Math 366 Practice Problems for Final Exam **Autumn 2005**

1. Each of the following statements has one of the forms

$$\sim p \qquad p \wedge q \qquad p \vee q \qquad p \rightarrow q \qquad p \leftrightarrow q$$

Find the appropriate form and indicate what each statement variable in your choice represents.
(a) If Archibald passes the first exam, then he will not drop the course.
(b) The moon is not made of green cheese.
(c) Harry got out of bed and brushed his teeth.

2. Use truth tables to verify each of the following logical equivalences.
(a) $p \vee (\sim p \wedge q) \;\equiv\; p \vee q$
(b) $p \leftrightarrow (p \wedge q) \;\equiv\; p \rightarrow q$
(c) $p \rightarrow (q \vee r) \;\equiv\; (p \wedge \sim q) \rightarrow r$

3. Show that each of the following arguments has a valid argument form by exhibiting such a form. Explain what each statement variable in your form represents.

(a) I'll either get a Christmas bonus or I'll sell my motorcycle.
If I get a Christmas bonus, then I'll buy a CD player.
If I sell my motorcycle, then I'll buy a CD player.
Therefore, I'll buy a CD player.

(b) If Christine intends to go to the party, then John will also.
John is not intending to go to the party.
Therefore, Christine is not intending to go to the party.

4. Determine which of the following argument forms are valid and which are not. Justify your answers. If the form is valid, verify that it is by two methods: truth tables and step by step derivations using theorem 1.1.1 and table 1.3.1 from the text.

(a) $p \vee q$
$\therefore\; p$

(b) $p \rightarrow (q \rightarrow r)$
$\sim r$
p
$\therefore\; \sim q$

(c) $\sim q$
$\therefore\; \sim (p \wedge q)$

(d) $\sim p \rightarrow q$
$\sim q$
$\therefore\; p$

(e) p

$$q$$
$$\therefore\ (p \wedge q) \vee r$$

(f) $\sim p \wedge q$

$$p \vee r$$
$$\therefore\ r$$

5. Make up an I/O table for three inputs P, Q, and R. Find a Boolean expression for your I/O table.

6. Build a circuit with inputs P, Q, and R using AND-gates, OR-gates, and NOT-gates.
(a) Find the I/O table for your circuit.
(b) Find a Boolean expression which is equivalent to your circuit.
Show your work.

7. For each of the following Boolean expressions, find an equivalent circuit.
(a) $(P \wedge Q) \vee (\sim P \wedge \sim Q)$
(b) $\sim ((P \wedge Q) \vee (\sim Q \wedge R))$

8. Each of the expressions below has one of the forms

$$\forall x \in D, P(x) \qquad \exists x \in D \ s.t.\ P(x)$$

Determine the appropriate form and indicate the interpretation of the domain D and the predicate $P(x)$.
(a) Everyone in the class who works hard will pass.
(b) Someone is sleeping.

9. Find a counterexample for each of the following universal statements.
(a) $\forall x \in \mathbf{R}, x^3 \neq -x$
(b) Everyone who lives in Columbus has been to Madagascar.

10. Show that each of the following arguments has a valid form in predicate logic by exhibiting such a form. Justify that your form is valid. Also indicate how to interpret any domain symbols or predicate symbols you use as well as any symbol used as a name.

(a) Every math 366 exam is simple.
This exam isn't simple.
Therefore, this exam is not a math 366 exam.

(b) Every math 366 exam is simple.
This exam is a math 366 exam.
Therefore, this exam is simple.

11. Give the first sentence of a direct proof of each of the following statements. Also indicate what remains to be proved.
(a) The product of two rational numbers is rational.
(b) If an integer is prime and different from 2 then it is odd.

12.Constructive Proof of an Existential Statement.

(a) Prove that there is an even integer n such that $n \, mod \, 3 = 1$.
(b) Prove that there exists a rational number q such that $9\,q^2 = 4$.
(c) Prove that there exist two real numbers whose product is less than their sum.
(d) Prove that there exist two real numbers which are not equal such that $xy = x + y$.
(e) Prove that there exist sets A and B such that $A - B = A \cup B$.
(f) Prove that there exist sets A and B such that $A - B \neq A \cup B$.

13. Direct Proof of a Universal Statement.
(a) Prove that if n is an integer which is divisible by 6 then n is divisible by 3.
(b) Prove that for any integers a, b, c, and d, if a divides b and c divides d then $a \cdot c$ divides $b \cdot d$.
(c) Prove that the square of every even integer is divisible by 4.
(d) Prove that for any sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.
(e) Prove that for any sets A, B, C and D, if $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$.

14. Proof by Cases.
(a) Prove that for every integer n, n and $n + 2$ have the same parity (i.e. either n and $n + 2$ are both even or n and $n + 2$ are both odd).
(b) Prove that for any integer n, $n^2 + n$ is even.
(c) Prove that for any integer n, if 3 divides $2n$ then 3 divides n. Hint: Use an argument by cases depending on what $n \, mod \, 3$ is.
(d) Prove that for any integer n, $n^2 \, mod \, 3 \neq 2$. Hint: Use an argument by cases depending on what $n \, mod \, 3$ is.
(e) Prove that for any sets A, B, and C, if $A \subseteq C$ then $A \cup (B \cap C) \subseteq C$.

15. Mathematical Induction.
(a) Prove that for any integer n, if $n \geq 0$ then 4 divides $5^n - 1$.
(b) Prove that for any integer n, if $n \geq 1$ then 4 divides $6^n - 2^n$.
(c) Show that $2n + 1 < 2^n$ for every integer n with $n \geq 3$.
(d) Using the fact that $2n + 1 < 2^n$ for every integer n with $n \geq 3$, show that for every integer n, if $n \geq 5$ then $n^2 < 2^n$.

16. Strong Mathematical Induction.
(a) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 0, c_1 = 1,$$

$$c_k = 2c_{k-1} - c_{k-2} + 2 \text{ for all integers } k \geq 2.$$

Prove that $c_n = n^2$ for all integers $n \geq 0$.
(b) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 2, c_1 = 5,$$

$$c_k = 5c_{k-1} - 6c_{k-2} \text{ for all integers } k \geq 2.$$

Prove that $c_n = 2^n + 3^n$ for all integers $n \geq 0$.

17. Proof by Contradiction or Contraposition.
(a) Prove that there is not a largest odd integer.
(b) For any integer n, if n^2 is odd then n is odd.

18. Computations with Sets.

(a) Let $A = \{a, c, d\}$ and $B = \{b, c, f\}$. Compute $A \cup B$, $A \cap B$, $A - B$ and $A \times B$ using "bracket" notation.

(b) Let $A = \{1, 3\}$ and $B = \{2, 3\}$. Compute $A \cup B$, $A \cap B$, $A - B$, and $A \times B$ using "bracket" notation.

19. More Proofs with Sets.
(a) Prove or disprove: For all sets A and B, if $A \subseteq B$ then $A \cap B = A$.
(b) Prove or disprove: For all sets A and B, $(A - B) \cup (B - A) = A \cup B$.
(c) Prove or disprove: For any sets A, B, and C, if $A \subseteq B$ then $A - (B \cap C) \subseteq A - C$.
(d) Prove or disprove: For any sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

20. Determine whether the following relations are functions. Find the inverse and determine whether it is a function. Justify your answers with proofs.
(a) R is the binary relation on $\mathbf{R}$ determined by xRy iff $x^2 + y^2 = 2$.
(b) R is the binary relation on $\mathbf{R}$ determined by xRy iff $x = y^2$.
(c) R is the following relation from $\{0, 1, 2\}$ to $\{0, 1, 2, 3\}$: $\{(0, 0), (1, 2), (2, 1)\}$

21. Determine whether each of the following functions is 1-1. Provide a proof of your answer.
(a) $f : \{0, 1, 2\} \to \{a, b, c, d\}$ where $f(0) = d$, $f(1) = b$, and $f(2) = c$.
(b) Make up a function $f : X \to Y$ where $X = \{a, b, c, d\}$ and $Y = \{0, 1, 2\}$ by drawing it's arrow diagram. Is it 1-1?
(c) The function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = -2x + 1$.
(d) The function $h : \mathbf{R} \to \mathbf{R}^{nonneg}$ given by $h(x) = x^2$.
(e) The function $h : \mathbf{N} \to \mathbf{N}$ given by $h(n) = n^3$.
(f) The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$.
(g) The function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^3 - x$.

22. Determine whether each of the functions in problem 21 is onto. Provide a proof of your answer. Hint: Part (g) is tricky. You may want to use the intermediate value theorem from calculus.

23. Each of the functions below is a 1-1 correspondence. Find a formula for the inverse.
(a) $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = -2x + 1$.
(b) $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = (x + 1)^3 + 2$.
(c) $f : X \to Y$ given by $f(x) = \frac{x+1}{x}$ where $X = \{x \in \mathbf{R} | x \neq 0\}$ and $Y = \{y \in \mathbf{R} | y \neq 1\}$.