

1. Constructive Proof of an Existential Statement.

- (a) Prove that there is an even integer n such that $n \bmod 3 = 1$.
- (b) Prove that there exists a rational number q such that $9q^2 = 4$.
- (c) Prove that there exist two real numbers whose product is less than their sum.
- (d) Prove that there exist two real numbers which are not equal to each other and whose product is equal to their sum.
- (e) Prove that there is an odd integer n such that $n > 1$ and $n \bmod 3 = 1$.

2. Direct Proof of a Universal Statement.

- (a) Prove that if n is an integer which is divisible by 6 then n is divisible by 3.
- (b) Prove that for any integers a, b, c , and d , if a divides b and c divides d then $a \cdot c$ divides $b \cdot d$.
- (c) Prove that the product of two odd integers is odd.
- (d) Prove that if n is an integer which is divisible by 5 then $3n$ is divisible by 15.
- (e) Prove that for any nonzero rational numbers a and b there is a rational number x such that $ax + b = 0$.
- (f) Show that the reciprocal of any nonzero rational number is rational.

3. Proof by Cases.

- (a) Prove that for every integer n , n and $n + 2$ have the same parity (i.e. either n and $n + 2$ are both even or n and $n + 2$ are both odd).
- (b) Prove that for any integer n , $n^2 + n$ is even.
- (c) Prove that for any integer n , $n^2 \bmod 3 \neq 2$. Hint: Use an argument by cases depending on what $n \bmod 3$ is.

4. Mathematical Induction.

- (a) Prove that for any integer n , if $n \geq 0$ then 4 divides $5^n - 1$.
- (b) Prove that for any integer n , if $n \geq 1$ then 4 divides $6^n - 2^n$.
- (c) Show that $2n + 1 < 2^n$ for every integer n with $n \geq 3$.
- (d) Using the fact that $2n + 1 < 2^n$ for every integer n with $n \geq 3$, show that for every integer n , if $n \geq 5$ then $n^2 < 2^n$.

5. Strong Mathematical Induction and the Well-Ordering Principle.

- (a) Suppose c_0, c_1, c_2, \dots is a sequence defined as follows:

$$c_0 = 0, c_1 = 1,$$

$$c_k = 2c_{k-1} - c_{k-2} + 2 \text{ for all integers } k \geq 2.$$

Prove that $c_n = n^2$ for all integers $n \geq 0$.

- (b) Suppose c_0, c_1, c_2, \dots is a sequence defined as follows:

$$c_0 = 2, c_1 = 5,$$

$$c_k = 5c_{k-1} - 6c_{k-2} \text{ for all integers } k \geq 2.$$

Prove that $c_n = 2^n + 3^n$ for all integers $n \geq 0$.

6. Proofs by Contradiction and Contraposition.

- (a) Prove that there is not a smallest real number x such that $1 < x < 2$.
- (b) For any integer n , if n^2 is even then n is even.

7. Computations with Sets.

(a) Let $A = \{a, c, d, f\}$ and $B = \{b, c, f\}$ be subsets of the universal set $U = \{a, b, c, d, e, f, g\}$. Compute $A \cup B$, $A \cap B$, $A - B$, A^c , and $A \times B$ using “bracket” notation.

(b) Let $A = \{1, 3\}$ and $B = \{2, 3\}$ be subsets of the universal set $U = \{0, 1, 2, 3, 4\}$. Compute $A \cup B$, $A \cap B$, $A - B$, A^c , and $A \times B$ using “bracket” notation.