Test 1, Math 415, Tanveer
Instructions: Closed book and notes. Show work

1a. Find solution $y(t)$ to the initial value problem:

$$y' = -3 \ \frac{y}{t} \ + \ \frac{\sin t}{t^3} \ ; \ y(1) \ = 2$$

Solution: In the standard form:

$$y' + \frac{3}{t} y = \frac{\sin t}{t^3}$$

IF $\mu = \exp\left[\int \frac{3}{t} dt\right] = e^{3 \ln t} = e^{\ln t^3} = t^3$. So,

$$t^3 y' + 3t^2 y = \sin t \ , \ \text{implies} \ \frac{d}{dt}\left[t^3 y\right] = \sin t$$

Integration gives

$$t^3 y = -\cos t + C$$

Using Initial conditions $2 = -\cos 1 + C$. So, $C = 2 + \cos 1$. So, solution to IVP

$$y = -\frac{\cos t}{t^3} + \frac{2 + \cos 1}{t^3}$$

1b. Determine largest t-interval over which solution in (1a) is valid ? State with reason if this interval can be predicted without actually finding solution.

Solution: On inspection solution in (1a) not valid at $t = 0$ where it blows up. So, largest interval containing initial $t_0 = 1$ over which solution is valid is $(0, \infty)$.

This can be predicted without solving for y since the largest t interval containing $t_0 = 1$ for which both $p(t) = \frac{3}{t}$ and $g(t) = \frac{\sin t}{t^3}$ are continuous is $(0, \infty)$. So, according to Theorem for linear 1st order ODE, this problem has a unique solution in that interval.

2a. Solve the initial value problem for $y(x)$

$$y' + \frac{3x^2}{2y} = 0 \ ; \ y(0) = 1$$

Solution: Note that this is separable and equation implies

$$2ydy = -3x^2 dx \ , \ \text{therefore } y^2 = -x^3 + C$$

Since $y(0) = 1$, $C = 1$. Therefore, solution to IVP

$$y = \sqrt{1 - x^3} \ , \ x < 1$$

Note only positive root relevant to satisfy $y(0) = 1$.

2b. Solve the ODE in (2a) with initial condition $y(1) = 0$ and find $y(x)$ explicitly. Determine if the solution is unique and if this is expected from known theorem(s).

Solution: As in (2a) general solution is $y^2 = -x^3 + C$. Since $y(1) = 0$, it follows that $C = 1$. So, solution is

$$y = \pm\sqrt{1 - x^3}$$

Notice both signs are possible since each satisfies $y(1) = 0$. So, solution is not unique in this case.

This is not unexpected from existence uniqueness Theorem about nonlinear first order ODE since in this case

$$y' = f(x, y) = -\frac{3x^2}{2y}$$

is not continuous at $y = 0$ (in this case f_y also fails to be continuous at $y = 0$, but theorem is invalid when either f or f_y is discontinuous.)

3. Suppose there are 10,000 fish in a pond initially. Assume new fish is born at a rate proportional to the existing population with birth rate of 0.5 baby fish/fish/year. A fisherman catches 200 fish per month. (a) What will be the fish population after 2 years? (b) What happens to fish population eventually?

Solution: Call the fish population $p(t)$ at time t (measured in years). Then, according to assumption $p(0) = 10,000$.

$$\frac{dp}{dt} = \frac{p}{2} - (200)(12) = \frac{p}{2} - 2400$$

So,

$$\frac{dp}{dt} - \frac{p}{2} = 2400$$

Integrating factor is $e^{-t/2}$. So,

$$\frac{d}{dt}\left[e^{-t/2}p\right] = -2400e^{-t/2} \text{ , implying } e^{-t/2}p = 4800e^{-t/2} + C$$

implying $p(t) = 4800 + Ce^{t/2}$ Since $p(0) = 10,000$, $C = 5200$ and

$$p(t) = 4800 + 5200e^{t/2}$$

So $p(2) = 4800 + 5200e \approx 18935$. Also, note population grows exponentially with t for large t.

4a. Find the solution to the initial value problem (IVP)

$$y'' + y' - 6y = 0\ ,\ \text{with } y(0) = 2\ ,\ y'(0) = 1$$

Is the solution unique?

Solution: Note that this is a constant coefficient linear homogenenous equation, with characteristic equation

$$r^2 + r - 6 = 0\ ,\ \text{implying } (r+3)(r-2) = 0\ ,\text{implying } r = -3,\ r = 2$$

Therefore, two independent solutions are $y_1 = e^{-3t}$, $y_2 = e^{2t}$. General solution is

$$y = c_1 e^{-3t} + c_2 e^{2t}\ ,\ \text{Note} : y' = -3c_2 e^{-3t} + 2c_2 e^{2t}$$

$$\text{So, } y(0) = 2 = c_1 + c_2\ ,\ y'(0) = 1 = -3c_1 + 2c_2$$

From the two equations, eliminating c_2, we have

$$5c_1 = 3\ ,\ \text{implying } c_1 = \frac{3}{5}$$

Therefore, $c_2 = 2 - c_1 = \frac{7}{5}$ and solution to IVP is

$$y = \frac{3}{5}e^{-3t} + \frac{7}{5}e^{2t}$$

This is the only possible for the IVP since uniqueness theorem applies since coefficients (constants) are continuous in any interval.

4b. Verify that t and t^2 are each solution to

$$t^2 y'' - 2ty' + 2y = 0,$$

and find the form of the general solution.

Solution: We note that for $y_1 = t$, $y_1' = 1$, $y_1'' = 0$, therefore

$$t^2 y_1'' - 2ty_1' + 2y_1 = -2t(1) + 2(t) = 0 = RHS$$

So, $y_1 = t$ is a solution. Again, if we call $y_2 = t^2$, we note $y_2' = 2t$, $y_2'' = 2$. Therefore,

$$t^2 y_2'' - 2ty_2' + 2y_2 = t^2(2) - 2t(2t) + 2(t^2) = 0 = RHS$$

Therefore, $y_2 = t^2$ is another solution. Now, checking to see if they are linearly independent, we have

$$W = y_1 y_2' - y_2 y_1' = (t)(2t) - t^2(1) = t^2 \neq 0$$

for any nonzero t. Hence the solutions y_1, y_2 are linearly independent. Therefore, general solution to the given second order linear homogeneous ODE is

$$y = C_1 t + C_2 t^2$$