**Quiz 1, Math 415, Tanveer, 25 minutes**
**Instructions: Closed book and notes. Show work. Each problem worth the same**

1. (a) Determine with reason if the following equation is linear or nonlinear ($x$ is the independent variable). (b) Determine the expression for the general solution.

$$y^{-2/3}y' = x$$

**Solution:** (a) In the standard form: $\frac{dy}{dx} - xy^{2/3} = 0$ is nonlinear equation since it is **not** of the form $y' + p(x)y = g(x)$ for known $p(x)$ and $g(x)$.

b) We note on separation of variable $y^{-2/3}dy = xdx$. So on integration, $3y^{1/3} = \frac{x^2}{2} + C$. So, general solution. $y = \left(\frac{x^2}{6} + \frac{C}{3}\right)^3$.

2. Solve the initial value problem

$$xy' = -y + e^x \quad , \quad y(1) = 2$$

**Solution:** In the standard form

$$y' + \frac{y}{x} = \frac{e^x}{x}$$

Integration factor $\mu = \exp\left[\int \frac{1}{x}dx\right] = \exp \ln|x| = |x|$. Since initial value is specified for $x > 0$, we will restrict ourselves to $x > 0$, in which case $\mu(x) = x$. So,

$$xy' + y = e^x \text{ implies } \frac{d}{dx}[xy] = e^x$$

Integrating, we have $xy = e^x + C$ Therefore, general solution is $y = \frac{e^x}{x} + \frac{C}{x}$. Since $y(1) = 2$, $2 = e + C$. So, $C = 2 - e$ and solution to initial value problem:

$$y = \frac{e^x}{x} + \frac{(2-e)}{x}$$

3. The rate of decay of a radioactive material is proportional to the amount present at any time. Derive a differential equation for the mass of a radioactive material. If it takes 1000 years for 1 gm to become 1/2 gram, determine the time elapsed before 1 gm becomes 1/5 gram.

**Solution:** Let the amount left at any time $t$ (measured in thousand year unit) be $m$ gms. $m$ is a function of $t$. Then, since decay rate is proportional to amount present at any time,

$$\frac{dm}{dt} = -rm$$

where $r$ is a constant. So, $\frac{dm}{dt} + rm = 0$. Integrating factor $\mu = e^{rt}$. So,

$$e^{rt}\left[\frac{dm}{dt} + rm\right] = 0 \quad , \quad \text{implying} \quad \frac{d}{dt}\left[e^{rt}m\right] = 0$$

So,

$$e^{rt}m = C$$

$m = Ce^{-rt}$. If $m(0) = 1$, $C = 1$. So, $m = e^{-rt}$. Since at the end of 1000 years, $m = 1/2 = e^{-r}$, it follows, $-r = \ln \frac{1}{2}$; implying $r = \ln 2 \approx 0.693/thousandyears$. Now, $\frac{1}{5} = e^{-\ln 2t}$. So, $\ln 5 = t \ln 2$. So, $t = \frac{\ln 5}{\ln 2}$ thousand years to decay to 1/5 gm.