Homework #2

14.5.39 At that instant t_0, increasing $\left.\frac{dl}{dt}\right|_{t_0} = \left.\frac{dw}{dt}\right|_{t_0} = + 2\,m/s$ (1), decreasing $\left.\frac{dh}{dt}\right|_{t_0} = - 3\,m/s$ (2)

$l = 1\,m, \quad w = h = 2\,m$

(a) $V = \text{volume} = lhw$ these are evaluated at $l(t_0), h(t_0), w(t_0)$.

$$\frac{dV}{dt} = V_l \left.\frac{dl}{dt}\right|_{t_0} + V_h \left.\frac{dh}{dt}\right|_{t_0} + V_w \left.\frac{dw}{dt}\right|_{t_0} \quad (1)$$

$$= hw(2) + lw(-3) + lh(2) \quad (2)$$

$$= (2)(2)(2) + (1)(2)(-3) + (1)(2)(2)$$

$$= 6\ m^3/s \quad (1)$$

(b) $S = \text{surface area} = 2(lh + lw + hw)$ these are evaluated at $l(t_0), h(t_0), w(t_0)$

$$\frac{dS}{dt} = S_l \left.\frac{dl}{dt}\right|_{t_0} + S_h \left.\frac{dh}{dt}\right|_{t_0} + S_w \left.\frac{dw}{dt}\right|_{t_0} \quad (1)$$

$$= 2(h+w)(2) + 2(l+w)(-3) + 2(l+h)(2) \quad (2)$$

$$= 2(2+2)(2) + 2(1+2)(-3) + 2(1+2)(2)$$

$$= 10\ m^2/s \quad (1)$$

(c) $L = \text{length of a diagonal}$ these are evaluated at $l(t_0), h(t_0), w(t_0)$

$$= \sqrt{h^2 + w^2 + l^2}$$

$$\frac{dL}{dt} = L_l \left.\frac{dl}{dt}\right|_{t_0} + L_h \left.\frac{dh}{dt}\right|_{t_0} + L_w \left.\frac{dw}{dt}\right|_{t_0} \quad (1)$$

$$= \frac{1}{\sqrt{h^2+w^2+l^2}} \left(l \left.\frac{dl}{dt}\right|_{t_0} + h \left.\frac{dh}{dt}\right|_{t_0} + w \left.\frac{dw}{dt}\right|_{t_0} \right) \quad (2)$$

this is also evaluated at $l(t_0), h(t_0), w(t_0)$

$$= \frac{1}{\sqrt{2^2+2^2+1^2}} \left[(1)(2) + (2)(-3) + (2)(2) \right]$$

$$= 0\ m/s \quad (1)$$

Alternatively

$$F = L^2 = h^2 + w^2 + l^2$$

$$\frac{dF}{dt} = 2\left(l\frac{dl}{dt} + h\frac{dh}{dt} + w\frac{dw}{dt} \right)$$

$$= 2\left[(1)(2) + (2)(-3) + (2)(2) \right]$$

$$= 0\ m^2/s$$

But $\frac{dF}{dt} = 2L\frac{dL}{dt}$ also

So $\frac{dL}{dt} = 0\ m/s$.

because $L \neq 0$ at that instant.

Warning !!

Geometrically, this means that even if two curves have the same tangent line at a point, the two curves can differ!!

Since the rates of change of l, h, and w are only given at a particular instant of time $t=t_0$, so that we have

$$\left.\frac{dl}{dt}\right|_{t_0} = \left.\frac{dw}{dt}\right|_{t_0} = 2, \qquad \left.\frac{dh}{dt}\right|_{t_0} = -3$$

it is incorrect to deduce from this that l, w, and h depend on t linearly, i.e.,

$$\left.\begin{aligned} l(t) &= 2t + l(0), \\ w(t) &= 2t + w(0), \\ h(t) &= -3t + h(0). \end{aligned}\right\} \quad (*)$$

Equations $(*)$ hold only if the rates of change are constant, i.e., $\frac{dl}{dt} = \frac{dw}{dt} = 2$ and $\frac{dh}{dt} = -3$ for all instants of time t.

A counterexample: Take $l = 2t^2 - 6t + 5$, $w = 2\sin(t-2) + 2$, $h = 2e^{-\frac{3}{2}(t-2)}$

Then at $t = t_0 = 2$,

$$l(2) = 1 \text{ m}, \quad w(2) = 2 \text{ m}, \quad h(2) = 2 \text{ m},$$

$$\left.\frac{dl}{dt}\right|_{t=2} = 4(2) - 6 = 2 \text{ m/s}, \quad \left.\frac{dw}{dt}\right|_{t=2} = 2\cos(2-2) = 2 \text{ m/s},$$

and

$$\left.\frac{dh}{dt}\right|_{t=2} = -3e^{-\frac{3}{2}(2-2)} = -3 \text{ m/s}$$

and clearly these also satisfy the hypothesis of the exercise.