Problem 7.4.6. Consider the vector functions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}.$$

The Wronskian of these functions is

$$\begin{aligned} W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) &= \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} \\ &= t(2t) - 1(t^2) \\ &= t^2. \end{aligned}$$

Since $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = t^2 \neq 0$ precisely when $t \neq 0$, the vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, i.e., the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ evaluated at t, are linearly independent precisely when $t \neq 0$. Hence the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent in any interval because any interval contains at least a nonzero t-value.

Suppose

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

or, more explicitly,

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

is the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. Then by Abel's formula, i.e. Eq.(14) on page 388, one has

$$\frac{dW}{dt} = [p_{11}(t) + p_{22}(t)]W,$$

where $W = W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$. Hence

$$W(t) = C \exp \int [p_{11}(t) + p_{22}(t)]\, dt$$

for some constant C that depends only on $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ but not on t. In particular, either $W(t)$ is identically zero or else never vanishes over the interval of continuity of $\mathbf{P}$, i.e., of p_{ij} for all $i, j = 1, 2$. (See theorem 7.4.3 on page 387). Comparing this observation to the fact that $W(t) = t^2$, it follows that $\mathbf{P}$ and thus at least one of p_{ij} for $i, j = 1, 2$ must be discontinous at $t = 0$.

To find $\mathbf{P}$ explicitly, one uses the assumption that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are (linearly independent) solutions of the corresponding system of homogeneous differential equations. Since

$$\frac{d\mathbf{x}^{(1)}}{dt} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \frac{d\mathbf{x}^{(2)}}{dt} = \begin{pmatrix} 2t \\ 2 \end{pmatrix},$$

one has

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}.$$

More explicitly,

$$tp_{11}(t) + p_{12}(t) = 1, \tag{1}$$
$$t^2 p_{11}(t) + 2tp_{12}(t) = 2t, \tag{2}$$
$$tp_{21}(t) + p_{22}(t) = 0, \tag{3}$$
$$t^2 p_{21}(t) + 2tp_{22}(t) = 2. \tag{4}$$

Now, (2)$-t\times$(1) gives $tp_{12}(t) = t$. Hence $p_{12}(t) = 1$ except possibly for $t = 0$. In fact, $p_{12}(0) = 1$ also as one can see by substituting 1 for t in (1). Therefore, by (1), one has $tp_{11}(t) = 0$, so that $p_{11}(t) = 0$ except possibly for $t = 0$. If $p_{11}(0) \neq 0$, then one has shown that p_{11} is discontinuous at $t = 0$. However, there is no information that one can get from (1)-(4) to ascertain that p_{11} is discontinuous at $t = 0$.

Let's proceed further. Now, (4)$-t\times$(3) yields $tp_{22}(t) = 2$ for $t \neq 0$. This implies the discontinuity of p_{22} at $t = 0$ as follows. Suppose, to the contrary, that p_{22} is continuous in an interval containing $t = 0$. Then, in particular, $\lim_{t\to 0} p_{22}(t) = p_{22}(0)$ exists as a finite number. It follows that

$$0 = 0 \cdot p_{22}(0) = \lim_{t\to 0} t \cdot \lim_{t\to 0} p_{22}(t) = \lim_{t\to 0} tp_{22}(t) = \lim_{t\to 0} 2 = 2$$

because $tp_{22}(t) = 2$ for all $t \neq 0$. This is absurd! Hence the assumption that p_{22} is continuous at $t = 0$ must be wrong! In fact, one gets from the equation $tp_{22}(t) = 2$ that $p_{22}(t) = 2/t$ for all $t \neq 0$. Hence there is no way p_{22} can be defined at $t = 0$ to make it continuous there. Likewise, one gets that $p_{21}(t) = -2/t^2$ for all $t \neq 0$ and there is no way to define it at $t = 0$ to make it continuous there either. The discontinuity of p_{21} and p_{22} at $t = 0$ verifies the conclusion one makes based on the consideration of the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Finally, one sees that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$