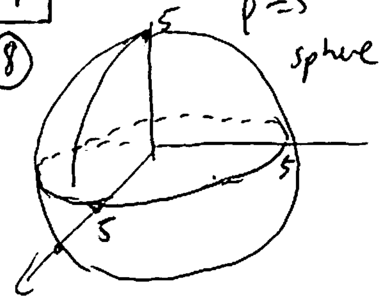
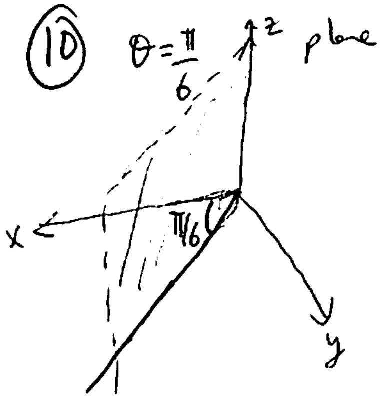
14.7

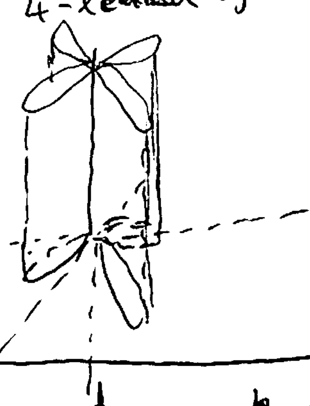
(8) $\rho = 5$ sphere



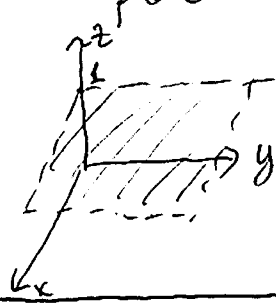
(10) $\theta = \frac{\pi}{6}$ plane



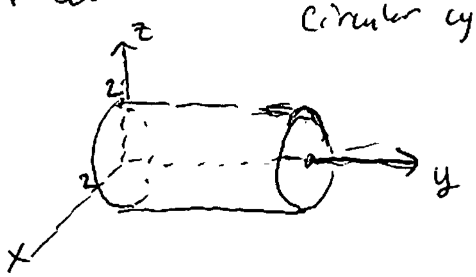
(12) $r = 2\sin 2\theta$
4-~~leaved~~ cylinder



(14) $\rho = \sec\phi \to \rho\cos\phi = 1$
$z = 1$
plane



(16) $r^2\cos^2\theta + z^2 = 4 \to x^2 + z^2 = 4$
circular cylinder.



(18) $r^2\cos^2\theta - r^2\sin^2\theta = 25$

$r^2\cos 2\theta = 25 \quad (\cos^2\theta - \sin^2\theta = \cos 2\theta)$

(20) $(x^2+y^2+z^2) + 3z^2 = 10$

$\rho^2 + 3\rho^2\cos^2\phi = 10$

$\Big\} \to \rho^2 = \dfrac{10}{1+3\cos^2\phi}$

(22) $\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - \cos^2\phi] = 1$

$\rho^2[\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - 1 + \sin^2\phi] = 1$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi(1-\sin^2\theta)] = 1$

$\rho^2[\sin^2\phi\cos^2\theta - 1 + \sin^2\phi\cos^2\theta] = 1$

$\rho^2[2\sin^2\phi\cos^2\theta - 1] = 1$

$\rho^2 = \dfrac{1}{2\sin^2\phi\cos^2\theta - 1}$

(24) $\rho^2 = 2\rho\cos\phi$

$r^2 + z^2 = 2z$

$r^2 = 2z - z^2$

$r = \sqrt{2z - z^2}$

15.1

(30) $144-16x^2-16y^2+9z^2 \geq 0 \rightarrow$ Points inside the hyperboloid of one sheet,
$\frac{x^2}{9}+\frac{y^2}{9}-\frac{z^2}{6}=1$ ~~wh~~ except the points for which $xyz=0$
(Namely, coordinate planes are excluded)

(32) $\ln(xy)$ is defined if $xy>0 \longrightarrow x>0, y>0$ or $x<0$ $y<0$

$\rightarrow D=\{(x,y,z): x,y,z\in\mathbb{R}$ and $[(x>0, y>0)$ or $(x<0, y<0)]\}$

(34) $100x^2+16y^2+25z^2=k \quad k>0$

$\rightarrow \frac{x^2}{\left(\frac{k}{100}\right)}+\frac{y^2}{\left(\frac{k}{16}\right)}+\frac{z^2}{\left(\frac{k}{25}\right)}=1$ $k>0$ is the set of all ellipsoids centered at origins such that their have their ratio $\left(\frac{1}{10}\right):\left(\frac{1}{4}\right):\left(\frac{1}{5}\right)$ or $2:5:4$

(36) $\frac{x^2}{\left(\frac{k}{9}\right)}-\frac{y^2}{\left(\frac{k}{4}\right)}-\frac{z^2}{1}=k$,

$k=0 \rightarrow \frac{y^2}{9}+\frac{z^2}{36}=\frac{x^2}{4}$ (elliptical cone) and

$k\neq 0 \rightarrow$ all hyperboloids (one or two sheets depending on the sign of k) with x-axis for axis such that $a:b:c$ is $\left(\frac{1}{3}\right):\left(\frac{1}{2}\right):1$
or $(2):(3):(6)$

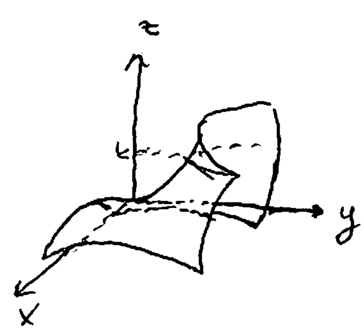
(40) If $z=0$ then $x=0$ or $x=\mp\sqrt{3}y$

If $z=k\neq 0$, $k=x^3-3y^2x$

$\rightarrow x^3-k=3y^2x$

$\frac{x^3-k}{3x}=y^2$

$\mp\sqrt{\frac{x^3-k}{3x}}=y$



15.2

18 $f_x(x,y) = 5(x^3+y^2)^4(3x^2)$

$f_{xy}(x,y) = 60x^2(x^3+y^2)^3(2y)$

$= 120x^2y(x^3+y^2)^3$

$f_y(x,y) = 5(x^3+y^2)^4(2y)$

$f_{yx}(x,y) = 40y(x^3+y^2)^3(3x^2)$

$= 120x^2y(x^3+y^2)^3$

Hence; $f_{xy} = f_{yx}$

20 $f_x(x,y) = y(1+x^2y^2)^{-1}$,

$f_{xy}(x,y) = (1-x^2y^2)(1+x^2y^2)^{-2}$

$f_y(x,y) = x(1+x^2y^2)^{-1}$

$f_{yx}(x,y) = (1-x^2y^2)(1+x^2y^2)^{-2}$

Hence, $f_{xy} = f_{yx}$

22 $F_x = (2x+y)(x^2+xy+y^2)^{-1}$

$F_x(-1,4) = 2/13$

$F_y = (2y+x)(x^2+xy+y^2)^{-1}$

$F_y(-1,4) = \frac{7}{13}$

26 Let $z = f(x,y) = \frac{1}{3}(36-9x^2-4y^2)^{1/2}$

$f_y = \left(-\frac{4}{3}\right)y(36-9x^2-y^2)^{-1/2}$

$f_y(1,-2) = \frac{8}{3\sqrt{11}}$ (slope)

32 $V \cdot P_V + TP_T$

$= V(-kTV^{-2}) + T(kV^{-1})$

$= -\frac{kT}{V} + \frac{kT}{V} = 0$

$PV = kT$

$P = \frac{kT}{V}$

$P_V = \frac{-kT}{V^2}$

$P_T = k/V$

$V = \frac{kT}{P}$

$V_T = \frac{k}{P}$

$T_P = \frac{V}{k}$

$P_V \cdot V_T \cdot T_P = \left(\frac{-kT}{V^2}\right)\left(\frac{k}{P}\right)\left(\frac{V}{k}\right)$

$= \frac{-kT}{VP} = -\frac{VP}{VP}$

$= -1$

34 $f_x = 2x(x^2+y^2)^{-1}$

$f_{xx} = -2(x^2-y^2)(x^2+y^2)^{-1}$

$f_y = 2y(x^2+y^2)^{-1}$

$f_{yy} = 2(x^2-y^2)(x^2+y^2)^{-1} = -f_{xx}$

So, $f_{xx} + f_{yy} = 0$

15.3

(6) $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{(x^2+y^2)} = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} \cdot \frac{1}{\cos(x^2+y^2)}$

Set $u = x^2+y^2$, $u \to 0$

$= \lim_{u\to 0}\left(\frac{\sin u}{u}\right) \cdot \lim_{u\to 0} \frac{1}{\cos u} = (1)(1) = \boxed{1}$

---

(8) $\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{(x^2+y^2)} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = \boxed{0}$

---

(16) $y=0 \longrightarrow \lim_{x\to 0} \frac{0}{x^2+0} = 0$

$y=x \longrightarrow \lim_{x\to 0} \frac{x^2+x^3}{x^2+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}(1+x)}{2\cancel{x^2}} = \frac{1}{2}$

$0 \neq \frac{1}{2} \longrightarrow$ limit does not exist

---

(18) Let $\varepsilon > 0$ be arbitrary. We wish to show that there is $\delta > 0$ such that for all x, y,
$0 < |(x,y) - (0,0)| = \sqrt{x^2+y^2} < \delta \longrightarrow |f(x,y) - L| = \left|\frac{xy^2}{x^2+y^2}\right| < \varepsilon$

[Scratch work: $\left|\frac{xy^2}{x^2+y^2}\right| \leq \frac{(x^2+y^2)^{3/2}}{x^2+y^2} = \sqrt{x^2+y^2}$.]

Set $\delta = \varepsilon$. Assume that (x,y) is s.t. $\sqrt{x^2+y^2} < \delta = \varepsilon$

$|f(x,y) - 0| = \left|\frac{xy^2}{x^2+y^2}\right| \leq \sqrt{x^2+y^2} < \delta = \varepsilon$ implies that $|f(x,y) - 0| < \varepsilon$

Hence, f is continuous at $(x,y) = (0,0)$

---

(25) $\frac{x^2-4y^2}{x-2y} = \frac{(x-2y)(x+2y)}{x-2y} = x+2y$ if $x \neq 2y$ $\quad (x = 2y \Rightarrow y = \frac{x}{2})$

We want $g(x) = x + 2\left(\frac{x}{2}\right) = 2x$ to have continuity

15.4

(12) $\nabla f = \langle 3x^2y + 3y^2, x^3 + 6xy \rangle \to \nabla f(2,-2) = \langle -12, -16 \rangle$

$z = f(2,-2) + \nabla f(2,-2) \cdot \langle x-2, y+2 \rangle$ (Tangent Plane)

$$\left.\begin{aligned} z &= 8 + \langle -12, -16 \rangle \langle x-2, y+2 \rangle \\ &= 8 + (-12x + 24) + (-16y + 32) \end{aligned}\right\} \Longrightarrow \boxed{z = -12x - 16y}$$

(14) $\nabla f = \left\langle \frac{2x}{y}, -\frac{x^2}{y^2} \right\rangle \to \nabla f(2,-1) = \langle -4, -4 \rangle$

$$\left.\begin{aligned} z = f(2,-1) + \langle -4, -4 \rangle \cdot \langle x-2, y+1 \rangle \end{aligned}\right\} \boxed{z = -4x - 4y}$$

(16) $\nabla f = \langle yz + 2x, xz, xy \rangle \longrightarrow \nabla f(2,0,-3) = \langle 4, -6, 0 \rangle$

$w = f(2,0,-3) + \langle 4, -6, 0 \rangle \cdot \langle x-2, y, z+3 \rangle = 4 + 4x - 8 - 6y + 0$

$\boxed{w = 4x - 6y + 4}$

(18) $\nabla(f^r) = \langle rf^{r-1} \cdot f_x, rf^{r-1} \cdot f_y \rangle = rf^{r-1} \langle f_x, f_y \rangle = rf^{r-1} \nabla f$

(20) Let $F(x,y,z) = x^3 - z = 0 \to \nabla F = \langle 3x^2, 0, -1 \rangle$

Tangent plane is horizontal if $\nabla F = \langle 0, 0, k \rangle$ where $k \neq 0$

Here we need to solve $3x^2 = 0$ only. But this means $x = 0$

Here there is a horizontal tangent plane at $(x,y) = (0,y)$.

(There is infinitely many points satisfying this)

(26) $\nabla f(p) = p \Rightarrow \nabla f(x,y) = \langle x, y \rangle \Rightarrow f_x = x, f_y = y$

$$\left.\begin{aligned} &\Rightarrow f = \int x\,dx = \frac{x^2}{2} + \alpha(y) \text{ for any function } \alpha(y). \\ &\Rightarrow f = \int y\,dy = \frac{y^2}{2} + \beta(x) \text{ for any function } \beta(x) \end{aligned}\right\} \to \boxed{f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C} \text{ for any } C \in \mathbb{R}.$$

15.5

(14) $-\nabla f(x,y) = \langle -3\cos(3x-y), \cos(3x-y)\rangle \rightarrow -\nabla f\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\langle -3, 1\rangle$

↳ most rapid decrease is at this direction

Unit vector: $-\frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{10}}\langle -3, 1\rangle$

~~(18) Vector from origin to $(0, \frac{\pi}{3})$: $\langle \frac{\pi}{3} - 0, 0 - 0\rangle = \langle 0, \frac{\pi}{3}\rangle \rightarrow$ unit vector $u = \langle 0, 1\rangle$~~

~~$D_u(x,y) = \langle -e^{-x}\cos y, -e^{-x}\sin y\rangle\langle 0, 1\rangle = -e^{-x}\sin y$~~

---

(18) Vector from $(0, \frac{\pi}{3})$ to origin: $\langle 0-0, 0-\frac{\pi}{3}\rangle = \langle 0, -\frac{\pi}{3}\rangle \rightarrow$ unit vector $u = \langle 0, -1\rangle$

$\rightarrow D_u(x,y) = \langle -e^{-x}\cos y, -e^{-x}\sin y\rangle\langle 0, -1\rangle = e^{-x}\sin y$

$\rightarrow D_u(0, \frac{\pi}{3}) = e^0 \cdot \sin\frac{\pi}{3} = 1\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

---

(24) Unit vector from $(2,4)$ toward $(5,0)$ is $\frac{\langle 5-2, 4-0\rangle}{\sqrt{(5-2)^2 + (4-0)^2}} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$

$\rightarrow D_u f(2,4) = \langle -3, 8\rangle\langle \frac{3}{5}, -\frac{4}{5}\rangle = \boxed{-8.2}$

---

(30) (9) $D_u f$ ~~$= \langle \frac{3}{5}, -\frac{4}{5}\rangle\langle f_x, f_y\rangle = -6 \rightarrow \begin{pmatrix} 3f_x = -30 \\ -4f_x = -30 \end{pmatrix} \rightarrow \begin{pmatrix} f_x = -10 \\ f_y \end{pmatrix}$~~ $\boxed{3f_x - 4f_y = -30}$

$\boxed{15.5, 30}$

a) $D_u f = \langle \frac{3}{5}, -\frac{4}{5} \rangle \langle f_x, f_y \rangle = -6 \rightarrow 3f_x - 4f_y = -30 \xrightarrow{(\times 3)} 9f_x - 12f_y = -90$

$D_v f = \langle \frac{4}{5}, \frac{3}{5} \rangle \langle f_x, f_y \rangle = 17 \rightarrow 4f_x + 3f_y = 85 \xrightarrow[(\times 4)]{} 16f_x + 12f_y = 340$ $+$

So $25f_x = 250$ $\left.\begin{matrix} \\ \end{matrix}\right\}$ $30 - 4f_y = -30$

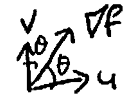
$\boxed{f_x = 10}$ $\quad 60 = 4f_y$

$\boxed{15 = f_y}$

$\nabla f = \langle 10, 15 \rangle$

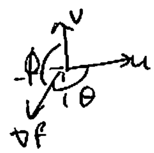
(b) Without loss of generality, let $u = \hat{i} = \langle 1, 0 \rangle$, $v = \hat{j} = \langle 0, 1 \rangle$. If θ and ϕ are the angles between u and ∇f, and between v and ∇f (respectively) then;

there are 4 cases:

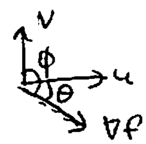


① : $\theta + \phi = \frac{\pi}{2}$ (if ∇f is in 1st quadrant)

② : $\theta = \phi + \frac{\pi}{2}$ (if ∇f is in 2nd ")



③ : $\theta + \phi = \frac{3\pi}{2}$ (if ∇f is in 3rd ")



④ : $\phi = \theta + \frac{\pi}{2}$ (if ∇f is in 4^{th} ")

In each case, $\cos\phi = \sin\theta$ or $\cos\phi = -\sin\theta$

Namely; $\cos^2\phi = \sin^2\theta$

Thus, $(D_u f)^2 + (D_v f)^2 = (u \cdot \nabla f)^2 + (v \cdot \nabla f)^2$ $\qquad |u| = 1 \quad |v| = 1$

$= |\nabla f|^2 \cdot \cos^2\theta + |\nabla f|^2 \cos^2\phi$

$= |\nabla f|^2 (\cos^2\theta + \cos^2\phi)$ $\qquad \cos^2\phi = \sin^2\theta$

$= |\nabla f|^2 (\cos^2\theta + \sin^2\theta)$

$= |\nabla f|^2$

15.6

(14) $\frac{\partial z}{\partial s} = (y+1)(1) + (x+1)(rt) = 1 + rt(1+2s+r+t)$

Hence, $\boxed{\left.\frac{\partial z}{\partial s}\right|_{(1,-1,2)} = 5}$

(16) $\frac{\partial w}{\partial \theta} = (2xy)(-\rho\sin\theta\sin\phi) + (x^2)(\rho\cos\theta\sin\phi) + 2z(0)$

$= \rho^3\cos\theta\sin^3\phi(-2\sin^2\theta + \cos^2\theta)$

$\left.\frac{\partial w}{\partial \theta}\right|_{(2,\pi,\frac{\pi}{2})} = -8$

(22) Let $F(x,y) = ye^{-x} + 5x - 17 = 0$ $\quad \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = \boxed{-\frac{(-ye^{-x}+5)}{e^{-x}}}$

(24) Let $F(x,y) = x^2\cos y - y^2\sin x = 0$

Then $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = \frac{-(2x\cos y - y^2\cos x)}{-x^2\sin y - 2y\sin x}$

(28) Use $z_r = \frac{\partial z}{\partial r}$ (as notation).

$z_r = z_x \cdot x_r + z_y \cdot y_r = z_x\cos\theta + z_y\sin\theta$

$z_\theta = z_x x_\theta + z_y y_\theta = z_x(-r\sin\theta) + z_y(r\cos\theta)$

$\frac{z_\theta}{r} = -z_x\sin\theta + z_y\cos\theta$

$(z_r)^2 + \frac{1}{r^2}(z_\theta)^2 = (z_x\cos\theta + z_y\sin\theta)^2 + (-z_x\sin\theta + z_y\cos\theta)^2$

$= z_x^2\cos^2\theta + z_y^2\sin^2\theta + 2z_x z_y\cos\theta\sin\theta$

$+ z_x^2\sin^2\theta + z_y^2\cos^2\theta - 2z_x z_y\cos\theta\sin\theta$

$= z_x^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + z_y^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}$

$= z_x^2 + z_y^2$

15.6

30) $w = f(x,y,z)$ $x = r-s$, $y = s-t$, $z = t-r$

$\frac{\partial w}{\partial r} = w_r$ (use this notation) $x_r = 1$ $x_s = -1$ $x_t = 0$ | $y_r = 0$, $y_s = 1$, $y_t = -1$

$z_r = -1$, $z_s = 0$, $z_t = 1$

$$w_r + w_s + w_t = (w_x \cdot x_r + w_y y_r + w_z \cdot z_r) + (w_x x_s + w_y y_s + w_z z_s) + (w_x x_t + w_y y_t + w_z z_t)$$

$$= w_x \underbrace{(x_r + x_s + x_t)}_{=0} + w_y \underbrace{(y_r + y_s + y_t)}_{=0} + w_z \underbrace{(z_r + z_s + z_t)}_{=0}$$

$$= \boxed{0}$$

---

32) If $f(tx,ty) = t f(x,y)$ then $\frac{d}{dt}(f(tx,ty)) = \frac{d}{dt}(t f(x,y))$

$$\Rightarrow f_{tx}(tx,ty) \cdot x + f_{ty}(tx,ty) \cdot y = f(x,y)$$

if $t=1$, $x f_x(x,y) + y f_y(x,y) = f(x,y)$

~~$(\nabla f(x,y)) \cdot (\langle x,y \rangle)$~~

~~15.7~~

16) ~~(1,1,1) satisfies each equation. So, the surfaces intersect at (1,1,1).~~

~~For $z = f(x,y) = x^2 y$, $\nabla f = \langle 2xy, x^2 \rangle \to \nabla f(1,1) =$~~

$\boxed{15.7}$ (1,1,1) satisfies each equations, so the surfaces intersect at (1,1,1)

(16) Define $F = \underbrace{f(x,y)-z}_{=0} = x^2y - z \rightarrow \nabla F = \langle 2xy, x^2, -1 \rangle$

$\rightarrow \nabla F(1,1,1) = \langle 2, 1, -1 \rangle$

Define $G = \underbrace{g(x,y) - y}_{=0} = \frac{1}{4}x^2 + \frac{3}{4} - y \rightarrow \nabla G = \langle \frac{1}{2}x, -1, 0 \rangle$

$\rightarrow \nabla G(1,1,1) = \langle \frac{1}{2}, -1, 0 \rangle$

$\nabla F(1,1,1) \cdot \nabla G(1,1,1) = \langle 2,1,-1 \rangle \cdot \langle \frac{1}{2}, -1, 0 \rangle = 0$

Hence, the normals of both target planes are perpendicular.

This means that tangent planes are perpendicular.

---

(18) Let $F = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \nabla F = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$

$\nabla F(x_0, y_0, z_0) = 2 \cdot \langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \rangle$

Tangent plane: $\nabla F(x_0,y_0,z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$$2\left(\frac{xx_0}{a^2} - \frac{x_0^2}{a^2} + \frac{yy_0}{b^2} - \frac{y_0^2}{b^2} + \frac{zz_0}{c^2} - \frac{z_0^2}{c^2} \right) = 0 \qquad \left(\times \frac{1}{2}\right)$$

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = \underbrace{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}}_{=1} = 1$$

---

(20) Let $f = x - z^2$, $g = y - z^3 \rightarrow \nabla f = \langle 1, 0, -2z \rangle$, $\nabla g = \langle 0, 1, -3z^2 \rangle$

$\rightarrow \nabla f(1,1,1) = \langle 1, 0, -2 \rangle$, $\nabla g(1,1,1) = \langle 0, 1, -3 \rangle$

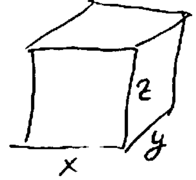
$$\langle 1,0,-2 \rangle \times \langle 0,1,-3 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} = \langle 2, 3, 1 \rangle$$

Line: $x = 1 + 2t$, $y = 1 + 3t$, $z = 1 + t$

1).+

(22) $V = xyz \quad dx = dy = \frac{1}{2}, \; dz = \frac{1}{4}, \; x = 72, \; y = 48, \; z = 36$

$dV = yz\,dx + xz\,dy + xy\,dz = 3024 \text{ in}^3$



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(26) Let $F = x^2 + y^2 + 2z^2 - 6 \rightarrow \nabla F = \langle 2x, 2y, 4z \rangle$

$\nabla F(1,2,1) = 2\langle 1, 2, 2 \rangle$

$\rightarrow \frac{\nabla F}{|\nabla F|} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle = u \qquad |\nabla F| = 6$

u is the unit vector at the direction of the flight and 4 being the speed of the bee, we get

$\langle x, y, z \rangle = \langle 1, 2, 1 \rangle + 4t \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$ is the location of bee along its line of flight t seconds after takeoff. Plugging in these x, y, z values in the plane formula we get:

$$2\left(1 + \frac{4t}{3}\right) + 3\left(2 + \frac{8t}{3}\right) + \left(1 + \frac{8t}{3}\right) = 49$$

$$9 + \frac{40t}{3} = 49 \longrightarrow \boxed{t = 3} \text{ seconds after takeoff}$$

The point on the plane is, then, given by:

$$(x, y, z) = (5, 10, 9).$$

(15.8) (16) d = distance from origin $= (x^2+y^2+z^2)^{1/2}$

$d^2 = x^2+y^2+z^2$ and $x+2y+3z=12$

$x = 12-2y-3z$

Minimize $f(y,z) = d^2 = y^2+z^2+(12-2y-3z)^2$

$\nabla f = \langle -48+12x+10y, -72+12y+20z \rangle = \langle 0,0 \rangle$ at $(x,y) = \left(\frac{12}{7}, \frac{18}{7}\right)$

$D = f_{yy}f_{zz} - f_{yz}^2 = 56 > 0 \qquad f_{yy} = 10 > 0$

By Second Partials Test, f has a local min at $\left(\frac{12}{7}, \frac{18}{7}\right)$

$\Rightarrow d^2 = \frac{504}{49} \Longrightarrow d = \frac{6\sqrt{14}}{7}$ is the shortest distance.

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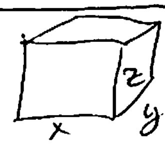
(18) Let $L = 4x+4y+4z \qquad xyz = V_0 \qquad z = \frac{V_0}{xy} \qquad \begin{Bmatrix} x \neq 0 \\ y \neq 0 \\ z \neq 0 \end{Bmatrix}$



$L(x,y) = 4x+4y+\frac{4V_0}{xy}$

$L_x = 4 - \frac{4V_0}{x^2y} \qquad L_y = 4 - \frac{4V_0}{xy^2} \qquad \nabla L = 0$ if $L_x = 0, L_y = 0$

$\rightarrow 4x^2y - 4V_0 = 0$ & $4xy^2 - 4V_0 = 0 \rightarrow 4x^2y = 4xy^2 \quad xy \neq 0$

$\rightarrow \cancel{4x} = y$

$CP = \{(x,x) : x \in \mathbb{R} - \{0\}\}$

$g(x) = L(x,x) = 8x + \frac{4V_0}{x^2} \rightarrow g'(x) = 8 - \frac{8V_0}{x^3}$

$g' = 0$ if $x = \sqrt[3]{V_0}$ and we have a (local min)* at this value.

Hence it gives the ~~min~~ $= 4\sqrt[3]{V_0} + 4\sqrt[3]{V_0} + 4\sqrt[3]{V_0} = 12\sqrt[3]{V_0}$.

(*) local min because $g''(\sqrt[3]{V_0}) > 0 \rightarrow g$ has a local min by 2nd Derivative Test.

$\boxed{15.8}$ 20 V = volume of the box / (x,y,z): first octant vertex
$(x>0, y>0, z>0)$

$$\Rightarrow \left.\begin{array}{l} V = (2x)(2y)(2z) = 8xyz \\ 24x^2 + y^2 + z^2 = 9 \end{array}\right\} \quad F(x,y) = V^2 = 64x^2y^2(9 - 24x^2 - y^2)$$
$$= 64(9x^2y^2 - 24x^4y^2 - x^2y^4)$$

$$\nabla F = \langle 18xy^2 - 96x^3y^2 - 2xy^4, 18x^2y - 48x^4y - 4x^2y^3 \rangle$$

$$= 128\langle xy^2(9 - 48x^2 - y^2), x^2y(9 - 24x^2 - 2y^2)\rangle$$
$$= \langle 0, 0 \rangle$$

if $9 - 48x^2 - y^2 = 0$ & $9 - 24x^2 - 2y^2 = 0$ (note $x=0$ or $y=0$ can easily dismissed since the $V=0$)

$y^2 = 9 - 48x^2 \longrightarrow 9 - 24x^2 - 2(9 - 48x^2) = 0$

$\rightarrow 72x^2 = 9 \rightarrow x^2 = \frac{1}{8} \Rightarrow x = \sqrt{\frac{1}{8}}$

$\rightarrow y^2 = 9 - 48\left(\frac{1}{8}\right) = 9 - 6 = 3 \rightarrow y = \sqrt{3}$

At $\left(\frac{1}{\sqrt{8}}, \sqrt{3}\right)$ $\left.\begin{array}{l} D = f_{xx}f_{yy} - f_{xy}^2 > 0 \\ f_{xx} < 0 \end{array}\right\}$ local max at $\left(\frac{1}{\sqrt{8}}, \sqrt{3}\right)$

Hence local max $\rightarrow$ global max at $\left(\frac{1}{\sqrt{8}}, \sqrt{3}\right)$

The greatest possible volume is $V = 8\left(\frac{1}{\sqrt{8}}\right)(\sqrt{3})(\sqrt{3}) = \boxed{6\sqrt{2}}$

$(z = \sqrt{3})$

1318

28

(a) $\frac{\partial f}{\partial m} = \sum_{i=1} \frac{\partial}{\partial m}(y_i - m x_i - b)^2 = 2\sum_{i=1}^{n}(y_i - m x_i - b)(-x$

$= -2\sum_{i=1}^{n}(x_i y_i - m x_i^2 - b x_i)$

Setting $\frac{\partial f}{\partial m} = 0$, $\sum_{i=1}^{n} x_i y_i = m\left(\sum_{i=1}^{n} x_i^2\right) + b\left(\sum_{i=1}^{n} x_i\right)$ (*)

$\frac{\partial f}{\partial b} = \sum_{i=1}^{n}\frac{\partial}{\partial b}(y_i - m x_i - b)^2 = 2\sum_{i=1}^{n}(y_i - m x_i - b)(-1)$

$= -2\sum_{i=1}^{n}(y_i - m x_i - b)$

Setting $\frac{\partial f}{\partial b} = 0$, $\sum_{i=1}^{n} y_i = m\left(\sum_{i=1}^{n} x_i\right) + bn$ (#)

(b) Multiplying (*) by $(-n)$ and (#) by $\left(\sum_{i=1}^{n} x_i\right)$, we get

$-m\left(\sum_{i=1}^{n} x_i^2\right)n - bn\left(\sum_{i=1}^{n} x_i\right) = -n\sum_{i=1}^{n} x_i y_i$

$+ \quad m\left(\sum_{i=1}^{n} x_i\right)^2 + bn\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)$

$m\left[\left(\sum_{i=1}^{n} x_i\right)^2 - n\left(\sum_{i=1}^{n}(x_i^2)\right)\right] = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - n\sum_{i=1}^{n} x_i y_i$

$$m = \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - n\sum_{i=1}^{n}(x_i y_i)}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\sum_{i=1}^{n}(x_i)^2}$$

(Using this we can find b)

~~Using this, we get $b = \sum_{i=1}^{n} y_i - m \sum x_i$~~

(c) $f_{mm} = 2\sum_{i=1} x_i^2$ $\quad f_{bb} = 2n$ $\quad f_{mb} = 2\sum_{i=1}^{n} x_i$

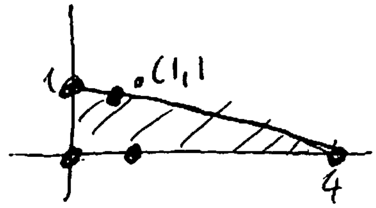
$D = 4n\left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right) > 0$ if all ~~[illegible]~~ x_i's are not the same (We have $f_{mm} > 0$)

Hence, the result in (b) gives a local min (hence global)

15.8 (30) $z = 2x^2 + y^2 - 4x - 2y + 5$

$\nabla z = \langle 4x - 4,\ 2y - 2 \rangle = 0$ if $(x,y) = (1,1)$

(1,1) is not in the given triangle.
Hence, not a critical pt.



on vertical side, $x = 0 \rightarrow z(y) = y^2 - 2y + 5 \quad y \in [0,1]$
$z'(y) = 2y - 2 = 0$, if $y = 1$. Hence no additional C.P. other than the corners. $z(0) = 5$, $z(1) = 4$

on horizontal side, $y = 0 \rightarrow z(x) = 2x^2 - 4x + 5 \quad x \in [0,4]$
$z'(x) = 4x - 4 = 0$ if $x = 1$. Hence, additional C.P at $(1,0)$. $z(1,0) = 3 \quad z(0) = 5 \quad z(4) = 21$

On hypotenuse, $x = 4 - 4y \rightarrow z(y) = 2(4-4y)^2 + y^2 - 4(4-4y) - 2y + 5$
$y \in [0,1]$
$\rightarrow z(y) = 33y^2 - 50y + 21$
$\rightarrow z'(y) = 66y - 50 = 0$, if $y = \frac{50}{66} = \frac{25}{33}$

Hence additional C.P at $\left(\frac{32}{33}, \frac{50}{66}\right)$.
$z\left(\frac{50}{66}\right) = 2.06$

Now:

x	y	z
0	0	5
4	0	21
0	1	4
1	0	3
$\frac{32}{33}$	$\frac{25}{33}$	2.06

The Max is 21 at (4,0)

The Min is 2.06 at $\left(\frac{32}{33}, \frac{25}{33}\right)$

(36) Global max $f(0,1) = 0.5$
min $f(0,-1) = -0.5$

(40) Global max
min (NONE)

(45) Global max $f(2.1, 2.1) = 3.5$

15.7 16 Let (x,y,z) be the point of intersection.

$f(x,y,z) = x^2+y^2+z^2$: square of distance from origin

Minimize f subject to $g(x,y,z) = x+y+z-8=0$ & $h(x,y,z) = 2x-y+3z-28$

Let $\nabla f = \lambda \nabla g + \mu \nabla h$

$\langle 2x, 2y, 2z\rangle = \lambda \langle 1,1,1\rangle + \mu \langle 2,-1,3\rangle$

$\langle 2x, 2y, 2z\rangle = \langle \lambda+2\mu, \lambda-\mu, \lambda+3\mu\rangle$

① $2x = \lambda + 2\mu$

② $2y = \lambda - \mu$

③ $2z = \lambda + 3\mu$

④ $x+y+z=8$

⑤ $2x-y+3z=28$

①+②+③ gives $2(\underbrace{x+y+z}_{8}) = 3\lambda + 4\mu \rightarrow$ ⑥ $16 = 3\lambda + 4\mu$

① $-\frac{1}{2}$② $+\frac{3}{2}$③ gives $2x-y+3z = 2\lambda + 7\mu \rightarrow$ ⑦ $28 = 2\lambda + 7\mu$

3·⑦ $-$ 2 ⑥ gives $84-32 = 21\mu - 8\mu$

$52 = 13\mu \rightarrow \boxed{\mu = 4}$

Using this in ⑥, we get $16 = 3\lambda + 16 \rightarrow \boxed{\lambda = 0}$

Using these values, we get $2x = 8 \rightarrow \boxed{x = 4}$

$2y = -4 \rightarrow \boxed{y=-2}$

$2z = 12 \rightarrow \boxed{z = 6}$

$f(4,-2,6) = 56$ and the nature of problem indicates that this is the minimum one.

Hence min distance $= \sqrt{56}$

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(20) Max: $f(-0.71, 0.71) = f(-0.71, -0.71)$

Min: $= 0.71$

(22) Max: $f(1.41, 1.41)$

$\parallel$

$f(-1.41, -1.41)$

$\parallel$

0.037