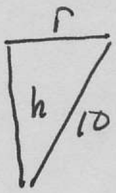
HW 4.1

4.1.30: The circumference of the top of the tank is the circumference of the circular sheet minus the arclength of the sector, $20\pi - 10\theta$ meters.

This is equal to the perimeter ~~radius~~ of the top of the tank: $2\pi r = 20\pi - 10\theta$



$$\Rightarrow r = \frac{20\pi - 10\theta}{2\pi} = \frac{5}{\pi}(2\pi - \theta)\left(= 10 - \frac{5\theta}{\pi}\right)$$

The slant height of the tank is 10 meters

So $h = \sqrt{10^2 - r^2} = \sqrt{100 - r^2} = \sqrt{100 - \left(10 - \frac{5\theta}{\pi}\right)^2}$

$$h = \frac{5}{\pi}\sqrt{4\pi\theta - \theta^2} \text{ meters.}$$

$$\text{So, } V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{\pi}(2\pi - \theta)\right)^2\left(\frac{5}{\pi}\sqrt{4\pi\theta - \theta^2}\right)$$

$$V(\theta) = V = \frac{125}{3\pi^2}(2\pi - \theta)^2\sqrt{4\pi\theta - \theta^2}$$

$$\frac{dV}{d\theta} = V'(\theta) = \frac{125}{3\pi^2}\left[2(2\pi - \theta)(-1)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2(4 - 2\theta)}{2\sqrt{4\pi\theta - \theta^2}}\right]$$

$$= \frac{125(2\pi - \theta)}{3\pi^2\sqrt{4\pi\theta - \theta^2}}\left[-2(4\pi\theta - \theta^2) + (2\pi - \theta)(2 - \theta)\right]$$

$$= \frac{125(2\pi - \theta)}{3\pi^2\left(\sqrt{4\pi\theta - \theta^2}\right)}\left(3\theta^2 - 12\pi\theta + 4\pi^2\right)$$

$\underline{V' = 0}$: if $2\pi - \theta = 0$ $\boxed{\theta = 2\pi}$ or $3\theta^2 - 12\pi\theta + 4\pi^2 = 0$

Using quadratic formula $\boxed{\theta = 2\pi - \frac{2\sqrt{6}}{3}\pi \text{ or } \theta = 2\pi + \frac{2\sqrt{6}}{3}\pi}$

$\underline{V' \text{ DNE}}$ if $\boxed{\theta = 0 \text{ or } \theta = 4\pi}$

Since $0 < \theta < 2\pi$, the only critical pt is $2\pi - \frac{2\sqrt{6}}{3}\pi$

and V has a local max at $\theta = 2\pi - \frac{2\sqrt{6}}{3}\pi$ which is unique

So, V has the global max when $\theta = 2\pi - \frac{2\sqrt{6}}{3}\pi$

HW 4.1

4.1.34: Note that $\cos t = \frac{h}{r} \Rightarrow h = r\cos t$ (r = constant)

$\sin t = \frac{\sqrt{r^2-h^2}}{r} \Rightarrow \sqrt{r^2-h^2} = r\sin t$

Area of submerged region $= tr^2 - h\sqrt{r^2-h^2} = tr^2 - r\cos t \, r\sin t$

$= r^2(t - \cos t \sin t)$

A = area of exposed wetted region $= \pi r^2 - \pi h^2 - r^2(t - \cos t \sin t)$

$= r^2(\pi - \pi\cos^2 t - t + \cos t \sin t)$

$$\frac{dA}{dt} = r^2\Big(-2\pi\cos t(-\sin t) \underbrace{-1 + \cos^2 t}_{-\sin^2 t} - \sin^2 t\Big)$$

$$= r^2(2\pi\cos t - 2\sin t)\sin t$$

Since $0 < t < \pi$ $\frac{dA}{dt} = 0$ only if $2\pi\cos t = 2\sin t$ $\left(\begin{array}{c} 0 < t < \pi \\ \Downarrow \\ \sin t \neq 0 \end{array}\right)$

$\pi = \tan t$

$$\tan t = \frac{\sin t}{\cos t} = \frac{r\sin t}{r\cos t} = \frac{\sqrt{r^2-h^2}}{h} = \pi$$

$$\Rightarrow \frac{r^2-h^2}{h^2} = \pi^2 \Rightarrow \left(\begin{array}{l} r^2 - h^2 = \pi^2 h^2 \\ r^2 = (\pi^2+1)h^2 \end{array}\right) \Rightarrow r = \sqrt{\pi^2+1}\,h$$

$$\boxed{h = \frac{r}{\sqrt{\pi^2+1}}}$$