Homework #1 (due Jan 15)

14.2.15

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \quad (3)$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(x^2+y^2+1)-1} \quad (3)$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} (\sqrt{x^2+y^2+1}+1) \quad (2)$$

$$= \sqrt{1}+1$$

$$= 2 \quad (2)$$

A student has an alternative calculation.

Let $x = r\cos\theta$, $y = r\sin\theta$ then $r^2 = x^2+y^2$

So $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

$$= \lim_{r\to 0} \frac{r^2}{\sqrt{r^2+1}-1}$$

$$\overset{\text{l'Hôpital}}{=} \lim_{r\to 0} \frac{2r}{r(r^2+1)^{-\frac{1}{2}}}$$

$$= \lim_{r\to 0} 2\sqrt{r^2+1} = 2.$$

14.3.51

$$u = e^{-s}\sin t$$

(1) $u_s = -e^{-s}\sin t = -u$ $\quad u_{st} = -e^{-s}\cos t$

(1) $u_{ss} = e^{-s}\sin t = u$ $\quad$ (1)

(1) $u_t = e^{-s}\cos t$ $\quad u_{ts} = -e^{-s}\cos t$

(1) $u_{tt} = -e^{-s}\sin t = -u$

The proof that $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = 2$ using the δ-ϵ definition

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For $(x,y)\neq(0,0)$, $\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \sqrt{x^2+y^2+1}+1.$

Now, let $\epsilon>0$ be given. We need to find a $\delta>0$ (that depends on ϵ)

such that if $0<\underbrace{\sqrt{(x-0)^2+(y-0)^2}}_{\sqrt{x^2+y^2}}<\delta$ then $\left|\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}-2\right|<\epsilon$

To this end, note that

$$\left|\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}-2\right| = \left|\sqrt{x^2+y^2+1}+1-2\right|$$

$$= \left|\sqrt{x^2+y^2+1}-1\right|$$

$$= \sqrt{x^2+y^2+1}-1 \quad \text{since } x^2+y^2+1>1$$

So $\left|\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}-2\right|<\epsilon$ if and only if $\sqrt{x^2+y^2+1}-1<\epsilon$

if and only if $x^2+y^2<(\epsilon+1)^2-1=\epsilon^2+2\epsilon$ if and only if

$\sqrt{x^2+y^2}<\sqrt{\epsilon^2+2\epsilon}$. This shows that if we choose δ to be

any positive number such that $0<\delta\le\sqrt{\epsilon^2+2\epsilon}$, then

$0<\sqrt{x^2+y^2}<\delta$ (so that $0<\sqrt{x^2+y^2}<\sqrt{\epsilon^2+2\epsilon}$

implies that $\left|\dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}-2\right|<\epsilon$. end of proof.