

10.7.9 $a^2 u_{xx} = u_{tt}, 0 < x < L, t > 0$

BCs $\left\{ \begin{array}{l} u(0,t) = 0 \\ u_x(L,t) = 0 \end{array} \right\} t \geq 0$

ICs $\left\{ \begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{array} \right.$

(1) $u(x,t) = \sum X(x) T(t)$

$a^2 \sum X'' T = \sum X T''$

$X'' + \lambda X = 0$ (1) $\frac{1}{2}$

$T'' + a^2 \lambda T = 0$ (2) $\frac{1}{2}$

(2) $\sum(0)T(t) = u(0,t) = 0 \Rightarrow \sum(0) = 0$ $\frac{1}{2}$
 $\sum'(L)T(t) = u_x(L,t) = 0 \Rightarrow \sum'(L) = 0$ $\frac{1}{2}$

So nonzero solutions of (1) are

$\sum_n(x) = \sin \frac{(2n-1)\pi x}{2L}, n=1,2,\dots$ $\frac{1}{2}$

with eigenfunctions with eigenvalues $\lambda_n = \left[\frac{(2n-1)\pi}{2L} \right]^2$ $\frac{1}{2}$

(3) $T'' + a^2 \lambda_n T = 0$

\vdots

$T_n(t) = k_1 \cos a\sqrt{\lambda_n}t + k_2 \sin a\sqrt{\lambda_n}t$

$u_t(x,0) = \sum X(x) T'(0) = 0 \rightarrow T'(0) = 0 \rightarrow k_2 = 0$ $\frac{1}{2}$

$T_n(t) = \cos \frac{(2n-1)\pi a t}{2L}$ $\frac{1}{2}$

$n=1,2,\dots$

(4) $u_n(x,t) = \sum X(x) T_n(t)$

$= \sin \frac{(2n-1)\pi x}{2L} \cos \frac{(2n-1)\pi a t}{2L}$ $\frac{1}{2}$

(5) $u(x,t) = \sum_{n=1}^{\infty} C_n u_n(x,t)$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{(2n-1)\pi x}{2L}$

for this to converge, one takes

$C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx$ $\frac{1}{2}$

(See problem 10.4.39)

