MATH 151

WEBWORK 7 SOLUTIONS

WINTER 2006

(1) $f(t) = \frac{1}{t} = t^{-1}$

$f'(t) = -t^{-2}$

$f''(t) = 2t^{-3} \Rightarrow f''(5) = 2(5)^{-3} = \boxed{\frac{2}{125} = 0.016}$

(2) B = function

C = 1st Derivative - increasing/decreasing

A = 2nd Derivative - concavity

*note - be able to explain why. Consider the easiest reason - look where B has max-mins → where C has roots - same for C and A.

(3) $f(x) = 4x^2 + 24x - 5 \quad [-6, 4]$ endpoints

$f(-6) = -5$ & $f(4) = 155$

$f'(x) = 8x + 24 = 0$

$8x = -24$

$x = -3$ $\Longrightarrow f(-3) = 4(-3)^2 + 24(-3) - 5 = -41$

$\boxed{\therefore x = -3,\ y = -41 \text{ is a minimum and } x = 4,\ y = 155 \text{ is a max.}}$

$f''(x) = 8 \neq 0 \Rightarrow$ no change in concavity - concave up everywhere (extra info, not required)

Critical Points: -3

*note - WW might also want the endpoints of the closed interval included as critical points. if so, then C.V. = $-6, -3, 4$

(4) $f(x) = 3x^2 + 12x - 5 \quad [5, 9]$

$f'(x) = 6x + 12 = 0$

$6x = -12$

$x = -2$

interval endpoints: $f(5) = 130$, $f(9) = 346$

-2 is not in $[5,9]$ so -2 is not a critical value in this case - it would be if it is in the interval of definition! so, CV's would be 5, 9 only.

so, the max is attained at $x = 9$, and the min is attained at $x = 5$

5. $f(x) = x^3 - 3x + 5 \qquad (-1.5, 4)$

$f'(x) = 3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

all refer to the open interval $(-1.5, 4)$ endpoints not considered

A. critical pts: $\boxed{-1, 1}$

$f(-1) = 7 \qquad f(1) = 3$

$(-1, 7)$ is a max $\qquad (1, 3)$ is a min

B. minimum value: $\boxed{3}$

C. maximum value: $\boxed{7}$

closed interval $[-1.5, 4]$

* Now, -1.5, 4 also must be considered as critical points

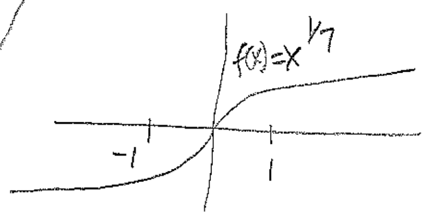
D. critical pts: $\boxed{-1.5, -1, 1, 4}$

$f(-1.5) = 6.125 \qquad f(-1) = 7 \qquad f(1) = 3 \qquad f(4) = 57$

$(1, 3)$ is the min. $\qquad (4, 57)$ is the max

E. minimum value $= \boxed{3}$ (from 1st Der.)

F. maximum value $= \boxed{57}$ (end point)



6. $f(x) = x^{1/7} \qquad [-1, 1]$

$f'(x) = \frac{1}{7}x^{-6/7}$

$= \dfrac{1}{7x^{6/7}}$

when $f'(x)$ is set to 0

we get $\dfrac{1}{7x^{6/7}} = 0$ but

$1 \neq 0$ so no CV

and $x \neq 0$ so what does this mean?

This kind of function is tricky. $f(x)$ is continuous everywhere. But $f'(x)$ is discontinuous at $x = 0 \Rightarrow$ usually this indicates a vertical asymptote, but this time it actually indicates a vertical tangent line at $x = 0$.

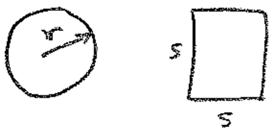
So accordingly, CV's are $-1, 1$ only

and $f(-1) = (-1)^{1/7} = -1$ min

$f(1) = (1)^{1/7} = 1$ max

7. $\ell = 19$

total area = $\text{area}_{\text{circle}} + \text{area}_{\text{square}}$



$$A = \boxed{\pi r^2 + s^2}$$

solve for $\underline{s}$ in terms of r:

Important - please note the comments

$\ell = 19 =$ circumference of circle + square

$19 = 2\pi r + 4\underline{s}$

$$\boxed{s = \frac{1}{4}(19 - 2\pi r)}$$

Note - this is How to substitute for s in the area function using the perimeter constraint.

max area:

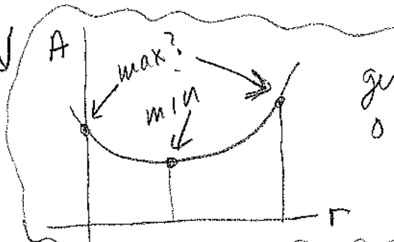
$$A = \pi r^2 + \left(\frac{1}{4}(19 - 2\pi r)\right)^2$$

$(s)^2$

This is the new Area function in terms of r alone using the perimeter constraint.

$$A' = 2\pi r + 2\left(\frac{1}{4}(19 - 2\pi r)\right)\left(\frac{1}{4}(0 - 2\pi)\right)$$

$$A' = 2\pi r + \frac{1}{2}(19 - 2\pi r)\left(-\frac{\pi}{2}\right) = 2\pi r - \frac{\pi}{4}(19 - 2\pi r) = 0$$



$$2\pi r - \frac{19\pi}{4} + \frac{2\pi^2 r}{4} = 0$$

$$r\left(2\pi + \frac{\pi^2}{4}\right) = \frac{19\pi}{4}$$

$$r = \frac{19\cancel{\pi}}{4\left(2\cancel{\pi} + \frac{\pi^{\cancel{2}}}{4}\right)}$$

$$r = \frac{19}{8 + \pi} \simeq 1.70532$$

$$A\left(\frac{19}{8+\pi}\right) = \pi\left(\frac{19}{8+\pi}\right)^2 + \left(\frac{1}{4}\left(19 - 2\pi\left(\frac{19}{8+\pi}\right)\right)\right)^2$$

$A = 13.4264$

other critical points:

$r = 0, \quad r = \frac{19}{2\pi}$ endpoints of r interval (domain)

$C = 2\pi r = 19$

$r = \frac{19}{2\pi}$

$$A(0) = 0 + \left(\frac{1}{4}(19 - 0)\right)^2$$

$$A = \left(\frac{19}{4}\right)^2 = 22.5625$$

$$A\left(\frac{19}{2\pi}\right) = \pi\left(\frac{19}{2\pi}\right)^2 + \left(\frac{1}{4}\left(19 - 2\pi\left(\frac{19}{2\pi}\right)\right)\right)^2$$

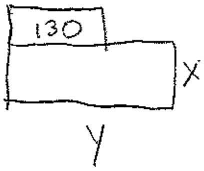
$A = 28.7275$

maxmal area $\Rightarrow \boxed{r = \frac{19}{2\pi}}$

minimal area $\Rightarrow \boxed{r = \frac{19}{8+\pi}}$

$\boxed{8}$

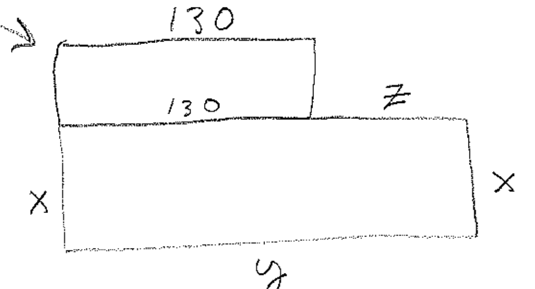
~~Max~~ perimeter is 300; $0 \leq x \leq 20$ $\quad y \geq 130$



$$2x + 2y = 300$$

i) $x + y = 150$

ii) Area $= xy$



*because $z + 2x + y = 170$

$\underline{130 + 170 = 300}$

Need an equation for area

- from (i), $y = 150 - x$

sub back into (ii) Area $= x(150 - x) = A(x)$

$$A(x) = 150x - x^2$$

Max area when slope approches 0

$$A'(x) = 150 - 2x$$

$$0 = 150 - 2x$$

$$150 = 2x$$

$$x = 75$$

* this is outside the range so since the function is a nice quadratic, the lowest slope within the range will be at the highest possible point, that being $x = 20$.

$$20 + y = 150$$

$$y = 150 - 20 = 130$$

$\boxed{\therefore \; x = 20 \quad y = 130}$

Fact - given any perimeter of a rectangular area, the max area is attained when the area is a square!

$\boxed{9}$ $f(x) = \frac{144}{x+2} + 36x \quad [-7, -3]$

fact - if $f(x)$ is undefined at some x, then $f'(x)$ will also be undefined at the same x! (and maybe at others)

Critical pts:

$f'(x) = 0$ or undefined

undefined when denominator $= 0$

$x + 2 = 0$

$x = -2$ * outside of domain of definition so we don't need to worry about this x value.

$$\cancel{\frac{144}{x+2} + 36x = 0}$$

Disregard

$$\cancel{\frac{144}{x+2} = -36x}$$

$$\cancel{\left(144 = -36x^2 - 72x\right)\frac{1}{36}}$$

$$\cancel{4 = -x^2 - 2x}$$

$$\cancel{-x^2 - 2x - 4 = 0}$$

$$\cancel{x = \frac{2 \pm \sqrt{2^2 - 4(-1)(-4)}}{2 \cdot (-1)} = \frac{2 \pm \sqrt{4 - 16}}{-2}} = \text{non real}$$

~~f(x) never goes to zero~~

~~the end points are critical points,~~ $\boxed{-7, -3}$

$f'(x) = 0 = 144(-1)(x+2)^{-2} + 36$

$$\frac{144}{(x+2)^2} = 36$$

$4 = (x+2)^2 = x^2 + 4x + 4$

$0 = x^2 + 4x$

$x = 0, -4$ 0 is also outside the domain of definition, so we don't need that.

$\therefore$ so the only critical points are $\boxed{-7, -4, -3}$

more on back

#9 con't

a max or min will occur at a critical point

$f'(x) = 0$ @ $x = -4$

@ $x = -4$, $y = -216$

* Please note - this does not necessarily indicate we have a max or min on the interval we must compare end points too

lets check the other critical points (ie endpoints) to determine absolute max and min:

@ $x = -3$ $y = -256$

@ $x = -7$ $y = -280.6$

Max: $y = -216$ where $x = -4$

Min: $y = -280.6$ where $x = -7$

global max & min on the interval

$\boxed{10.}$ $f(x) = 4x(x-3)^{2/3}$ $[-5, 10]$

Critical pts:

End points are critical pts; $\underline{-5, 10}$

$f'(x) = 4(x-3)^{2/3} + 4x\,\frac{2}{3}(x-3)^{-\frac{1}{3}} = 0$

$(x-3)^{2/3} = -\frac{2}{3}x(x-3)^{-\frac{1}{3}}$

$(x-3) = -\frac{2}{3}x$

$x + \frac{2}{3}x - 3 = 0$

$\frac{5}{3}x - 3 = 0$

$\frac{5}{3}x = 3$

$x = \frac{3}{5/3} = \frac{9}{5}$

* common term of $x-3$ on each side, so if $x-3 \to 0$ the the function goes to 0

$x - 3 = 0$

$\underline{x = 3}$

Critical pts are @ $\boxed{-5, \frac{9}{5}, 10}$

Max value / min value:

$\boxed{@\ x = -5 \quad y = -80 \quad \Leftarrow \text{min}}$

@ $x = \frac{9}{5}$ $y = 8.13$

~~@ $x = 3$ $y = 0$~~

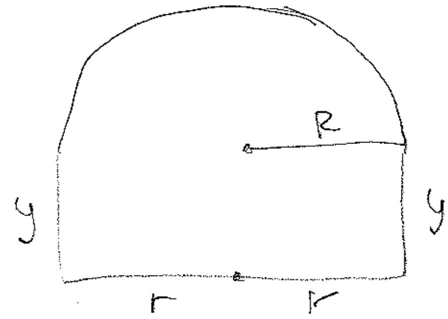
$\boxed{@\ x = 10 \quad y = 146.37 \quad \Leftarrow \text{max}}$

(11) $A = 2yr + \frac{\pi r^2}{2}$

$P = 2y + 2r + \frac{\cancel{2}\pi r}{\cancel{2}}$

$29 = 2y + 2r + \pi r$ — constraint

$y = \frac{29 - 2r - \pi r}{2}$



$A = \cancel{2}r\frac{(29 - 2r - \pi r)}{\cancel{2}} + \frac{\pi r^2}{2}$

$A = 29r - r^2(2+\pi) + \frac{\pi r^2}{2}$

$\frac{\partial A}{dr} = 29 - r(4 + 2\pi) + \pi r$

$0 = 29 - r(4 + 2\pi - \pi)$

$r = \frac{29}{4+\pi} \approx 4.0607$

$A = \left(\frac{29}{4+\pi}\right)\left(29 - 2\left(\frac{29}{4+\pi}\right) - \pi\left(\frac{29}{4+\pi}\right)\right) + \frac{\pi\left(\frac{29}{4+\pi}\right)^2}{2}$

$A = \frac{841}{2(\pi+4)} \approx 58.88$ TV

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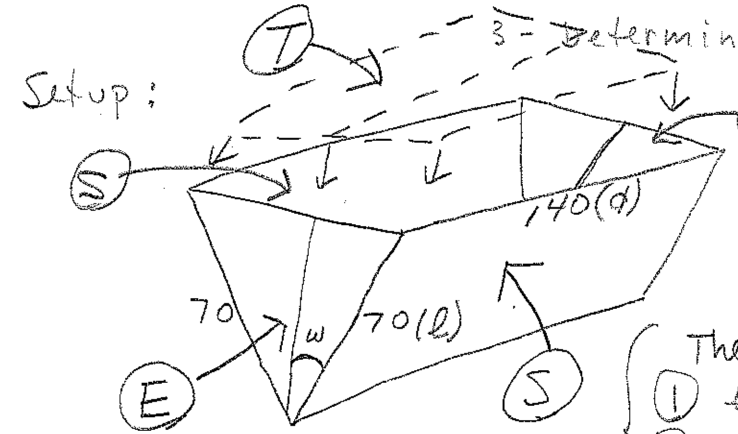
WW#7 P-12

Goals: 1 - Find a formula for the surface area of a closed triangular trough in terms of an angle
2 - Determine the natural domain of the formula
3 - Determine the global max & min of the formula.

Setup:

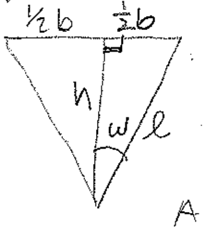


we need to find a surface area formula for this trough in terms of the angle w, let's call it SA(w).

The surface area consists of 5 parts:
1. the rectangular top (T) - depends on w
2. two rectangular sides (S) - does not depend on w
3. Two triangular ends (E) - depends on w

So, $SA(w) = T + 2S + 2E$ Find each of T, S & E and then add.
Note that a couple of labels have been added so this can be done more generally

(A) Since the angle w is actually embedded in (E), WORK THIS OUT first.



The area E is an isocelese triangle, but w only applies to exactly $\frac{1}{2}$ of it. so find the area of $\frac{1}{2}$ of it in terms of w and multiply by 2.

Area $= \frac{1}{2}bh$ and $\sin w = \frac{\frac{1}{2}b}{l}$ so $l \sin w = \frac{1}{2}b \Rightarrow 2l \sin w = b$

$\cos w = \frac{h}{l}$ so $l \cos w = h$

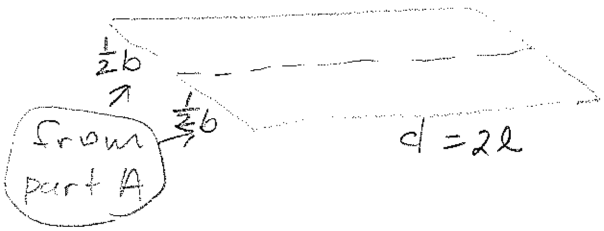
$\Rightarrow$ Area $= l\sin w \cdot l \cos w = l^2 \sin w \cos w$

and there are 2 ends, so that give $\boxed{2E = 2l^2 \sin w \cos w}$

*recall $2\sin\theta\cos\theta = \sin 2\theta \Longrightarrow \quad = \boxed{l^2 \sin 2w}$

(B) Now for the top (T), which also depends on w. But we've already done what we need.

$b = 2l \sin w$ $d = 2l$ see why below



$\boxed{T = 4l^2 \sin w}$

(C) Now the sides. There are 2 sides, 2S; and the nice thing is that the sides are formed from constant lengths.

So $\boxed{2S = 2ld.}$ OR since the sides are formed from a square, (con't) ⟶

$2S = d^2$ because $2l = d$

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(D) Now we put it all together!

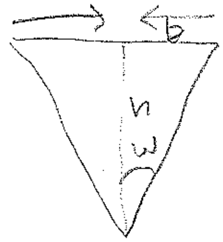
$SA(w) = T + 2E + 2S = 4\ell^2 \sin w + 2\ell^2 \sin w \cos w + d^2$

—OR— $= 4\ell^2 \sin w + \ell^2 \sin 2w + d^2$

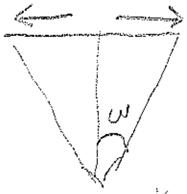
either of these will work!

notice there is only one variable, the angle w.

By the way, we also want to determine the domain of w



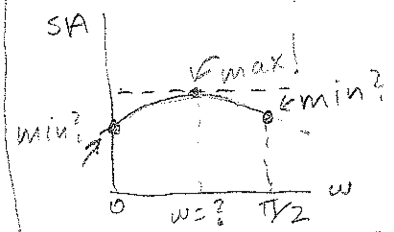
if we "close" the triangle, it follows that $w \to 0$ radians and $b \to 0$ while $h \to \ell$



if we "open" the triangle, it follows that $w \to \frac{\pi}{2}$ (not π!) and $b \to \ell$ while $h \to 0$

* so the domain of w is $0 \le w \le \pi/2$

The graph of $SA(w)$ looks like this:



Interesting note— notice the tangent line where SA is max – the y-intercept is the max value!

Now to find the min & max, we need the derivative of SA and set that to 0

using $SA = 4\ell^2 \sin w + 2\ell^2 \sin w \cos w + d^2$

$$\frac{dSA}{dw} = 4\ell^2 \cos w + 2\ell^2(\cos^2 w - \sin^2 w)$$

where $\cos^2 w - \sin^2 w = 2\cos^2 w - 1$

$$= 4\ell^2 \cos w + 2\ell^2(2\cos^2 w - 1)$$

$$= 4\ell^2 \cos w + 4\ell^2 \cos^2 w - 2\ell^2 = 0$$

$$\Rightarrow 2\cos^2 w + 2\cos w - 1 = 0$$

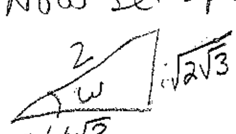
a quadratic! think $\cos w = x$

$$\cos w = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2 \cdot 2} = \boxed{\frac{-1 \pm \sqrt{3}}{2}}$$

take the positive side

when $w = 0$, we have the min. see why?

Now set up $\triangle$



$\sin w = \frac{2\sqrt{3}}{2} = \frac{\sqrt{2\sqrt{3}}}{2}$

$\cos w = \frac{-1+\sqrt{3}}{2}$

* NOTE – Nowhere in this problem are we ever asked for the derivative. Graph SA and find the max!

so

$$SA = 4\ell^2\left(\frac{\sqrt{2\sqrt{3}}}{2}\right) + 2\ell^2\left(\frac{\sqrt{2\sqrt{3}}}{2}\right)\left(\frac{-1+\sqrt{3}}{2}\right) + d^2$$

so for $\ell = 70$, $d = 140$

$$SA = 4 \cdot 70^2\left(\frac{\sqrt{2\sqrt{3}}}{2}\right) + 2 \cdot 70^2\left(\frac{\sqrt{2\sqrt{3}}}{2}\right)\left(\frac{-1+\sqrt{3}}{2}\right) + 140^2$$

which is the max

13) $f(x) = 4x^2 - \frac{4}{x^2} = 4x^2 - 4x^{-2}$

$f'(x) = 8x + 8x^{-3}$

$f''(x) = 8 - 24x^{-4} = 8 - \frac{24}{x^4} = \frac{8x^4 - 24}{x^4}$

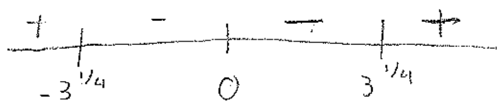
Inflection Points: $8x^4 - 24 = 0 \Rightarrow 8x^4 = 24 \Rightarrow x^4 = 3 \Rightarrow x = \pm(3^{1/4})$

$x^4 = 0 \Rightarrow x = 0 \Leftarrow$ because discontinuous, not an inflection point

Inf. Pts $\in -(3^{1/4}), \cancel{0}, 3^{1/4}$

Concave Up: $(-\infty, -3^{1/4}) \cup (3^{1/4}, \infty)$

Concave Down: $(-3^{1/4}, 0) \cup (0, 3^{1/4})$



14) $F(x) = 15x^2 + 9\sin^2 x$

$F'(x) = 30x + 18\sin x\cos x = 30x + 9\sin 2x$

$F''(x) = 30 + 9\cos 2x \cdot 2 = 30 + 18\cos 2x$

Inflection Points: $30 + 18\cos 2x = 0 \Rightarrow \cos 2x = -\frac{30}{18} = -\frac{5}{3}$ is impossible so DNE

so Concave Up: $(-\infty, \infty)$

Concave Down: DNE