Quiz 2, Math 415A, Tanveer, 25 minutes
Instructions: Closed book and notes. Show work. Each problem worth the same

1. For the equation
$$t^2y'' + 2ty' - 2y = 0,$$
one independent solution is $y_1(t) = t$. Use this to find a second independent solution and thereby determine general solution for $t > 0$.

Solution: $y_1(t) = t$. Seek second solution $y_2(t) = tv(t)$. Note $y_2' = tv' + v$, $y_2'' = tv'' + 2v'$. So,
$$t^2y'' + 2ty' - 2y = t^3v'' + 4t^2v' = 0$$
Substitute $v' = u$, Then $v'' = u'$ Equation above becomes
$$t^3u' + 4t^2u = 0 \quad \text{, implying } \frac{du}{u} = -\frac{4}{t}dt,$$
$$\ln u = -4\ln t + c \quad , \quad \text{implying } u = e^{-4\ln t}e^c = C/t^4$$
Without loss of generality, $C = 1$. So, $v' = \frac{1}{t^4} = t^{-4}$, so $v = -\frac{1}{3}t^{-3}$. A second solution is $v(t)y_1(t)$, which is $y_2 = -\frac{1}{3t^2}$. So, general solution is
$$y = c_1t - \frac{c_2}{3}t^{-2} = c_1t + \tilde{c}_2t^{-2}$$

2. Find general solution $y(t)$ for
$$y'' - y = t^2 + 1$$

Solution: Since characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$ and none of these are zero, we seek particular solution
$$y_p = A_0t^2 + A_1t + A_2$$
$$y_p' = 2A_0t + A_1$$
$$y_p'' = 2A_0$$
So,
$$L[y_p] = -A_0t^2 - A_1t + (-A_2 + 2A_0) = t^2 + 1$$
So, $A_0 = -1$, $A_1 = 0$, $2A_0 - A_2 = 1$, implying $A_2 = -1 + 2A_0 = -3$. So, $y_p = -t^2 - 3$. Since homogeneous solutions are $\{e^t, e^{-t}\}$, it follows that general solution is
$$y = -t^2 - 3 + c_1e^t + c_2e^{-t}$$

3. An LCR electrical circuit has no initial voltage across the capacitor but an initial current of 1 *Amp*. Assume inductance to be 1 Henry, resistance 2 Ohm and capacitance 0.5 Farad.

a. Formulate an ODE initial value problem describing the charge on the capacitor plate as a function of time.

b. Calculate charge on a capacitor plate as a function of time.

Solution: a. Let $Q(t)$ be the charge in Coulombs on the capacitor plate at time t seconds. Here $C = 0.5$, $L = 1$, $R = 2$ and the voltage drop across the circuit is

$$LQ'' + RQ' + \frac{Q}{C} = Q'' + 2Q' + 2Q = 0 \quad , \quad \text{or}$$

$$Q'' + 2Q' + 2Q = 0$$

Since initial voltage across capacitor $\frac{Q(0)}{C} = 0$, we have $Q(0) = 0$. Since initial current is $I(0) = Q'(0) = 1$. So, the IVP is

$$Q'' + 2Q' + 2Q = 0 \quad , \quad Q(0) = 0 \quad , \quad Q'(0) = 1$$

b.: Solving above ODE, characteristic equation is $r^2 + 2r + 2 = 0$; solving quadratic $r = -1 \pm i$ So, general solution is

$$Q(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$$

Since $Q(0) = 0 = C_1$, we have

$$Q(t) = C_2 e^{-t} \sin t$$

Since

$$Q'(t) = -C_2 e^{-t} \sin t + C_2 e^{-t} \cos t$$

So, $Q'(0) = 1 = C_2$. So charge in Coulombs on a capacitor plate at time t seconds is

$$Q(t) = e^{-t} \sin t$$