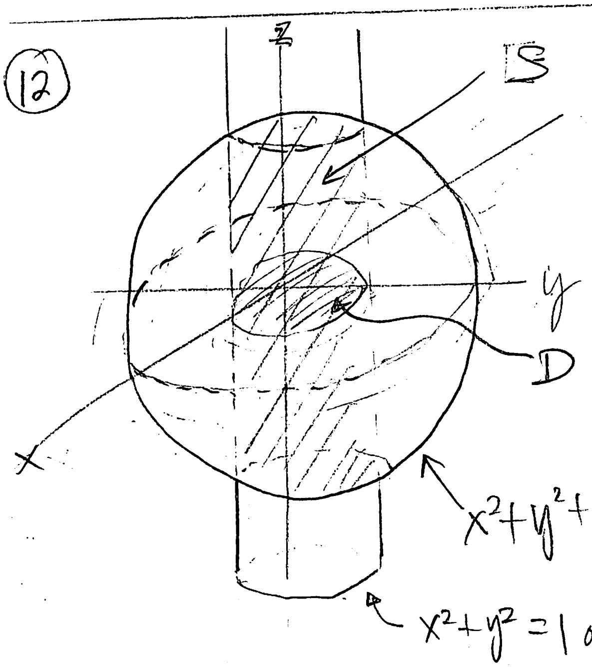


$x^2 + y^2 = 1$

Note: The identity is needed at some point: $\cos 2\theta = 2\cos^2\theta - 1$

---

(12)



$x^2 + y^2 + z^2 = 4$ or $r^2 + z^2 = 4$

$x^2 + y^2 = 1$ or $r = 1$

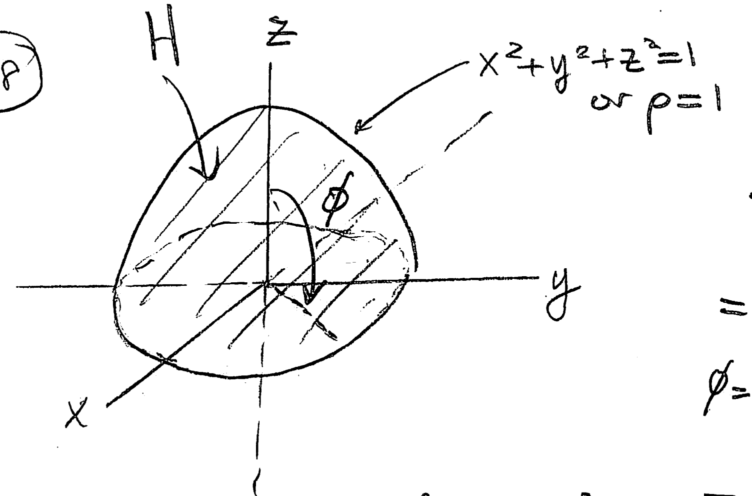
$$V(S) = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

or by symmetry

$$V(S) = 2\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

$$= \frac{4}{3}\pi(8 - 3^{3/2})$$

(18)



$$\iiint_H (x^2+y^2)\,dV$$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \rho^2 \sin^2\phi \; \underbrace{\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}_{dV}$$

$$= \frac{4\pi}{15}$$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$x^2+y^2+z^2 = \rho^2$

$x^2+y^2 = \rho^2 \sin^2\phi$

<u>Note:</u> To find $\int \sin^3\phi \, d\phi$, write

$$\int \sin^3\phi \, d\phi = \int \sin^2\phi \sin\phi \, d\phi$$

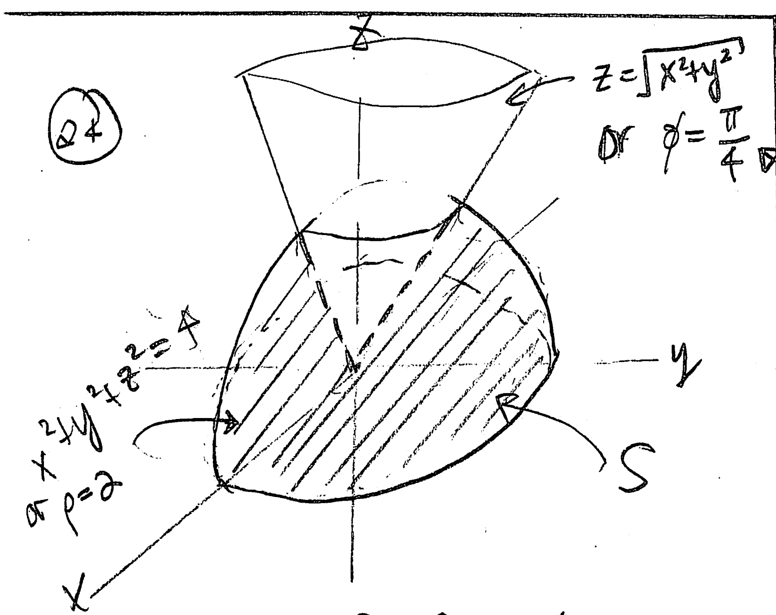
$$= \int (1-\cos^2\phi)\sin\phi \, d\phi$$

$$= \int (\sin\phi - \cos^2\phi \sin\phi)\, d\phi$$

$$= -\cos\phi - \underbrace{\int \cos^2\phi \sin\phi \, d\phi}_{\text{Use substitution } u=\cos\phi}$$

$\vdots$

(24)



$$z = \sqrt{x^2+y^2}$$

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi}$$

$$\rho\cos\phi = \rho|\sin\phi|$$

$$\cos\phi = \sin\phi \longleftarrow \text{for } 0 \le \phi \le \frac{\pi}{2}$$

$$\tan\phi = 1$$

$$\phi = \frac{\pi}{4}$$

$$V(S) = \iiint_S dV$$

$$= \int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{8\sqrt{2}\pi}{3}$$