Solutions to Practice Final
March 2004 exam

1 Problem 1

1.1 Part A

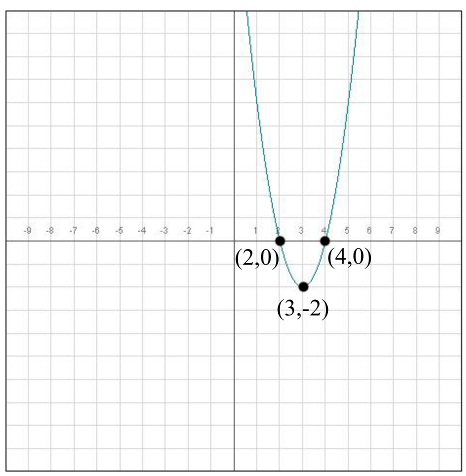
Part A asks us to "Determine the basic function and the sequence of transformations needed to transform the graph of the basic function into the graph of f". So, first we must find the basic function. Since f is the graph of a parabola, I know my basic function will be $b(x) = x^2$. Now, I need to transform the graph of the basic function into the graph of f. I accomplish this with the following transformations:

1. Horizontal shift right by 3 units
2. Vertical stretch by 2 units
3. Vertical shift down by 2 units

Remember that the order of the transformations is relevant here, so be careful. The safest order is as follows: horizontal shifts, reflections about the x-axis, vertical/horizontal compressions/stretches, vertical shifts.

1.2 Part B

We now must graph f and plot three different points.



Remember to plot the points *on the graph*.

2 Problem 2

We are given the function $g(x) = (x^2 - 2x - 8)^2$.

2.1 Part A

Here we are to determine the x-intercepts and determine whether the graph of g crosses or touches at each x-intercept. And also we must determine whether the graph has any type of symmetry. So we begin by finding the x-intercepts. To do so we set $g(x) = 0$ and solve for x.

$$\begin{aligned} g(x) &= 0 \\ (x^2-2x-8)^2 &= 0 \\ ((x-4)(x+2))^2 &= 0 \\ (x-4)^2(x+2)^2 &= 0 \\ x &= 4,\ -2 \end{aligned}$$

And since the multiplicity of both roots is 2, we know that at both of the x-intercepts the graph of g touches the x-axis. Now we must determine whether the graph has any type of symmetry. First let's check for evenness. To do so we compute $g(-x)$ and compare the result to the original function $g(x)$. If the two are equal, then the graph is even or in other words is symmetric about the y-axis. So we have that

$$g(-x) = ((-x)^2 - 2(-x) - 8)^2 = (x^2 + 2x - 8)^2 \neq g(x)$$

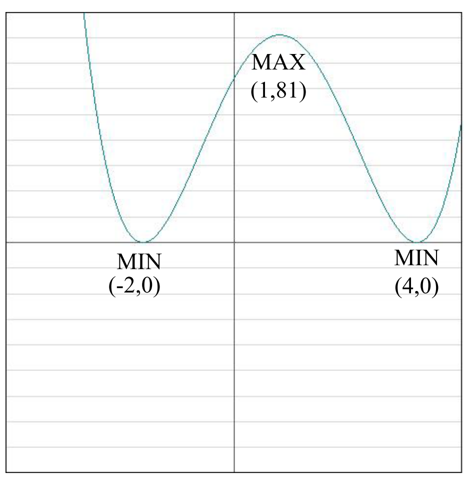
We know that $g(-x)$ is not equal to $g(x)$; for example $g(-1) = 25$ whereas $g(1) = 81$. And so g is not even. Now we must check for oddness. To do so, we check to see if $g(-x) = -g(x)$. We already computed $g(-x)$ and upon inspection we see that

$$g(-x) = (x^2 + 2x - 8)^2 \neq -(x^2 - 2x - 8)^2$$

Or we could have simply noted that $g(-x)$ is always positive while $-g(x)$ is always negative, so they can't possibly be equal. So g is not odd, and we are done with this part of the problem.

2.2 Part B

Here we are asked to find all the turning points of the graph of g. By turning points, they mean all the "hills" and "valleys". Here's where you need to be comfortable using the MAX and MIN functions on your calculator. You should find that we have a max at (1,81), a min at (-2,0), and another min at (4,0).



3 Problem 3

Let R be the rational function defined by $R(x) = \dfrac{6}{x^2 - 2x - 3}$.

3.1 Part A

Find the x and y intercepts and all asymptotes. We'll begin here by find the x-intercepts. We set $R(x) = 0$ and solve for x:

$$\begin{aligned} R(x) &= 0 \\ \frac{6}{x^2 - 2x - 8} &= 0 \\ \frac{6}{x^2 - 2x - 8} \times (x^2 - 2x - 8) &= 0 \times (x^2 - 2x - 8) \\ 6 = 0 \end{aligned}$$

And so we've reached a contradiction. Therefore $R(x)$ must not have any x-intercepts. Now we'll check for y-intercepts. To do so we compute $R(0)$:

$$R(0) = \frac{6}{0^2 - 2 \cdot 0 - 3} = -2$$

So we have a y-intercept at (0, -2).

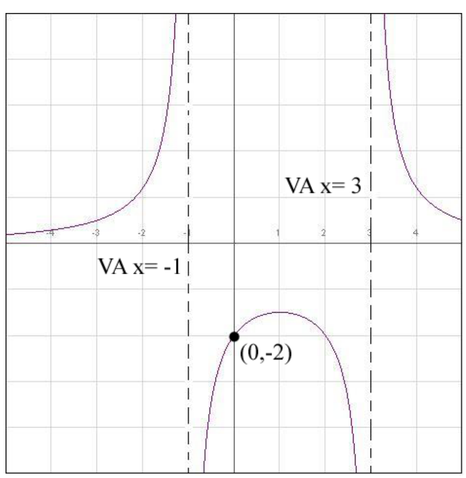
So we've found all the intercepts, now lets move on and find the asymptotes. First, let us check for vertical asymptotes. To do so, we set the bottom part of $R(x)$ equal to 0 and solve for x. So in our case we have

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \\ x &= 3, \ -1 \end{aligned}$$

And so $R(x)$ has vertical asymptotes at $x = 3$ and $x = -1$. Now we must find horizontal or oblique asymptotes. To do so we use long division. Note that $x^2 - 2x - 3$ divides into 6 zero times, so we must have a horizontal asymptote at $y = 0$.

3.2 Part B

Here we must sketch the graph of R making sure to include all the information about R that we found in part A:



4 Problem 4

4.1 Part A

We are asked to solve algebraically $5^{2x-1} = 3^{2-x}$. We accomplish this by taking the natural log of both sides of the equation and then using the following property of logs to solve for x: $ln(a^b) = b \cdot ln(a)$. The work unfolds as such:

$$\begin{aligned}
5^{2x-1} &= 3^{2-x} \\
ln(5^{2x-1}) &= ln(3^{2-x}) \\
(2x-1)ln(5) &= (2-x)ln(3) \\
2x \cdot ln(5) - ln(5) &= 2 \cdot ln(3) - x \cdot ln(3) \\
2x \cdot ln(5) + x \cdot ln(3) &= 2ln(3) + ln(5) \\
x\Big(2ln(5) + ln(3)\Big) &= 2ln(3) + ln(5) \\
x &= \frac{2ln(3) + ln(5)}{2ln(5) + ln(3)}
\end{aligned}$$

4.2 Part B

Here we are to express the following in terms of sums and differences of logarithms:

$$ln\left(\frac{\sqrt[3]{x^2-4}}{5x^4(x+1)^2}\right)$$

We use the following rules of logarithms to achieve this: $ln(a^b) = b \cdot ln(a)$, $ln(a) + ln(b) = ln(a \cdot b)$, and $ln(a) - ln(b) = \ln(\frac{a}{b})$. Let's begin:

$$\begin{aligned}
ln\left(\frac{\sqrt[3]{x^2-4}}{5x^4(x+1)^2}\right) &= ln(\sqrt[3]{x^2-4}) - ln(5x^4(x+1)^2) \\
&= \frac{1}{3}ln(x^2-4) - \big(ln(5) + ln(x^4) + ln(x+1)^2\big) \\
&= \frac{1}{3}ln\left((x-2)(x+2)\right) - ln(5) - 4ln(x) - 2ln(x+1) \\
&= \frac{1}{3}ln(x-2) + \frac{1}{3}ln(x+2) - ln(5) - 4ln(x) - 2ln(x+1)
\end{aligned}$$

4.3 Part C

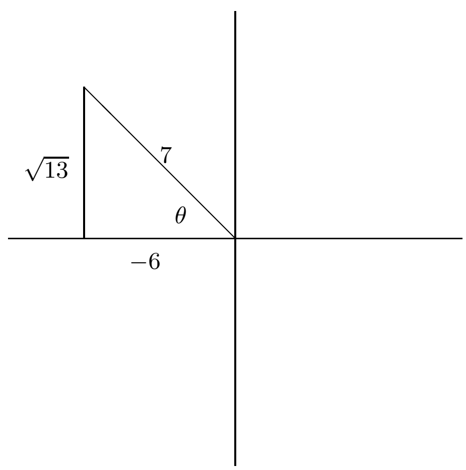
Here we are to solve algebraically: $log_5(x) + log_5(x-4) = 1$. Again we use the properties of logarithms to assist in the solving of this equality:

$$\begin{aligned}
log_5(x) + log_5(x-4) &= 1 \\
log_5(x(x-4)) &= 1 \\
5^{log_5(x(x-4))} &= 5^1 \\
x(x-4) &= 5 \\
x^2 - 4x &= 5 \\
x^2 - 4x - 5 &= 0 \\
(x-5)(x+1) &= 0 \\
x &= 5, -1
\end{aligned}$$

But the work isn't quite done yet. We still have to do a "reality" check, that is make sure the solutions we got for x actually make sense in our log_5 function. We notice that $log_5(-1)$ is undefined so $x = -1$ is not a solution to this equation. On the other hand $log_5(5)$ and $log_5(5-4)$ are perfectly well defined, so $x = 5$ is a solution, moreover, it is our only solution.

5 Problem 5

Here we are given that $\sec(\theta) = -\frac{7}{6}$ and that $\tan(\theta) < 0$. And we are to find what quadrant θ lies in and also determine the remaining 5 trigonometric functions. We have that $\sec(\theta) = -\frac{7}{6}$ so $\cos(\theta) = -\frac{6}{7}$ But cos is negative only in the II and III quadrants, so θ must lie in either quadrant II or quadrant III. Now we use the second peice of information that was given, namely that $\tan(\theta) < 0$. We know that tan is negative only in the II and IV quadrants, so I know θ must lie somewhere in either quadrant II or quadrant IV. But I know θ can't be in quadrant IV since $\cos(\theta) < 0$, so θ must be in quadrant II:



Of course, we know the sides of the triangle since $\cos(\theta) = \frac{-6}{7} = \frac{adjacent}{hypotenuse}$ and using the famed Pythagorean Theorem to find the "opposite" side, i.e. $\sqrt{13}$. Now the remaining trigonometric functions can be read right off the picture:

$$\sin(\theta) = \frac{\sqrt{13}}{7} \qquad \csc(\theta) = \frac{7}{\sqrt{13}}$$
$$\cos(\theta) = \frac{-6}{7} \qquad \sec(\theta) = \frac{-7}{6}$$
$$\tan(\theta) = \frac{-\sqrt{13}}{6} \qquad \cot(\theta) = \frac{-6}{\sqrt{13}}$$

6 Problem 6

Let $y = -3\cos(4x + \pi)$.

6.1 Part A

Here we must find the amplitude, period, and phase shift of the equation given. To do this we simply compare the given equation to the general equation $A\cos(\omega x - \phi)$

where $|A| = amplitude$, $T = \frac{2\pi}{\omega} = period$, and $\frac{\phi}{\omega} = phase\ shift$. Easily enough we see that $A = -3$, $\omega = 4$ and $\phi = -\pi$, so we have that

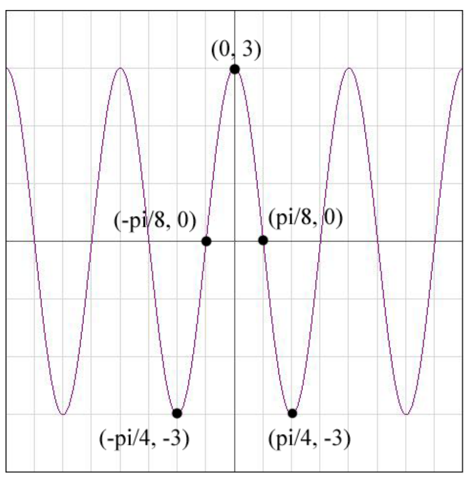
$$\begin{aligned} |A| &= 3 \\ T &= \frac{2\pi}{4} = \frac{\pi}{2} \\ PS &= \frac{-\pi}{4} \end{aligned}$$

So the amplitude is 3, the period is $\frac{\pi}{2}$ and the phase shift is $\frac{-\pi}{4}$.

6.2 Part B

Here we are to graph the equation on the interval $[-\pi, \pi]$. Before we begin to graph the equation, we first must find the sub-interval length, as these sub-intervals give us the "5-key points" of the graph, i.e. peaks, troughs, and zeroes. We find the subinterval length by taking the period T and dividing it by 4. So in our situation here, we see that the subinterval length is given by $\frac{T}{4} = \frac{\pi/2}{4} = \frac{\pi}{8}$.

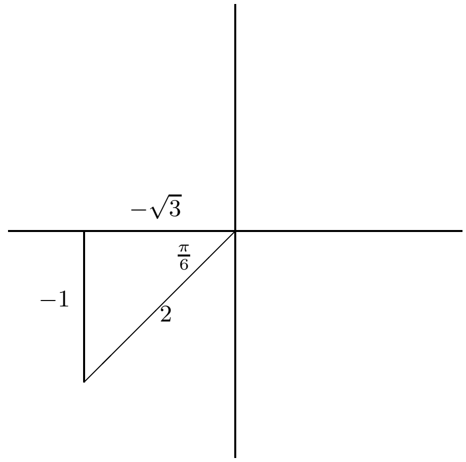
Now we are ready to begin graphing. We first begin at the phase shift. We do this because we know we will have a "key point" at the phase shift, i.e. a peak, trough, or zero depending upon the trig function. Since we have a cos function and the fact that there is a negative 3 in front of it, we know we will have a trough at $\frac{-\pi}{4}$, specifically we have the point $(\frac{-\pi}{4}, -3)$. We find the next "key point" by adding one subinterval length to the phase shift: $\frac{-\pi}{4} + \frac{\pi}{8} = \frac{-\pi}{8}$. Since we had a trough at the previous key point, we know at this next key point we will have a zero, specifically we have the point $(\frac{-\pi}{8}, 0)$. we find the next key point by adding the subinterval length again, and continue to work in this way until the whole graph is filled out:



You need only label the "5 key points" for the "first" interval.

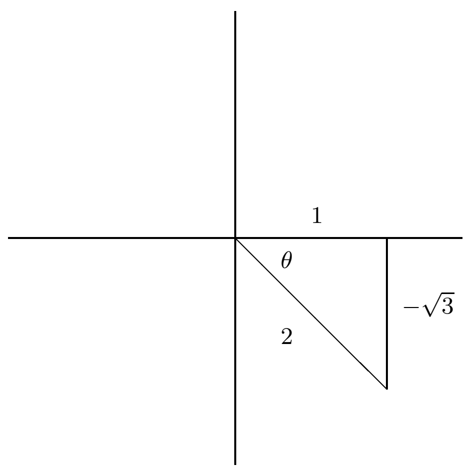
7 Problem 7

Here we are to find the exact value of $\sin^{-1}\big(\cos\big(\frac{7\pi}{6}\big)\big)$. These types of problems are done in two parts. Specifically, we work from the inside-out. So let's first concern ourselves with computing $\cos\left(\frac{7\pi}{6}\right)$. We know that $\frac{7\pi}{6}$ is 210 degrees so $\frac{7\pi}{6}$ lies in quadrant III and has $\frac{\pi}{6}$ as a reference angle. So we can draw $\frac{\pi}{6}$ and label the sides:



And so we see that $\cos\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}}{2}$.

Now we move onto the second phase of the problem, that is computing the value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$. We know that $\sin^{-1}$ has a restricted range, namely between $[\frac{\pi}{2}, \frac{\pi}{2}]$. So if $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \theta$, then θ lies somewhere in either quadrant I or quadrant IV. But $\sin(\theta) < 0$ so we know that θ must be in quadrant IV. Knowing that $\sin(\theta) = \frac{-\sqrt{3}}{2}$ we can draw θ and label all the sides of the resulting triangle with the help of the Pythagorean Theorem:



And so we immediately notice that θ is a special angle considering the ratios of the sides, specifically we notice that $\theta = \frac{-\pi}{3}$.

8 Problem 8

Here we are given that $\tan(\theta) = 2$ and we are to find the exact value of $\tan(\theta - \frac{\pi}{6})$. Since $\tan\left(\frac{\pi}{6}\right)$ can be readily computed to be $\frac{1}{\sqrt{3}}$, we now only need to apply the difference formula for tan:

$$
\begin{aligned}
\tan\left(\theta - \frac{\pi}{6}\right) &= \frac{\tan(\theta) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan(\theta)\tan\left(\frac{\pi}{6}\right)} \\
&= \frac{2 - \frac{1}{\sqrt{3}}}{1 + 2 \cdot \frac{1}{\sqrt{3}}} \\
&= \frac{2 - \frac{\sqrt{3}}{3}}{1 + 2 \cdot \frac{\sqrt{3}}{3}} \\
&= \frac{\left(\dfrac{6 - \sqrt{3}}{3}\right)}{\left(\dfrac{3 + 2\sqrt{3}}{3}\right)} \\
&= \frac{6 - \sqrt{3}}{3 + 2\sqrt{3}} \\
&= \frac{6 - \sqrt{3}}{3 + 2\sqrt{3}} \times \frac{3 - 2\sqrt{3}}{3 - 2\sqrt{3}} \\
&= \frac{24 - 15\sqrt{3}}{-3} \\
&= 5\sqrt{3} - 8
\end{aligned}
$$

9 Problem 9

Here we are to prove the identity:

$$\cos(\theta) + \frac{\sin^2(\theta)}{1 + \cos(\theta)} = 1$$

We shall manipulate the left side of the equation using algebra and trigonometric identities to prove that the left side is actually equal to the right side:

$$
\begin{aligned}
\cos(\theta) + \frac{\sin^2(\theta)}{1 + \cos(\theta)} &= \frac{\cos(\theta) \cdot (1 + \cos(\theta))}{1 + \cos(\theta)} + \frac{\sin^2(\theta)}{1 + \cos(\theta)} && (1) \\
&= \frac{\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)}{1 + \cos(\theta)} && (2) \\
&= \frac{\cos(\theta) + 1}{1 + \cos(\theta)} && (3) \\
&= 1 && (4)
\end{aligned}
$$

And so the identity is proven. From (2) to (3) we applied the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

10 Problem 10

Here we are to find the exact value of $\sin\left(\frac{11\pi}{8}\right)$. Unfortunately, there is no real way to choose which trig formula to use. It's more or less a trial and error process. Although if you notice that the denominator of the angle is double that of a denominator of an angle we know how to find, in this case $8 = 4 \cdot 2$, then it's always a good thing to try the half angle formula. So, let's set $\frac{11\pi}{8} = \frac{\theta}{2}$ and solve for θ to see if we get a "nice" angle for θ.

$$\begin{aligned} \frac{11\pi}{8} &= \frac{\theta}{2} \\ \frac{11\pi}{8} \cdot 2 &= \frac{\theta}{2} \cdot 2 \\ \frac{11\pi}{4} &= \theta \end{aligned}$$

Since $\frac{11\pi}{4}$ is an angle that we can find trig values for, we know that if we use the half angle formula we should be successful. So we see that:

$$\begin{aligned} \sin \frac{11\pi}{8} &= \sin \left(\frac{\left(\frac{11\pi}{4}\right)}{2} \right) \\ &= \pm \sqrt{\frac{1 - \cos\left(\frac{11\pi}{4}\right)}{2}} \\ &= \pm \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}} \\ &= \pm \sqrt{\frac{\left(\frac{2+\sqrt{2}}{2}\right)}{2}} \\ &= \pm \sqrt{\frac{2+\sqrt{2}}{4}} \\ &= \pm \frac{\sqrt{2+\sqrt{2}}}{2} \end{aligned}$$

And so now we need only decide whether we want to use the positive or the negative value. To decide this we look at the *original* angle $\frac{11\pi}{8}$. We know that $\frac{11\pi}{8}$ lies in quadrant III, and since sin is always negative in quadrant III, we know to use the negative value. And so we have that

$$\sin\left(\frac{11\pi}{8}\right) = -\frac{\sqrt{2+\sqrt{2}}}{2}$$

11 Problem 11

Here we are to give a general formula for all the solutions of the equation: $\cos\left(\frac{\theta}{3}\right) = \frac{1}{2}$. First we let $\frac{\theta}{3} = \varphi$, and we consider $\cos(\varphi) = \frac{1}{2}$. We know that $\cos(\varphi) = \frac{1}{2}$ when $\varphi = \frac{\pi}{3}$ or when $\varphi = \frac{-\pi}{3}$. These are the only two solutions between 0 and 2π, so to find the general solution, we simply add $2k\pi$ to each solution, that is, we have $\varphi = \frac{\pi}{3} + 2k\pi$ or $\varphi = \frac{-\pi}{3} + 2k\pi$.

Now we need only substitute $\frac{\theta}{3}$ in for φ:

$$\begin{aligned} \varphi = \frac{\theta}{3} &= \pm\frac{\pi}{3} + 2k\pi \\ \theta &= \pm\pi + 6k\pi \end{aligned}$$

12 Problem 12

12.1 Part A

We are to rewrite the equation $r = 2\cos(\theta)$ in terms of rectangular coordinates:

$$\begin{aligned}
r &= 2\cos(\theta) && (5)\\
r^2 &= 2r\cos(\theta) && (6)\\
x^2 + y^2 &= 2x && (7)\\
(x^2 - 2x) + y^2 &= 0 && (8)\\
(x^2 - 2x + 1) + y^2 &= 1 && (9)\\
(x-1)^2 + y^2 &= 1 && (10)
\end{aligned}$$

So we see that $r = 2\cos(\theta)$ is the equation for a circle with radius 1 and center at (1,0). From line (5) to (6) we simply multiplied both sides by r. From line (6) to (7) we use the identities $r^2 = x^2 + y^2$ and $x = r\cos(\theta)$. And then we completed the square with respect to the variable x to get the equation in a more general form.

12.2 Part B

Here we are to evaluate the complex number $\left(\sqrt{2}\big(\cos(15) + i\sin(15)\big)\right)^8$ and express our answer in the standard form $a + bi$. We use De Moivre's Theorem to solve this problem. We have that:

$$\begin{aligned}
\left(\sqrt{2}\big(\cos(15) + i\sin(15)\big)\right)^8 &= (\sqrt{2})^8\left(\cos(15\cdot 8) + i\sin(15\cdot 8)\right)\\
&= 16\left(\cos(120) + i\sin(120)\right)\\
&= 16\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\\
&= -8 + 8\sqrt{3}i
\end{aligned}$$

And that does it.