Math 415A | Sec 3.4 : Complex Roots of the Characteristic Equations

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$y'' - 2y' + 6y = 0$

$r^2 - 2r + 6 = 0$

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-20}}{2}$$

$$= \frac{2 \pm \sqrt{(-1)(4)(5)}}{2}$$

$$= \frac{2 \pm 2i\sqrt{5}}{2}$$

$$= 1 \pm i\sqrt{5}$$

$$y = C_1 e^{1t} \cos \sqrt{5}\, t + C_2 e^{1 \cdot t} \sin \sqrt{5}\, t$$

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$y'' + 6y' + 13y = 0$

$r^2 + 6r + 13 = 0$

$$r = \frac{-6 \pm \sqrt{6^2 - 4(13)}}{2}$$

$$= \frac{-3 \pm \sqrt{-16}}{2}$$

$$= \frac{-3 \pm \sqrt{(-1)(16)}}{2}$$

$$= -3 \pm i\,\frac{4}{2}$$

$$= -3 \pm 2i$$

$$y = C_1 e^{-3t} \cos 2t + C_2 e^{-3t} \sin 2t$$

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$4y'' + 9y = 0$

$4r^2 + 9 = 0$

$$r^2 = -\frac{9}{4}$$

$$= (-1)\left(\frac{9}{4}\right)$$

$$r = \pm\sqrt{(-1)\left(\frac{9}{4}\right)}$$

$$= \pm i\,\frac{3}{2}$$

$$= 0 \pm \frac{3}{2} i$$

$$y = C_1 e^{0t} \cos \tfrac{3}{2} t + C_2 e^{0t} \sin \tfrac{3}{2} t$$

$$= C_1 \cos \tfrac{3}{2} t + C_2 \sin \tfrac{3}{2} t$$

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$y'' + 4y = 0, \; y(0) = 0, \; y'(0) = 1$

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$$y = C_1 \cos 2t + C_2 \sin 2t$$

$y(0) = 0 : \quad C_1 = 0$

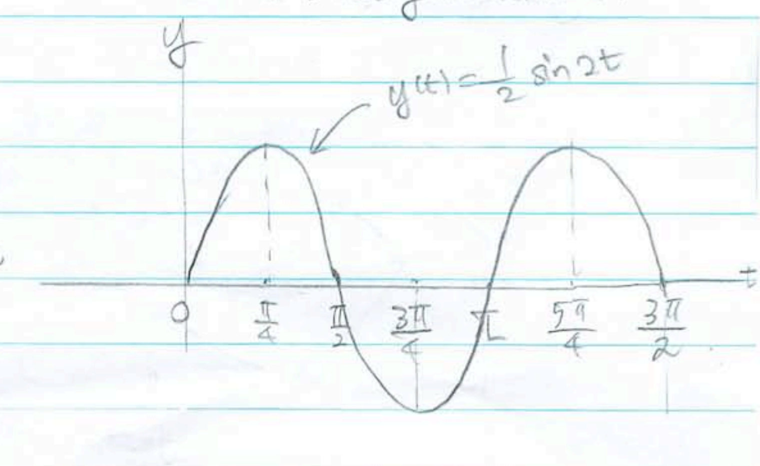
So $y = C_2 \sin 2t$

$y' = 2C_2 \cos 2t$

$y'(0) = 1 : \quad 2C_2 = 1, \; C_2 = \frac{1}{2}$

Thus $y = \frac{1}{2} \sin 2t$,

a steady oscillation.



18 $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$

$r^2 + 4r + 5 = 0$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2}$$

$$= -2 \pm i$$

$$y = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$$

$y(0) = 1$: $C_1 = 1$

So $y = e^{-2t} \cos t + C_2 e^{-2t} \sin t$

$$y' = -2e^{-2t} \cos t - e^{-2t} \sin t - 2C_2 e^{-2t} \sin t + C_2 e^{-2t} \cos t$$

$y'(0) = 0$: $-2 + C_2 = 0$

$$C_2 = 2$$

Thus $y = e^{-2t} \cos t + 2e^{-2t} \sin t$

$$= \underbrace{e^{-2t}}_{\text{Contributes to the decay}} \underbrace{(\cos t + 2 \sin t)}_{\text{contributes to oscillation}}$$

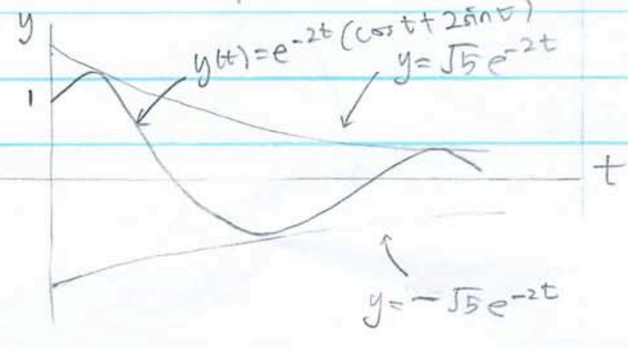
So y is a decaying oscillation

In fact,

$$y = e^{-2t} \sqrt{5} \cos(t - \tau)$$

where $\tau = \tan^{-1} 2$.

(We did this in problem 2.3.18)



19 $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.

$r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{2^2 - 4(5)}}{2} = 1 \pm 2i$$

$$y = C_1 e^t \cos 2t + C_2 e^t \sin 2t$$

$y(\frac{\pi}{2}) = 0$:

$$C_1 e^{\frac{\pi}{2}} \underbrace{\cos \pi}_{-1} + C_2 e^{\frac{\pi}{2}} \underbrace{\sin \pi}_{0} = 0$$

So $C_1 = 0$.

$$y = C_2 e^t \sin 2t$$

$$y' = C_2 e^t \sin 2t + 2C_2 e^t \cos 2t$$

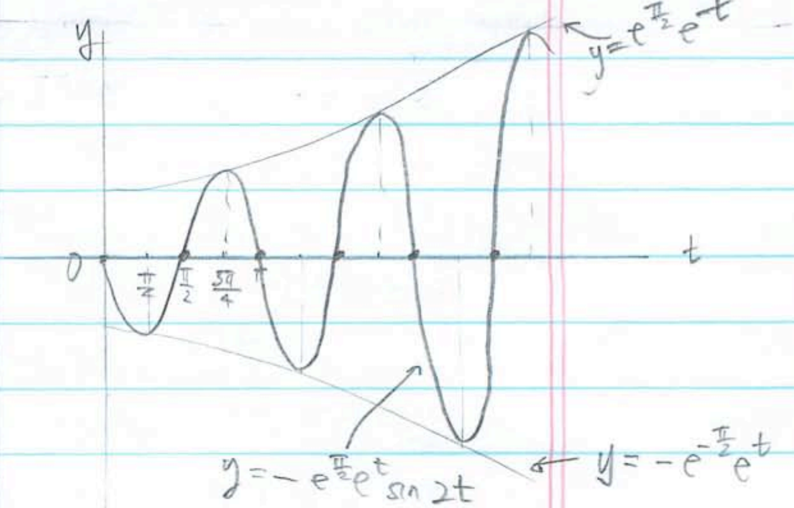
$y'(\frac{\pi}{2}) = 2$:

$$C_2 e^{\frac{\pi}{2}} \sin \pi + 2C_2 e^{\frac{\pi}{2}} \underbrace{\cos \pi}_{-1} = 2$$

$$C_2 = -e^{-\frac{\pi}{2}}$$

Thus $y = -e^{-\frac{\pi}{2}} \underbrace{e^t}_{\text{contributes to the growth}} \underbrace{\sin 2t}_{\text{contributes to the oscillation}}$

So y is a growing oscillation.



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$\ast$ $\boxed{22}$ $y'' + 2y' + 2y = 0$,
$y(\frac{\pi}{4}) = 2$, $y'(\frac{\pi}{4}) = -2$

$r^2 + 2r + 2 = 0$

$r = \dfrac{-2 \pm \sqrt{2^2 - 4(2)}}{2}$

$= -1 \pm i$

$y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$

$y(\frac{\pi}{4}) = 2$: $C_1 e^{-\frac{\pi}{4}} \underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} + C_2 e^{-\frac{\pi}{4}} \underbrace{\sin\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} = 2$

So $C_1 + C_2 = 2\sqrt{2}\, e^{\frac{\pi}{4}}$ (1)

$y' = -C_1 e^{-t}\cos t - C_1 e^{-t}\sin t$
$\quad - C_2 e^{-t}\sin t + C_2 e^{-t}\cos t$

$y'(\frac{\pi}{4}) = -2$:
$- C_1 e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} - C_1 e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}$
$- C_2 e^{-\frac{\pi}{4}}\sin\frac{\pi}{4} + C_2 e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -2$ (2)

So $C_1 = \sqrt{2}\, e^{\frac{\pi}{4}}$

Using (1) gives

$C_2 = \sqrt{2}\, e^{\frac{\pi}{4}}$ also

Thus $y = \sqrt{2}\, e^{\frac{\pi}{4}} \underbrace{e^{-t}}_{\text{contributes to the decay}} \underbrace{(\cos t + \sin t)}_{\text{contributes to oscillation}}$

Thus y is a decaying oscillation

Set

$\cos t + \sin t$
$= R\cos(t - \tau)$
$= R\cos\tau\cos t + R\sin\tau\sin t$

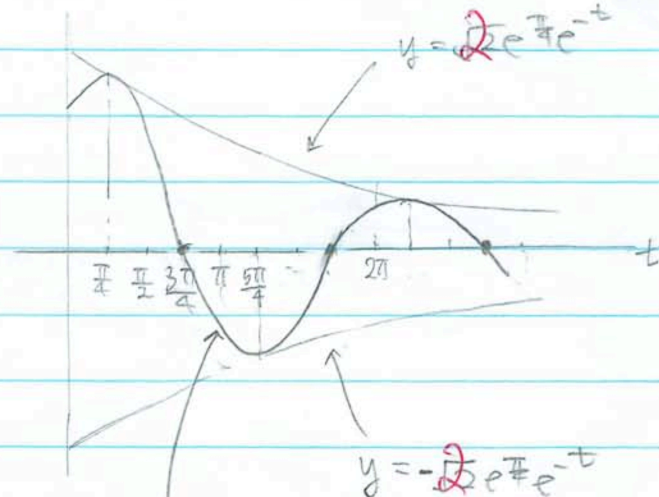
yields

$\left.\begin{array}{l} R\cos\tau = 1 \\ R\sin\tau = 1 \end{array}\right\}$ $R = \sqrt{2}$, $\tau = \tan^{-1} 1 = \frac{\pi}{4}$

$R^2\cos^2\tau = 1^2$
$R^2\sin^2\tau = 1^2$
So $R^2 = 1 + 1 = 2$
$R = \sqrt{2}$

Thus we can also write

$y = \cancel{\sqrt{2}}\, 2\, e^{\frac{\pi}{4}} e^{-t}\cos(t - \frac{\pi}{4})$



$y = \sqrt{2}\, e^{\frac{\pi}{4}} e^{-t}(\cos t + \sin t)$
$= 2\, e^{\frac{\pi}{4}} e^{-t}\cos(t - \frac{\pi}{4})$