Solution to Test 2, Math 415A, Tanveer

1a. Find solution $y(t)$ satisfying the initial value problem:

$$y'' + 4y' + 4y = 0 \ , \ y(0) = 0 \ , \ y'(0) = 1$$

Solution: Since constant coefficient, homogeneous solution, we look at the characterestic equation $r^2+4r+4 = 0$, or $(r+2)^2 = 0$ or $r = -2$ twice. Since roots are coincident, two independent solutions are e^{-2t} and te^{-2t}. So,

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Since $y(0) = 0$, implies $c_1 = 0$. Since $y'(t) = c_2 e^{-2t} - 2c_2 t e^{-2t}$, $y'(0) = 1 = c_2$. Therefore, solution to IVP

$$y(t) = te^{-2t}$$

1b. Find **form** of the particular solution in each of the two cases below (**Note:** not asking to find the coefficients):

i. $y'' + 4y' + 4y = e^{-2t}(1 + t^2)$

Solution: Since $r = -2$ is root of the characteristic equation twice, it follows that we must have

$$y_p = t^2 e^{-2t}(A_0 t^2 + A_1 t + A_2)$$

ii. $y'' + y = e^t \sin(t) + t$

Solution: Since $1 + i$ is not a root of characteristic equation $r^2 + 1 = 0$ (which has roots $\pm i$), it follows $y_{p,1} = A_0 e^t \sin t + B_0 e^t \cos t$ is a particular solution to

$$y'' + y = e^t \sin(t)$$

Again since $r = 0$ is not a root of the characteristic equation $r^2+1 = 0$, it follows that particular solution to

$$y'' + y = t$$

is of the form $y_{p,2} = A_1 t + B_1$. So particular solution to the problem (**ii**) is

$$y_p = y_{p_1} + y_{p_2} = A_0 e^t \sin t + B_0 e^t \cos t + A_1 t + B_1$$

2. A mass of 1 kgm stretches a spring downwards by 9.8 meters under the action of gravity. The mass is subjected to downward external force of $2\cos(3t)$ Newtons. Assume negligible friction, an initial downwards displacement of 0.25 m and no initial velocity. Determine displacement as a function of time.

Solution: Problem statement implies $k(9.8) = 9.8$; so $k = 1$. So equation for downward displacement is simply

$$u'' + u = 2\cos(3t)$$

Note homogeneous solutions are $\cos t$ and $\sin t$. So, seek a particular solution of the form

$$u_p = A\cos(3t) + B\sin(3t)$$

to obtain

$$(1-9)A\cos(3t) + (1-9)B\sin(3t) = 2\cos(3t)$$

So, $B = 0$, $A = -\frac{1}{4}$. General solution is therefore

$$u(t) = -\frac{1}{4}\cos(3t) + c_1\cos t + c_2\sin t$$

Since $u'(0) = 0$, it follows $c_2 = 0$. Also, $u(0) = 0.25$, implies

$$0.25 = -\frac{1}{4} + c_1 \text{ , implying } c_1 = 0.5$$

So,

$$u(t) = -\frac{1}{4}\cos(3t) + \frac{1}{2}\cos t$$

3a. Suppose

$$f(x) = x \text{ for } -0 \le x < \pi, \text{ and } f(x) = -x \text{ for } -\pi \le x < 0,$$

and $f(x + 2\pi) = f(x)$. Determine Fourier Series of f. What does the Fourier Series converge to at $x = \pi$?

Solution: $L = \pi$ since period is 2π. and $f(x)$ is an even function in $(-\pi, \pi)$; so $b_m = 0$ and for integer $m \ge 1$,

$$a_m = \frac{2}{\pi}\int_0^\pi x\cos(mx)dx = \frac{2}{\pi}\int_0^\pi x\cos(mx)dx,$$

Calculate indefinite integral:

$$\int x\cos(mx)dx = \frac{1}{m}\int xd\sin(mx) = \frac{x}{m}\sin(mx) - \frac{1}{m}\int \sin(mx)dx$$
$$= \frac{x}{m}\sin(mx) + \frac{1}{m^2}\cos(mx)$$

No contribution from sine term above at the end point $x = 0$ or $x = 2$ as it is zero. So,

$$a_m = \frac{2}{m^2\pi}\cos(mx)|_0^\pi = -\frac{2}{\pi m^2}[1 - \cos(m\pi)]$$

On the otherhand,

$$a_0 = \frac{2}{\pi}\int_0^\pi xdx = \pi$$

Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1,3,5..}^{\infty}\frac{1}{m^2}\cos(mx)$$

Periodic extension of f is continuous $x = \pi$, since $f(\pi^-) = 2$ and $f(\pi^+) = f(-\pi^+) = 2$, follows from theory that Fourier series converges at $x = \pi$ to $f(x) = \pi$.

3b. Express the following $f(x)$ as a Fourier sine series with period 4. Sketch the function to which Fourier series converges to in (-4, 4).

$$f(x) = 1\ ,\ \text{for } 0 < x < 1\ ,\ f(x) = 0\ ,\ \text{for } 1 < x < 2$$

Solution: In this case, we do odd extension followed by period 4 extension, as shown in Figure 1. This is what the Fourier sine series with period 4 converges pointwise (except at discontinuities, where it converges to mean value on two sides.)

Fourier since coefficients are

$$b_n = \int_0^1 \sin\frac{n\pi x}{2}dx = \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$$

So,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}$$

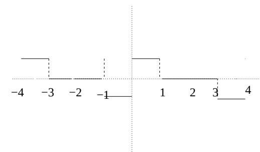

FIGURE 1. Odd extension and function to which Fourier sine series converges

4a. Consider the heat equation $u_t = u_{xx}$ for $0 < x < \pi$ for $t > 0$, with boundary conditions $u(0,t) = 0 = u_x(\pi, t)$. Use separation of variable to show that there are simple solutions $u(x,t) = X(x)T(t)$, with $X(x) = \sin\left[(n-\frac{1}{2})x\right]$ for integer $n \geq 1$. Determine corresponding $T(t)$?

Solution: Separation of variable leads to

$$X'' + \lambda X = 0 \ , \ X(0) = 0 \ = X'(\pi)$$
$$T' + \lambda T = 0$$

The first is an eigen value problem. In the range $\lambda = \beta^2 > 0$, general solution X to given ODE is

$$X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$$

Since $X(0) = 0$, implies $c_1 = 0$, so $X(x) = c_2 \sin(\beta x)$. $X'(\pi) = 0$, implies $c_2 \beta \cos(\beta \pi) = 0$. Nonzero c_2 possible, when $\beta\pi = (2n-1)\frac{\pi}{2}$ for integer $n \geq 1$. or for $\beta = \left(n - \frac{1}{2}\right)$. So, eigenvalue $\lambda = \left(n-\frac{1}{2}\right)^2 \equiv \lambda_n$ and corresponding eigenfunction $X(x) = \sin\left\{\left(n-\frac{1}{2}\right)x\right\}$ as given. So, solving ODE for T, $T(t) = c_n e^{-\lambda_n t}$.

4b. Use (**4a.**) to determine solution to heat equation $u_t = u_{xx}$ for $0 < x < \pi$, $t > 0$ satisfying boundary and initial conditions:

$$u(0,t) = 0 = u_x(\pi, t) \ , \ u(x,0) = \sin\frac{x}{2} - \frac{1}{4}\sin\frac{3x}{2}$$

Solution: Using a linear combinatin of solution as in **4a** we have a more general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin\left[\left(n - \frac{1}{2}\right)x\right]$$

$$u(x,0) = c_1 \sin\frac{x}{2} + c_2 \sin\frac{3x}{2} \ + \ .. = \sin\frac{x}{2} - \frac{1}{4}\sin\frac{3x}{2}$$

provided $c_1 = 1$, $c_2 = -\frac{1}{4}$ and all other $c_n = 0$. Using $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{9}{4}$,

$$u(x,t) = e^{-t/4}\sin\frac{x}{2} - \frac{1}{4}e^{-9t/4}\sin\frac{3x}{2}$$