# Stability of Dirac Concentrations in an Integro-PDE Model for Evolution of Dispersal

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**Abstract**

We consider an integro-PDE model from evolutionary biology. The solution $u_\epsilon(x, \alpha)$ structured by two variables $x \in D \subset \mathbb{R}^k$ and $\alpha \in (\underline{\alpha}, \overline{\alpha}) \subset\subset \mathbb{R}_+$. The diffusion coefficient in the $x$ direction depends on $\alpha$ and the diffusion coefficient in the $\alpha$ direction is a constant $\epsilon^2$. A special feature of this model is the appearance of the integral $\hat{u}_\epsilon(x)$ of the solution in the $\alpha$ variable, which can be viewed as an infinite dimensional parameter of the problem. In a previous work, the existence of a steady state that exhibits Dirac-concentration in one of the variables yet remains regular in the other variables was proved independently by [Lam and Lou, *J. Funct. Anal.* (2017)] and [Perthame and Souganidies, arXiv:1505.03420 (2015)].

In this paper, we tackle the long-time dynamics of solutions. When the environment function is non-constant, we show that the steady state is linearly stable by considering the corresponding nonlocal eigenvalue problem. Uniqueness of steady state is obtained from the stability result via a degree argument. When the environment function is constant, the global asymptotic stability result is obtained. This problem can be regarded as a competition of infinitely many species parameterized by $\alpha$. As with the competition model for three or more species, the integro-PDE model does not generate a monotone dynamical system so that it is necessary to consider all (real or complex) eigenvalues in determining its linear stability.

## 1 Introduction

Elliptic and parabolic differential equations and systems arise in studies and models related to population dynamics, combustion theory and nerve conduction. See, e.g. the survey of Aronson and Weinberger [5] and the book of Perthame [44]. The associated elliptic eigenvalue problem, on the other hand, plays a decisive role when dealing with problems connected to the existence, uniqueness and stability of solutions to these systems [3]. For instance, a steady state solution is said to be linearly stable if the spectrum of the linearized eigenvalue problem lies entirely in $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$. In the case of a single elliptic equation or a system of elliptic equation with cooperativity, the stability is completely determined by a principal eigenvalue $\lambda_1$ which is real and equals the infimum of the real part of the spectrum [34]. However, for general elliptic systems (e.g. competition systems of three or more species), or elliptic equations possessing nonlocal (or integral) dependence on solution, the associated eigenvalue problem does not in general possess such a principal eigenvalue.

In this paper, we will prove stability and uniqueness property of the positive solution of an elliptic equation with nonlocal dependence by studying the linearized eigenvalue problem associated with the positive solution. A novel feature of the model is the appearance of the integral of the solution with respect to a subset of the independent variables. Roughly speaking, this model describes the competition of infinitely many species parameterized by the variable $\alpha$, and the selection of the optimal phenotype.

### 1.1 The mutation-selection model

The model concerns a population structured simultaneously by a spatial variable $x \in D$ and the spatial motility trait $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ of the species, where $D$ be a bounded smooth domain in $\mathbb{R}^k$ and $0 < \underline{\alpha} < \overline{\alpha}$. The population comprises a family of phenotypes differentiated by the spatial motility rate $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, i.e. Individuals belong to the same phenotype if they have the same spatial motility rate. We assume that the population has overlapping generation, such that the mutation is modeled by a diffusion process with constant rate $\epsilon^2 > 0$ acting on the phenotypic trait variable $\alpha$. The resource $m(x)$ is heterogeneously distributed in space, and individuals compete with all other individuals at the same spatial location. Precisely, consider

$$\begin{cases} \partial_t u = \alpha \Delta_x u + u(m(x) - \hat{u}(x,t)) + \epsilon^2 \partial_\alpha^2 u & \text{in } D \times (\underline{\alpha}, \overline{\alpha}) \times (0,\infty), \\ \partial_n u = 0 & \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}) \times (0,\infty), \\ \partial_\alpha u = 0 & \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\} \times (0,\infty), \\ u(x,\alpha,0) = u_0(x,\alpha) & \text{on } \Omega := D \times (\underline{\alpha}, \overline{\alpha}). \end{cases} \tag{1.1}$$

Here $\Delta_x = \sum_{i=1}^k \partial_{x_i}^2$ denotes the Laplace operator in the spatial variables; $\epsilon > 0$ is a constant; $n$ is the outer unit-normal vector on the boundary $\partial D$ of $D$; $\partial_n = n \cdot \nabla_x$ is the outer normal derivative with respect to the boundary portion $\partial D \times [\underline{\alpha}, \overline{\alpha}]$ of $\partial\Omega$. The density of the total population at the spatial location $x \in D$ and time $t > 0$ is given by

$$\hat{u}(x,t) := \int_{\underline{\alpha}}^{\overline{\alpha}} u(x,\alpha,t)\, d\alpha,$$

so that the competition is nonlocal in the trait variable $\alpha$. i.e. each individual competes with all other individuals across phenotypes that are present at the same location $x$ for resources.

In a previous paper, the following result concerning the asymptotic behavior of positive steady states of (1.1) was proved: (We refer the readers to Theorem 3.2 for details and also to the work of Perthame and Souganidis [46] where an alternative approach is presented for a closely related problem.)

**Theorem A** ([36]). *Suppose $\int_D m(x)\,dx > 0$, then for all $\epsilon > 0$, (1.1) has at least one positive steady state $u_\epsilon$. Moreover, if $m(x)$ is non-constant then any positive steady state of (1.1) satisfies*

$$u_\epsilon(x,\alpha) \to \delta_0(\alpha - \underline{\alpha})\theta_{\underline{\alpha}}(x) \quad \textit{in distribution as } \epsilon \to 0,$$

*where $\theta_{\underline{\alpha}} = \theta_{\underline{\alpha}}(x)$ is the unique positive solution to (1.4) when $\alpha = \underline{\alpha}$.*

We investigate in this paper the uniqueness and stability of Dirac-concentrated steady states of (1.1).

## 1.2 Motivations from evolutionary biology

The model (1.1) can be viewed as a continuum (in trait) version of the following mutation-selection model considered by Dockery et al. [24], concerning the competition of $N$ species with different dispersal rates but are otherwise identical:

$$\begin{cases} \partial_t U_j = \alpha_j \Delta_x U_j + U_j \left[ m(x) - \sum_{i=1}^N U_i \right] + \epsilon^2 \sum_{i=1}^N M_{ij} U_i \\ \qquad\qquad\qquad\qquad\qquad \text{for } x \in D, t > 0, \text{ and } 1 \le j \le N, \\ \partial_n U_j = 0 \qquad\qquad\quad\ \text{for } x \in \partial D, t > 0, \text{ and } 1 \le j \le N, \\ U_j(x,0) = U_{j,0}(x) \qquad \text{for } x \in D, \text{ and } 1 \le j \le N, \end{cases} \tag{1.2}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$ are constants, $m(x) \in C^2(\overline{D})$ is non-constant, $M_{ij}$ is an irreducible real $N \times N$ matrix that models the mutation process so that $M_{ii} < 0$ for all $i$, and $M_{ij} \ge 0$ for $i \ne j$ and $\epsilon^2 \ge 0$ is the mutation rate.

Model (1.2) was introduced to address the question of evolution of unconditional dispersal. The case when there is no mutation, i.e. $\epsilon = 0$, was considered in [28] where it was shown that in a competition system of two species with different diffusion rates but are otherwise identical, a rare competitor can invade the resident species if and only if the rare invader has a lower motility rate. Dockery et al. [24] generalized the work of Hastings [28] to $N$ species situation, and proved that no two species can coexist at equilibrium, i.e. the set of non-trivial, non-negative steady states of the system (1.2) is given by

$$\{(\theta_{\alpha_1}, 0, ..., 0), (0, \theta_{\alpha_2}, 0, ..., 0), ..., (0, ..., 0, \theta_{\alpha_N})\}, \tag{1.3}$$

where $\theta_\alpha$ is the unique positive solution of ([15, Propositions 3.2 and 3.3])

$$\alpha \Delta_x \theta + \theta(m(x) - \theta) = 0 \quad \text{in } D, \qquad \text{and} \qquad \partial_n \theta = 0 \quad \text{on } \partial D. \tag{1.4}$$

Moreover, among the non-trivial steady states, only $(\theta_{\alpha_1}, 0, ..., 0)$, the steady state where the slowest diffuser survives, is stable and the rest of the steady states, including the trivial steady state, are all unstable. Furthermore, when $N = 2$, the steady state $(\theta_{\alpha_1}, 0)$ is globally asymptotically stable among all non-negative, non-trivial solutions. Whether such a result holds for three or more species remains an interesting and important open question (see, however, Theorem 1.3 for some progress for continuum model (1.1)).

Dockery et al. [24] further inquired the effect of small mutation. More precisely, when $0 < \epsilon \ll 1$, it is shown that (1.2) has a *unique* non-negative, non-trivial steady state $(\tilde{U}_1, ..., \tilde{U}_N)$, such that $\tilde{U}_i > 0$ for all $i$, and $(\tilde{U}_1, ..., \tilde{U}_N) \to (\theta_{\alpha_1}, 0, ..., 0)$ as $\epsilon \to 0$. In a sense, the vanishing mutation limit is selecting the correct candidate among all non-negative steady states of (1.2) with an evolutionary advantage. This result is proved by exploiting the fact that for (1.2), $0 < \epsilon \ll 1$ is a *regular perturbation* of the case $\epsilon = 0$, which permits the use of the implicit function theorem to keep track of the $N + 1$ non-degenerate steady states in (1.3).

The continuum model (1.1) takes a more intrinsic point of view. Formally, if we set $\epsilon = 0$ and consider initial conditions of the form $u_0(x, \alpha) = \sum_{i=1}^{N} \delta_0(\alpha - \alpha_i) U_{i,0}(x)$, then (1.1) contains (1.2) as a subsystem for arbitrary number $N$ and motility rates $0 < \alpha_1 < ... < \alpha_N$. In particular, when $\epsilon = 0$, $\tilde{u}(x, \alpha) = \delta_0(\alpha - \alpha_0)\theta_{\alpha_0}(x)$ can be regarded as a steady state of (1.1) for all $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$. It is therefore quite surprising that, as presented in Theorem A, the Dirac-mass corresponding to the phenotype of the lowest motility rate is selected when mutation is switched on.

While the selection of the phenotype with minimal motility has long been observed in the evolution of unconditional dispersal in spatially heterogeneous but temporally constant environments, there are situations when an intermediate motility is favored, see, e.g. [30] for time-periodic environments; and [35, 37] for the situations when the species adopts some directed movement or when there is an environmental drift. In those cases when it is difficult to determine the exact value (or even multiplicity) of the optimal traits, continuum in trait models similar to (1.1) has the potential of being able to single out the optimal trait. E.g. by considering the long-time dynamics of the system and simply identifying the support of positive Dirac-concentrated steady states for small value of $\epsilon$, e.g. by numerical computation. The present paper together with [36] can be considered as a proof of concept to this approach.

**Remark 1.** *We remark here that Theorem A, while consistent with the known results of* (1.2)*, is based on entirely different mathematical arguments. In contrast to* (1.2)*, there are formally infinitely many positive steady states of* (1.1) *in the form* $\delta_0(\alpha - \alpha_0)\theta_{\alpha_0}(x)$ *when* $\epsilon = 0$*, and there is no simple way to determine the non-degeneracy of such steady states. Even if this can be accomplished, there would be no "spectral gap" between the slowest* *diffuser and the "second slowest" diffuser to carry out the implicit function argument.*

The evolution of dispersal is a classical and important topic in biology. See for instance [17, 49, 50] and reference therein. The study of reaction-diffusion models of population structured by trait has received considerable attention [11, 18, 19, 31, 33, 39]. See also [12, 22] for the relation with the theory of adaptive dynamics. The stability of steady states of such models was previously studied in the case of finite-dimensional (e.g. competition or predator-prey interaction without space) interactions in [10, 13, 20]. The involvement of the spatial variable, a form of infinite-dimensional interaction, is more recent. See for instance [32]. It is worth mentioning that while the minimal motility is selected in a bounded domain, the reverse happens in unbounded domains, in connection with the accelerating wave observed in the Australian invasive cane toad species [47]. See [7, 8, 9] for some recent mathematical results.

### 1.3 Statements of main results

We begin our mathematical discussion by presenting some standard persistence and extinction critieria of (1.1). To this end, we define $\mu_1(\underline{\alpha}\Delta_x+m; D)$ to be the principal eigenvalue of

$$\underline{\alpha}\Delta_x\phi + m\phi + \mu\phi = 0 \quad \text{in } D, \quad \text{and} \quad \partial_n\phi = 0 \quad \text{on } \partial D. \tag{1.5}$$

**Theorem 1.1.** *(a) If $\mu_1(\underline{\alpha}\Delta_x + m; D) \geq 0$, then $\lim_{t\to\infty} \|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})} = 0$.*

*(b) If $\mu_1(\underline{\alpha}\Delta_x + m; D) < 0$, then there is $\epsilon^* \in (0, +\infty]$ such that*

*(i) If $\epsilon^* < +\infty$ and $\epsilon \geq \epsilon^*$, then $\lim_{t\to\infty} \|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})} = 0$;*

*(ii) If $0 < \epsilon < \epsilon^*$, then there is $\delta_1 = \delta_1(\epsilon) > 0$ independent of initial condition $u_0$ such that any solution $u$ of (1.1) satisfies*

$$\delta_1 \leq \liminf_{t\to\infty} \left[\inf_{\Omega} u(\cdot,\cdot,t)\right] \leq \limsup_{t\to\infty} \left[\sup_{\Omega} u(\cdot,\cdot,t)\right] \leq 1/\delta_1.$$

*Furthremore (1.1) has at least one positive steady state.*

Our main result concerns the uniqueness and asymptotic stability of Dirac-concentrated steady states $u_\epsilon$ of (1.1), for small values of $\epsilon$.

**Theorem 1.2.** *Assume $m(x)$ is non-constant and $\mu(\underline{\alpha}\Delta_x + m; D) < 0$. Then for all $\epsilon$ sufficiently small, (1.1) has a unique positive steady state $u_\epsilon$. Moreover, $u_\epsilon$ is locally asymptotically stable.*

Making essential use of the detailed asymptotic estimates for $u_\epsilon$ obtained in [36], the proof of Theorem 1.2 is based on the analysis of the following nonlocal eigenvalue problem:

$$\begin{cases} \epsilon^2\partial_\alpha^2\varphi + \alpha\Delta_x\varphi + \varphi(m(x)-\hat{u}_\epsilon) - \hat{\varphi}u_\epsilon + \lambda\varphi = 0 & \text{in } D\times(\underline{\alpha},\overline{\alpha}), \\ \partial_n\varphi = 0 \quad \text{on } \partial D\times(\underline{\alpha},\overline{\alpha}), \quad \text{and} \quad \partial_\alpha\varphi = 0 & \text{on } D\times\{\underline{\alpha},\overline{\alpha}\}, \end{cases} \tag{1.6}$$

where

$$\hat{u}_\epsilon = \hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\,d\alpha, \quad \text{and} \quad \hat{\varphi} = \hat{\varphi}(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi(x,\alpha)\,d\alpha \tag{1.7}$$

are integrals of the positive steady state $u_\epsilon$ and the eigenfunction $\varphi$ in the trait variable respectively. In particular, the linear operator on the left-hand side of (1.6) does not generate a positive semigroup and the Krein-Rutman theorem is not applicable. Thus, all possible eigenvalues need to be taken into account in the analysis.

Based on our result, we propose the following conjecture:

**Conjecture.** *Suppose, for some $\epsilon > 0$, that* (1.1) *has a positive steady state $u_\epsilon$. Then $u_\epsilon$ is unqiue and globally asymptotically stable among all non-negative, non-trivial solutions of* (1.1).

As already pointed out in [24], the lack of comparison principle presents a major hurdle in understanding the dynamical properties of the solutions of $N$ species competition models when $N$ is greater than 2. The same holds true for our structured-population model (1.1), where there is a coninuum of interacting phenotypes. Nonetheless, the conjecture holds in the special case that $m(x)$ is a constant. Note that all previous theorems concerns the case when $m(x)$ is *nonconstant*.

**Theorem 1.3.** *Assume $m(x) \equiv m_0$ in $D$.*

*(a) If $m_0 \leq 0$, then $0$ is globally asymptotically stable for all $\epsilon > 0$.*

*(b) If $m_0 > 0$, then $u_\epsilon \equiv \frac{m_0}{\overline{\alpha}-\underline{\alpha}}$ is globally asymptotically stable for all $\epsilon > 0$.*

This work is one of the first attempts to study linear stability of steady states in this class of mutation-selection models, which belongs to a new class of nonlocal reaction-diffusion equations. In some previous works, the nonlocal term appears as a constant [12, 13, 54], which can then treated as a one-dimensional parameter when considering the existence and stability of steady states. In comparison, the nonlocalities $\hat{u}_\epsilon(x)$ and $\hat{\varphi}(x)$ in (1.6) present infinite-dimensional parameters, which give a kind of "indefinite" nature to the eigenvalue problem. In this paper, the trouble comes from the "spatial" element, while in other works (e.g. [23]) the nonlocality is a consequence of the asymmetric competition kernel (which in our case is a constant).

In the proof of our main results concerning the nonlocal eigenvalue problem (1.6), we have combined, in a novel way, the comparison principle, the spectral theory of sectorial operator (Subsect. 4.1), elliptic $L^p$ estimates and the local maximum principle. The key observation seems to be the fact that eigenfunctions corresponding to $O(1)$ eigenvalues exhibits the same concentration as the steady state $u_\epsilon(x,\alpha)$ as $\epsilon \to 0$. This allows the reduction of the problem to a well-known eigenvalue problem (4.14) concerning the stability of a logistic model of a single species! It is worth mentioning that this seems to be connected to the notions of inner (the stability among the one or several dominant species) and outer (the dominant species versus the rest of the species) stability in evolutionary biology literature [53].

We organize the rest of the paper as follows: In Section 2, we prove Theorem 1.1 and in particular the existence of positive steady states of (1.1) via the Leray-Schauder degree after proving some apriori estimates. In Section 3, we state some asymptotic estimates (as $\epsilon \to 0$) from the previous paper. Section 4 is devoted to the linear stability analysis of steady states when $\epsilon \to 0$. This in particular yields asymptotic stability, as the semiflow of concern is generated by a sectorial operator. Uniqueness is proved in Section 5 via degree considerations. Theorem 1.3 is proved in Section 6 by applying LaSalle's invariance principle. Finally, we defer the more technical estimates of eigenfunction $\varphi$ of (1.6) to Appendix B.

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# 2 Persistence theory

## 2.1 Two eigenvalue problems in the characterization of persistence

Denote $\mu_1 = \mu_1(\underline{\alpha}\Delta_x + m; D)$ to be the principal eigenvalue of (1.5) with principal eigenfunction $\phi_1(x)$, and $\mu_{1,\epsilon}$ to be the principal eigenvalue of

$$\begin{cases} \epsilon^2 \partial_\alpha^2 \phi + \alpha \Delta_x \phi + m\phi + \mu\phi = 0 & \text{in } \Omega = D \times (\underline{\alpha}, \overline{\alpha}), \\ \partial_n \phi = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), & \text{and} \quad \partial_\alpha \phi = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}. \end{cases} \tag{2.1}$$

**Lemma 2.1.** *(i) $\mu_{1,\epsilon} \geq \mu_1$ for all $\epsilon > 0$ and $\mu_{1,\epsilon}$ is increasing in $\epsilon$.*

*(ii) If $\mu_1 < 0$, then there exists $\epsilon^* \in (0, +\infty]$ such that $\mu_{1,\epsilon} < 0$ for $\epsilon \in (0, \epsilon^*)$ and $\mu_{1,\epsilon} \geq 0$ for $\epsilon \geq \epsilon^*$.*

*Proof.* Since $\mu_1$ is also the principal eigenvalue (with principal eigenfunction $\Phi(x,\alpha) = \phi_1(x)$) of

$$\epsilon^2 \partial_\alpha^2 \phi' + \underline{\alpha} \Delta_x \phi' + m\phi' + \mu\phi' = 0 \quad \text{in } \Omega = D \times (\underline{\alpha}, \overline{\alpha})$$

with Neumann boundary condition on $\partial\Omega = [\partial D \times (\underline{\alpha}, \overline{\alpha})] \cup [D \times \{\underline{\alpha}, \overline{\alpha}\}]$, we deduce by variational characterization that

$$\mu_1 = \inf_{\phi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (\epsilon^2 |\partial_\alpha \phi|^2 + \underline{\alpha} |\nabla_x \phi|^2 - m|\phi|^2)\, dx d\alpha}{\int_\Omega |\phi'|^2 \, dx d\alpha}$$

$$\leq \inf_{\phi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (\epsilon^2 |\partial_\alpha \phi|^2 + \alpha |\nabla_x \phi|^2 - m|\phi|^2)\, dx d\alpha}{\int_\Omega |\phi'|^2 \, dx d\alpha} = \mu_{1,\epsilon}$$

It is also standard to deduce the monotonicity of $\mu_{1,\epsilon}$ in $\epsilon > 0$ from the last equality. This proves (i).

For (ii), it suffices to show that $\lim_{\epsilon\to 0} \mu_{1,\epsilon} = \mu_1$ from which the rest of the claim follows from the monotonicity of $\mu_{1,\epsilon}$ with respect to $\epsilon$. In fact, by (i) it suffices to show $\limsup_{\epsilon\to 0} \mu_{1,\epsilon} \leq \mu_1$.

For each $\delta > 0$, choose a test function in the form $\phi_1(x)\eta(\alpha)$, where $\phi_1$ is the principal eigenfunction of (1.5) and $\eta(\alpha) \in C^\infty(\mathbb{R})$ is positive and supported in $[\underline{\alpha}, \underline{\alpha} + \delta]$, then

$$\mu_{1,\epsilon} \leq \frac{\int_\Omega [\epsilon^2 |\partial_\alpha (\phi_1 \eta)|^2 + (\underline{\alpha} + \delta)|\nabla_x (\phi_1 \eta)|^2 - m|(\phi_1\eta)|^2]\, dx d\alpha}{\int_\Omega |(\phi_1 \eta)|^2 \, dx d\alpha}$$

$$= \frac{\int_\Omega [\epsilon^2 \phi_1^2 |\eta'|^2 + (\underline{\alpha} + \delta)|\nabla_x \phi_1|^2 \eta^2 - m\phi_1^2 \eta^2]\, dx d\alpha}{[\int_D |\phi_1|^2 \, dx][\int_{\underline{\alpha}}^{\underline{\alpha}+\delta} |\eta|^2 \, d\alpha]}.$$

Letting $\epsilon \to 0$,

$$\limsup_{\epsilon\to 0} \mu_{1,\epsilon} \leq \frac{\int_\Omega (\underline{\alpha} + \delta)|\nabla_x \phi_1|^2 \eta^2 - m\phi_1^2\eta^2]\, dx d\alpha}{[\int_D |\phi_1|^2 \, dx][\int_{\underline{\alpha}}^{\underline{\alpha}+\delta} |\eta|^2 \, d\alpha]}$$

$$= \frac{\int_D (\underline{\alpha} + \delta)|\nabla_x \phi_1|^2 - m|\phi_1|^2 \, dx}{\int_D |\phi_1|^2 \, dx}$$

$$= \mu_1 + \delta \frac{\int_D |\nabla_x \phi_1|^2 \, dx}{\int_D |\phi_1|^2 \, dx}$$

where the last equality follows from the definition of $(\mu_1, \phi_1)$ being the principal eigenpair of (1.5). And the desired conclusion follows after letting $\delta \to 0$. This proves (ii). □

## 2.2 Standard Persistence and Extinction Results

In this section, we consider a family of parabolic problem parameterized by $s \in [0, 1]$.

$$\begin{cases} \partial_t u = \epsilon^2 \partial_\alpha^2 u + \alpha \Delta_x u + u[m(x) - su - (1-s)\hat{u}] = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_n u = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}) \times (0,\infty), \\ \partial_\alpha u = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\} \times (0,\infty), \\ u(x,\alpha,0) = u_0(x,\alpha) \quad \text{in } \Omega. \end{cases} \tag{2.2}$$

Since the arguments in this section are rather standard, we refer the interested readers to Appendix A for the proofs of Lemma 2.2 and Proposition 2.3.

**Lemma 2.2.** *Fix $\epsilon > 0$, there exists $C_1$ independent of $s \in [0,1]$ and initial condition $u_0$ such that any solution $u$ to (2.2) satisfies*

$$\limsup_{t\to\infty} \left[\sup_{\Omega} u(\cdot,\cdot,t)\right] \leq C_1. \tag{2.3}$$

*Furthermore, if we define for each $t_0 \geq 0$, $u^{t_0}(x,\alpha,t) = u(x,\alpha,t+t_0)$, then (by parabolic estimates) $\{u^{t_0}(\cdot,\cdot,t)\}_{t_0\geq 0}$ is precompact in $C(\overline{\Omega}\times[0,T])$ for any $T > 0$.*

**Proposition 2.3.** *(a) Suppose $\mu_{1,\epsilon} \geq 0$, then for any $s \in [0,1]$ and any solution $u$ to (2.2), $\lim_{t\to\infty} \|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})} = 0$.*

*(b) Suppose $\mu_{1,\epsilon} < 0$, then there exists $\delta_1 = \delta_1(\epsilon) > 0$ independent of $s \in [0,1]$ and initial condition $u_0 \geq 0$ such that for any non-trivial solution $u$ to (2.2),*

$$\delta_1 < \liminf_{t\to\infty} \left[\inf_{\Omega} u(\cdot,\cdot,t)\right] \leq \limsup_{t\to\infty} \left[\sup_{\Omega} u(\cdot,\cdot,t)\right] < 1/\delta_1.$$

Consider now the existence and uniqueness of positive solutions of the following slightly more general problem:

$$\begin{cases} \epsilon^2\partial_\alpha^2 u + \Delta_x u + u[m(x) - su - (1-s)\hat{u}] = 0 & \text{in } \Omega = D\times(\underline{\alpha},\overline{\alpha}), \\ \partial_n u = 0 \quad \text{on } \partial D\times(\underline{\alpha},\overline{\alpha}), & \text{and} \quad \partial_\alpha u = 0 \quad \text{on } D\times\{\underline{\alpha},\overline{\alpha}\}, \end{cases} \tag{2.4}$$

where $s \in [0,1]$ is a homotopy parameter. A direct consequence of Proposition 2.3 is the following apriori estimate of positive solutions of (2.4).

**Corollary 2.4.** *Fix $\epsilon > 0$, and suppose $\mu_{1,\epsilon} < 0$, where $\mu_{1,\epsilon}$ is the p.e.v. of (2.1). Then there exists $\delta_1 > 0$ independent of $s \in [0,1]$ such that if $u$ is a positive solution of (2.4) for some $s \in [0,1]$, then*

$$\delta_1 < u(x) < 1/\delta_1 \quad \text{in } \overline{\Omega}.$$

For the existence of positive solution to (2.4) and other purposes, we also prove a degree-theoretic result. Fix $\epsilon > 0$ and let $X = C(\overline{\Omega})$ and define, for $s \in [0,1]$,

$$A_s(u) := (-\epsilon^2\partial_\alpha^2 - \Delta_x + I)^{-1}[u(m + 1 - su - (1-s)\hat{u})]$$

with $(-\epsilon^2\partial_\alpha^2 - \Delta_x + I)^{-1} : X \to X$ being the inverse operator subject to Neumann boundary condition on $\partial\Omega$. Note that $u \in X$ is a solution of (2.4) if and only if it is a fixed point of $A_s$. We calculate the Leray-Schauder degree in the following result.

**Lemma 2.5.** *Under the assumption of Corollary 2.4,*

$$\deg(A_0, X_1, 0) = 1,$$

*where*

$$X_1 = \{u \in X : \delta_1 < u(x) < 1/\delta_1 \text{ in } \bar{D}\} \tag{2.5}$$

*and $\delta_1$ is given by Corollary 2.4.*

*Proof.* By Corollary 2.4, we see that for all $s \in [0,1]$, $A_s$ has no fixed point on $\partial X_1$. By the homotopy invariance of the Leray-Schauder degree, we deduce that

$$\deg(A_0, X_1, 0) = \deg(A_1, X_1, 0). \tag{2.6}$$

It remains to calculate $\deg(A_1, X_1, 0)$. For that purpose, recall the well-known result that, assuming $\mu_{1,\epsilon} < 0$ where $\mu_{1,\epsilon}$ is the p.e.v. of (2.1), then

$$\begin{cases} \epsilon^2 \partial_\alpha^2 u + \alpha \Delta_x u + u(m(x) - u) = 0 & \text{in } \Omega, \\ \partial_n u = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}) \quad \text{and} \quad \partial_\alpha u = 0 & \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases} \tag{2.7}$$

has a unique, non-degenerate, positive solution $u_1$ (see, e.g. [15, Propositions 3.2 and 3.3]). By Corollary 2.4, we must have $u_1 \in X_1$. Therefore, we deduce from (2.6) that

$$\deg(A_0, X_1, 0) = \text{index}_X(A_1'(u_1), 0) = (-1)^\beta, \tag{2.8}$$

where $\text{index}_X$ denotes the fixed point index, and

$$\beta = \sum_{\mu > 1} \dim \left[ \cup_{k=1}^{\infty} \ker (\mu I - A_1)^k \right], \tag{2.9}$$

where the summation is taken over all real eigenvalues $\mu$ of $A_1$ such that $\mu > 1$. It remains to show the following.

**Claim 2.6.** $\beta = 0$.

Let $u_1$ be the unique positive solution to (2.7). By variational characterization, we have

$$0 = \inf_{\phi \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega [\epsilon^2 |\partial_\alpha \phi|^2 + \alpha |\nabla_x \phi|^2 + (u_1 - m)\phi^2] \, dx d\alpha}{\int_\Omega \phi^2 \, dx d\alpha}. \tag{2.10}$$

It suffices to show, for each $\mu > 1$, that there is no non-trivial solution $\phi$ to

$$\begin{cases} -\mu(\epsilon^2 \partial_\alpha^2 \phi + \alpha \Delta_x \phi - \phi) + (2u_1 - m - 1)\phi = 0 & \text{in } \Omega, \\ \text{with Neumann boundary condition on } \partial\Omega. \end{cases}$$

Multiply the above equation by the complex conjugate of $\phi$, and integrate by parts, we have

$$
\begin{aligned}
0 &= \int_\Omega [\mu(\epsilon^2|\partial_\alpha\phi|^2 + \alpha|\nabla_x\phi|^2) + (2u_1 - m + \mu - 1)|\phi|^2] \\
&\geq \int_\Omega [\epsilon^2|\partial_\alpha\phi|^2 + \alpha|\nabla_x\phi|^2 + (2u_1 - m)|\phi|^2] \\
&\geq \int_\Omega u_1|\phi|^2 \geq 0,
\end{aligned}
$$

where the second last inequality follows from (2.10). Now, every term in the above calculation vanishes and in particular $\int_\Omega u_1|\phi|^2 = 0$. This implies that $\phi \equiv 0$ and proves Claim 2.6. The lemma follows from (2.6) and (2.8). □

An immediate corollary is the following existence result for (1.1).

**Corollary 2.7.** *Fix $\epsilon > 0$, and suppose $\mu_{1,\epsilon} < 0$ where $\mu_{1,\epsilon}$ is the p.e.v. of* (2.1)*. Then* (1.1) *has at least one positive steady state $u_\epsilon \in X_1$ where $X_1$ is defined in* (2.5)*.*

*Proof.* By Lemma 2.5, $\deg(A_0, X_1, 0) = 1$. This implies that $I - A_0$ has at least one fixed point $u \in X_1$, i.e. there exists at least one positive solution $u$ satisfying $u = (-\epsilon^2\partial_\alpha^2 - \Delta_x + I)^{-1}[u(m+1-\hat{u})]$. By definition of $(\epsilon^2\partial_\alpha^2 + \Delta_x + I)^{-1}$ being an inverse operator subject to homogeneous Neumann boundary condition, we see that $u$ is a positive steady state of (1.1). □

*Proof of Theorem 1.1.* First, suppose $\mu_1 \geq 0$ (where $\mu_1$ is the p.e.v. of (1.5)) we prove (a). By Lemma 2.1, $\mu_{1,\epsilon} \geq \mu_1 \geq 0$ for all $\epsilon$. Hence by Proposition 2.3(a), we have $\|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})} \to 0$ as $t \to \infty$. Next, assume $\mu_1 < 0$. Then by lemma 2.1(ii), there exists $\epsilon^* \in (0, +\infty]$ such that (i) $\mu_{1,\epsilon} < 0$ for $\epsilon \in (0, \epsilon^*)$, and (ii) $\mu_{1,\epsilon} \geq 0$ for $\epsilon \geq \epsilon^*$. Then assertion (b)(i) follows from Proposition 2.3(a), while assertion (b)(ii) follows from Proposition 2.3(b) and Corollary 2.7. □

# 3 Estimates

Unless stated otherwise, we assume for the remainder of this paper that (i) $m(x)$ is non-constant in $D$, (ii) $\mu_1 < 0$ and (iii) $\epsilon \in (0, \epsilon_*)$, so that (1.1) has a positive steady state $u_\epsilon(x, \alpha)$.

Let $\hat{u}_\epsilon(x)$ be given in (1.7). For each $\epsilon, \alpha > 0$, consider the eigenvalue problem in the domain $D$ of $x$:

$$
\begin{cases} -\alpha\Delta\psi + (\hat{u}_\epsilon(x) - m(x))\psi = \sigma\psi & \text{in } D, \\ \partial_n\psi = 0 \quad \text{on } \partial D \quad \text{and} & \int_D \psi^2\,dx = \int_D \theta_{\underline{\alpha}}^2\,dx. \end{cases} \tag{3.1}
$$

**Lemma 3.1.** *For each $\epsilon, \alpha > 0$, denote the principal eigenpair of* (3.1) *by $(\sigma_\epsilon(\alpha), \psi_\epsilon(\cdot, \alpha))$.*

(i) *$\sigma_\epsilon$ is bounded in $C^3([\underline{\alpha},\overline{\alpha}])$ independent of $\epsilon$ small.*

(ii) *There exist $a_0, a_1 > 0$ independent of $\epsilon$ small such that*

$$\lim_{\epsilon\to 0} \epsilon^{-2/3}\sigma_\epsilon(\underline{\alpha}) \to -a_0, \quad \textit{and} \quad \lim_{\epsilon\to 0} \partial_\alpha \sigma_\epsilon(\underline{\alpha}) = a_1.$$

(iii) *There exists $a_2 > 0$ independent of $\epsilon$ small, such that $\partial_\alpha\sigma_\epsilon(\alpha) \geq a_2$ for all $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$.*

(iv) *There exists $C > 1$ independent of $\epsilon$ small, such that*

$$1/C \leq \psi_\epsilon(x,\alpha) \leq C \quad \textit{in } \Omega \quad \textit{and} \quad \|\partial_\alpha\psi_\epsilon\|_{L^\infty(\Omega)} + \|\partial^2_\alpha\psi_\epsilon\|_{L^\infty(\Omega)} \leq C.$$

*Proof.* Assertions (i), (iii) and (iv) follow from [36, Lemma 4.1 (i), (iii), (iv)] respectively. Assertion (ii) follows from [36, Corollary 5.7]. □

Let $\eta^*(s) > 0$ be a rescaled Airy function that is uniquely determined by

$$\begin{cases} \eta'' + (a_0 - a_1 s)\eta = 0 \quad \text{on } (0,\infty), \\ \eta'(0) = \eta(\infty) = 0 \quad \text{and} \quad \int_0^\infty \eta\, ds = 1, \end{cases} \tag{3.2}$$

where $a_0, a_1$ are given by Lemma 3.1(ii). We collect some properties of $u_\epsilon$.

**Lemma 3.2.** *Suppose $u_\epsilon$ is a positive steady state of* (1.1)*.*

(i) *For each $\beta > 0$, there exists $C_\beta > 0$ such that*

$$0 < u_\epsilon(x,\alpha) \leq C_\beta \epsilon^{-2/3} \exp(-\beta\epsilon^{-2/3}(\alpha - \underline{\alpha})) \quad \textit{in } \Omega.$$

(ii) *Let $\theta_{\underline{\alpha}}(x)$ be the unique positive solution to* (1.4) *(with $\alpha = \underline{\alpha}$) and $\eta^*(s)$ be uniquely determined by* (3.2)*. Then*

$$\left\| \epsilon^{2/3} u_\epsilon(x,\alpha) - \theta_{\underline{\alpha}}(x)\eta^*\left(\frac{\alpha - \underline{\alpha}}{\epsilon^{2/3}}\right)\right\|_{L^\infty(\Omega)} \to 0 \quad \textit{as } \epsilon \to 0.$$

(iii) *As $\epsilon \to 0$, $\hat{u}_\epsilon \to \theta_{\underline{\alpha}}$ in $C(\overline{D})$ and $\|\hat{u}_\epsilon\|_{L^2(D)} \to \|\theta_{\underline{\alpha}}\|_{L^2(D)} > 0$.*

(iv) *There exists a constant $C > 0$ independent of $\epsilon$ so that*

$$\epsilon \left\|\frac{\partial_\alpha u_\epsilon}{u_\epsilon}\right\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_x u_\epsilon}{u_\epsilon}\right\|_{L^\infty(\Omega)} \leq C.$$

*Proof.* Assertions (i), (ii) and (iii) are contained in [36, Theorem 2.3], while assertion (iv) is proved in [36, Proposition 3.7]. □

Let $\lambda$ be a given eigenvalue of (1.6) with a corresponding eigenfunction $\varphi(x,\alpha)$ normalized by

$$\int_D \left( \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi(x,\alpha)|\, d\alpha \right)^2 dx = 1. \tag{3.3}$$

The following properties of $\varphi$ are proved in Appendix B.

**Proposition 3.3.** *Let $M > 0$ be fixed. Suppose $(\lambda, \varphi)$ is an eigenpair of (1.6) such that $\operatorname{Re}\lambda \le 0$ and $|\lambda| \le M$, and (3.3) holds, then*

(i) *For each $q \in (0, 2/3)$, there exists $r_\epsilon$ such that $|r_\epsilon| \le o(\epsilon^p)$ for all $p > 1$ and*

$$\int_D \left( \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha \right)^2 dx \le 2\epsilon^q \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|^2\, d\alpha dx + r_\epsilon,$$

*where $\varphi^\perp$ is given by the decomposition*

$$\varphi = a_\epsilon u_\epsilon + \varphi^\perp \quad \text{with} \quad a_\epsilon = \frac{\iint_\Omega u_\epsilon \varphi\, d\alpha dx}{\iint_\Omega u_\epsilon^2\, d\alpha dx}. \tag{3.4}$$

(ii) *For each $q \in (0, 2/3)$, there exists $C > 0$ independent of $\epsilon$ such that*

$$\left\| \int_{\underline{\alpha}}^{\overline{\alpha}} \alpha \varphi(x,\alpha)\, d\alpha - \underline{\alpha} \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi(x,\alpha)\, d\alpha \right\|_{L^2(D)} \le C\epsilon^q.$$

(iii) *There exists $C > 0$ independent of $\epsilon$ such that*

$$\iint_\Omega |\varphi^\perp(x,\alpha)|^2\, d\alpha dx \le C\epsilon^{-4/3} \int_D |\hat{\varphi}(x)|^2\, dx,$$

*where $\varphi^\perp$ is given in (3.4) and $\hat{\varphi}(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi(x,\alpha)\, d\alpha$.*

*Proof.* Assertions (i) and (ii) are proved in Proposition B.3 (iv) and (iii) respectively. For (iii), see Corollary B.5. □

# 4 Linear Stability of $u_\epsilon$

For the rest of the paper, we assume $\mu_1 < 0$, so that by Theorem 1.1(b)(ii), the steady state equation of (1.1)

$$\begin{cases} \epsilon^2 \partial_\alpha^2 u + \alpha \Delta_x u + (m - \hat{u})u = 0 & \text{in } \Omega = D \times (\underline{\alpha}, \overline{\alpha}), \\ \partial_n u = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), & \text{and} \quad \partial_\alpha u = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases} \tag{4.1}$$

has at least one positive solution $u_\epsilon$ for all $\epsilon \in (0, \epsilon^*)$. The main result of this section is

**Proposition 4.1.** *For all $\epsilon > 0$ sufficiently small, any positive steady state $u_\epsilon$ of* (1.1) *is linearly stable in the sense that every eigenvalue $\lambda$ of* (1.6) *has strictly positive real part.*

In subsection 4.1, we show that (1.6) has no eigenvalues in $\{\lambda' \in \mathbb{C} : \operatorname{Re}\lambda' \le 0, \quad |\lambda'| \ge \omega_1\}$ for some $\omega_1 > 0$, and a resolvent estimate asserting that the linearized operator of (1.6) is sectorial. This part is essentially independent of smallness of $\epsilon$. In subsection 4.2, we show that for $\epsilon$ small, (1.6) has no eigenvalue in $\{\lambda' \in \mathbb{C} : \operatorname{Re}\lambda' \le 0, \quad |\lambda'| \le \omega_1\}$. This completes the proof of Proposition 4.1.

## 4.1 $\lambda \gg 1$

By the transformation $\tilde{\varphi}(x,\alpha) = \varphi(x,\alpha)/u_\epsilon(x,\alpha)$, we observe that the eigenvalue problem (1.6) is equivalent to

$$\begin{cases} L_\epsilon\tilde{\varphi} + \lambda\tilde{\varphi} - S_\epsilon\tilde{\varphi} = 0 \quad \text{in } \Omega, \\ \text{with Neumann boundary condition on } \partial\Omega, \end{cases} \tag{4.2}$$

where

$$L_\epsilon := \epsilon^2\partial_\alpha^2 + \alpha\Delta_x + 2\epsilon^2\frac{u_{\epsilon,\alpha}}{u_\epsilon}\partial_\alpha + 2\alpha\frac{\nabla_x u_\epsilon}{u_\epsilon}\nabla_x \tag{4.3}$$

and $S_\epsilon : L^\infty(\Omega) \to L^\infty(\Omega)$ is given by $S_\epsilon[v] = \int_{\underline{\alpha}}^{\overline{\alpha}} v(x,\alpha)u_\epsilon(x,\alpha)\,d\alpha$.

**Lemma 4.2.** *There exist $C_1, \omega_0 > 0$ such that if $\operatorname{Re}\lambda \le 0$ and $|\lambda| \ge \omega_0$, then for every $v \in C^2(\bar{\Omega})$ satisfying the Neumann boundary condition on $\partial\Omega$, we have*

$$|\lambda|\|v\|_{L^\infty(\Omega)} \le C_1\|L_\epsilon v + \lambda v\|_{L^\infty(\Omega)}.$$

*Proof.* First, extend the definition of $u_\epsilon$ and $v$ to $D\times(\underline{\alpha}, 2\overline{\alpha}-\underline{\alpha})$ by reflecting along $D\times\{\overline{\alpha}\}$, then extend to all of $D\times\mathbb{R}$ periodically. Second, let $V(x,\tau) = v(x,\epsilon\tau)$, then we have

$$L_\epsilon v + \lambda v = \tilde{L}_\epsilon V + \lambda V$$

where $\tilde{L}_\epsilon = \partial_{\tau\tau} + \alpha\Delta_x + 2\epsilon\frac{u_{\epsilon,\alpha}}{u_\epsilon}\partial_\tau + 2\alpha\frac{\nabla_x u_\epsilon}{u_\epsilon}\nabla_x$ and $\alpha = \alpha(\tau)$ is a Lipschitz continuous function such that $\underline{\alpha} \le \alpha(\tau) \le \overline{\alpha}$ for all $\tau$. By Lemma 3.2(iv), the coefficients of $\tilde{L}_\epsilon$ (as an operator on $D\times\mathbb{R}$) are bounded in $L^\infty$. Third, apply [41, P.77, (3.1.26)] to $V$ so that for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \le 0$ and $|\lambda| \ge \omega_0$,

$$|\lambda|\|V\|_{L^\infty(D\times\mathbb{R})} \le C_1\|\tilde{L}_\epsilon V + \lambda V\|_{L^\infty(D\times\mathbb{R})}.$$

(Though stated only for $\lambda$ with $\operatorname{Re}\lambda \le -\omega_0$ and on the entire space $\mathbb{R}^k$, the proof of [41, P.77, (3.1.26)] actually works for all $\lambda$ with $\operatorname{Re}\lambda \le 0$ and $|\lambda| \ge \omega_0$ on the cylinder $D\times\mathbb{R}$ with Neumann b.c. on $\partial D\times\mathbb{R}$.) Hence,

$$\begin{aligned} |\lambda|\|v\|_{L^\infty(\Omega)} &= |\lambda|\|V\|_{L^\infty(D\times\mathbb{R})} \\ &\le C_1\|\tilde{L}_\epsilon V + \lambda V\|_{L^\infty(D\times\mathbb{R})} = C_1\|L_\epsilon v + \lambda v\|_{L^\infty(\Omega)}. \end{aligned} \tag{4.4}$$

This proves the lemma. □

**Proposition 4.3.** *There exists a constant $\omega_1 > 0$ independent of $\epsilon$ so that if $\operatorname{Re}\lambda \leq 0$ and $|\lambda| \geq \omega_1$, then $L_\epsilon + \lambda I - S_\epsilon$ is invertible from $L^\infty(\Omega)$ to $L^\infty(\Omega)$. In particular, $\{\lambda' \in \mathbb{C} : \operatorname{Re}\lambda' \leq 0, |\lambda'| \geq \omega_1\}$ contains no eigenvalue of (1.6). Furthermore, $L_\epsilon - S_\epsilon$ is a sectorial operator.*

*Proof.*

**Claim 4.4.** *There exists $\omega_1 > 0$ such that if $\operatorname{Re}\lambda \leq 0$ and $|\lambda| \geq \omega_1$, then $\|S_\epsilon R_\lambda(L_\epsilon)\| \leq \frac{1}{2}$. In particular, the operator $I - S_\epsilon R_\lambda(L_\epsilon) : L^\infty(\Omega) \to L^\infty(\Omega)$ is invertible.*

It suffices to show that the operator $S_\epsilon R_\lambda(L_\epsilon) : L^\infty(\Omega) \to L^\infty(\Omega)$ has norm less than 1/2. Let $f \in L^\infty(\Omega)$ be given. Let $C_1$ be given by Lemma 4.2, we can find by Lemma 3.2(iii) a constant $\omega_1 > \omega_0$ independent of $\epsilon$ such that $\omega_1 > 2C_1\|\hat{u}_\epsilon\|_{L^\infty(D)}$. Then by Lemma 4.2,

$$\|R_\lambda(L_\epsilon)f\|_{L^\infty(\Omega)} \leq \frac{C_1}{|\lambda|}\|f\|_{L^\infty(\Omega)} \leq \frac{C_1}{\omega_1}\|f\|_{L^\infty(\Omega)} \leq \frac{1}{2\|\hat{u}_\epsilon\|_{L^\infty(D)}}\|f\|_{L^\infty(\Omega)}.$$

for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \leq 0$ and $|\lambda| \geq \omega_1$. Hence,

$$\|S_\epsilon R_\lambda(L_\epsilon)f\|_{L^\infty(\Omega)} \leq \Big\| \|R_\lambda(L_\epsilon)f\|_{L^\infty(\Omega)}\, \hat{u}_\epsilon \Big\|_{L^\infty(\Omega)} \leq \frac{1}{2}\|f\|_{L^\infty(\Omega)}.$$

This proves Claim 4.4. The next observation is due to [13].

**Claim 4.5.** *Suppose $\lambda \in \rho(L_\epsilon)$ and that $I - S_\epsilon R_\lambda(L_\epsilon)$ is invertible, then $L_\epsilon + \lambda I - S_\epsilon$ is invertible.*

To see the claim, we rewrite

$$L_\epsilon v + \lambda v - S_\epsilon v = f \tag{4.5}$$

into (denote $R_\lambda(L_\epsilon) = (L_\epsilon + \lambda I)^{-1}$)

$$v = R_\lambda(L_\epsilon)S_\epsilon v + R_\lambda(L_\epsilon)f. \tag{4.6}$$

Apply $S_\epsilon$ on both sides, and put the terms involving $S_\epsilon[v]$ on the left, we derive

$$(I - S_\epsilon R_\lambda(L_\epsilon))S_\epsilon v = S_\epsilon R_\lambda(L_\epsilon)f. \tag{4.7}$$

Given $f$, if $I - S_\epsilon R_\lambda(L_\epsilon)$ is invertible, then $S_\epsilon v$ is uniquely determined by (4.7) and hence $v$ is uniquely given by (4.6). This proves Claim 4.5.

Now consider (4.5). Given $f$, $S_\epsilon v$ is uniquely solvable, by Claims 4.5 and 4.4. Substitute into (4.6), we obtain $v$. Moreover, by Claim 4.4 and (4.7),

$$\begin{aligned}\|S_\epsilon v\|_{L^\infty(\Omega)} &\leq \|(I - S_\epsilon R_\lambda(L_\epsilon))^{-1}\|\|S_\epsilon R_\lambda(L_\epsilon)f\|_{L^\infty(\Omega)} \\ &\leq \frac{1}{1 - 1/2} \cdot \frac{1}{2}\|f\|_{L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)}\end{aligned}$$

and by (4.4),

$$\|v\|_{L^\infty(\Omega)} \le \|R_\lambda(L_\epsilon)\|(\|S_\epsilon v\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}) \le \frac{C\|f\|_{L^\infty(\Omega)}}{|\lambda|}, \qquad (4.8)$$

for $\lambda$ with non-negative real part and $|\lambda| \ge \omega_1$. Finally, note that the resolvent estimate (4.8) implies that $L_\epsilon + S_\epsilon$ is a sectorial operator (see, e.g. [48, Theorem 12.31]). This proves Proposition 4.3. □

## 4.2 Proof of Theorem 1.2

In this subsection, we prove the linear stability of positive steady states $u_\epsilon$ of (1.1), announced in the beginning of this section. We will proceed by a contradiction argument, assuming the existence of an eigenvalue $\lambda$ with non-positive real part and eigenfunction $\Phi$ for (a sequence of) $\epsilon \to 0$. The proof consists of three main steps: (Step 1) $\|\hat{\varphi}(x)\|_{L^2(D)} \ge c\epsilon^p$ for all $p \in (\frac{1}{3}, \frac{2}{3})$; (Step 2) the normalized $\hat{\varphi}/\|\hat{\varphi}\|_{L^2(D)} \to \Phi_0 \neq 0$ strongly in $L^2(D)$, and $\Phi_0$ is an eigenfunction of a limit eigenvalue problem corresponding to a eigenvalue with non-positive real part; (Step 3) all eigenvalues of the limit eigenvalue problem are real and positive, contradictions.

*Proof of Proposition 4.1.* Suppose $(\lambda, \varphi)$ is an eigenpair of (1.6) such that $\operatorname{Re}\lambda \le 0$ and $\varphi$ is normalized by (3.3). By Proposition 4.3, there is $M > 0$ independent of $\epsilon$ such that $\lambda \le M$.
Step 1: We claim that for each $p \in (1/3, 2/3)$, there is $c > 0$ such that

$$\|\hat{\varphi}\|_{L^2(D)} \ge c\epsilon^p. \qquad (4.9)$$

Suppose to the contrary that

$$\int_D |\hat{\varphi}|^2\, dx \le o(\epsilon^{4/3-q}) \quad \text{for some } q \in (0, 2/3). \qquad (4.10)$$

Recall the definition of $a_\epsilon$ and $\varphi^\perp$ from (3.4), and define

$$\widehat{\varphi^\perp}(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi^\perp(x,\alpha)\, d\alpha \quad \text{for } x \in D.$$

**Claim 4.6.** *Suppose* (4.10) *holds. Then* $\left\|\int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha\right\|_{L^2(D)} \to 0$ *as* $\epsilon \to 0$. *In particular,* $\left\|\widehat{\varphi^\perp}\right\|_{L^2(D)} = \left\|\int_{\underline{\alpha}}^{\overline{\alpha}} \varphi^\perp\, d\alpha\right\|_{L^2(D)} \to 0$ *as* $\epsilon \to 0$.

To see the claim, we use Proposition 3.3 (i) and (iii) to get

$$\begin{aligned}\int_D \left(\int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha\right)^2 dx &\le 2\epsilon^q \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|^2\, d\alpha dx + o(1) \\ &\le C\epsilon^{q-4/3} \int_D |\hat{\varphi}|^2\, dx + o(1)\end{aligned}$$

where the last expression tends to zero by assumption (4.10). This proves Claim 4.6.

**Claim 4.7.** *Suppose* (4.10) *holds. Then* $a_\epsilon \to 0$ *as* $\epsilon \to 0$*, where* $a_\epsilon$ *is given by* (3.4).

If we integrate (3.4) over $\alpha \in (\underline{\alpha}, \overline{\alpha})$, then we have $a_\epsilon \hat{u}_\epsilon(x) = \hat{\varphi}(x) - \widehat{\varphi^\perp}(x)$ in $D$. By (4.10) and Claim 4.6, the right-hand side tends to zero in $L^2(D)$. In view of Lemma 3.2(iii) and the fact that $\|\hat{u}_\epsilon\|_{L^2(D)} \not\to 0$, we must have $a_\epsilon \to 0$. This proves Claim 4.7.

Next, integrate the inequality $|\varphi| \le |a_\epsilon| u_\epsilon + |\varphi^\perp|$ over $\alpha$ to obtain

$$\int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi|\, d\alpha \le |a_\epsilon| \hat{u}_\epsilon + \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha.$$

Since $a_\epsilon \to 0$ (Claim 4.7), $\|\hat{u}_\epsilon\|_{L^2(D)}$ stays bounded (Lemma 3.2(iii)), and that $\left\| \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha \right\|_{L^2(D)} \to 0$ (Claim 4.6), we must have $\left\| \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi|\, d\alpha \right\|_{L^2(D)} \to 0$. This contradicts with the normalization (3.3), and proves (4.9).

<u>Step 2:</u> Define

$$\Psi(x) := \frac{\int_{\underline{\alpha}}^{\overline{\alpha}} \alpha\varphi\, d\alpha}{\underline{\alpha}\|\hat{\varphi}\|_{L^2(D)}}, \quad \Phi(x) := \frac{\hat{\varphi}}{\|\hat{\varphi}\|_{L^2(D)}} = \frac{\int_{\underline{\alpha}}^{\overline{\alpha}} \varphi\, d\alpha}{\|\hat{\varphi}\|_{L^2(D)}}.$$

Then by (4.9) and Proposition 3.3(ii), we have

$$\|\Psi - \Phi\|_{L^2(D)} \le o(1), \quad \|\Phi\|_{L^2(D)} = 1 \tag{4.11}$$

from which we also deduce

$$\|\Psi\|_{L^2(D)} \to 1 \quad \text{as } \epsilon \to 0. \tag{4.12}$$

Next, we integrate (1.6) over $\alpha \in (\underline{\alpha}, \overline{\alpha})$, and divide by $\|\hat{\varphi}\|_{L^2(D)}$, we have

$$\begin{cases} \underline{\alpha}\Delta_x \Psi = (2\hat{u}_\epsilon - m - \lambda)\Phi & \text{in } D, \\ \frac{\partial}{\partial n}\Psi = 0 & \text{on } \partial D. \end{cases} \tag{4.13}$$

Then standard elliptic estimate gives

$$\|\Psi\|_{W^{2,2}(D)} \le C(\|\Psi\|_{L^2(D)} + \|(2\hat{u}_\epsilon - m - \lambda)\Phi\|_{L^2(D)}) \le C'.$$

Hence we may pass to a sequence so that for some $\Phi_0 \in W^{2,2}(D)$,

$$\Psi \rightharpoonup \Phi_0 \text{ (weakly) in } W^{2,2}(D) \quad \text{and} \quad \Psi \to \Phi_0 \text{ (strongly) in } L^2(D).$$

And by (4.11), also $\Phi \to \Phi_0$ (strongly) in $L^2(D)$. Note that $\Phi_0 \ne 0$ as $\|\Phi_0\|_{L^2(D)} = \lim_{\epsilon\to 0} \|\Phi\|_{L^2(D)} = 1$ (by (4.11)). Now, if we multiply (4.13) by a test function $\rho(x) \in C^\infty(\bar{D})$ such that $\frac{\partial}{\partial n}\rho = 0$ on $\partial D$, and integrate by parts so that

$$\int_D \underline{\alpha}\Psi \Delta_x \rho \, dx = \int_D (2\hat{u}_\epsilon - m - \lambda)\Phi\rho \, dx,$$

we may pass to the limit so that $\Phi_0$ is a non-trivial weak solution of

$$\begin{cases} \underline{\alpha}\Delta_x \Phi_0 + (m - 2\theta_{\underline{\alpha}} + \lambda_0)\Phi_0 = 0 & \text{in } D, \\ \frac{\partial}{\partial n}\Phi_0 = 0 & \text{on } \partial D. \end{cases} \tag{4.14}$$

<u>Step 3:</u> Since $\Phi_0 \neq 0$, we may multiply (4.14) by the complex conjugate of $\Phi_0$, and integrate by parts, to deduce that $\lambda_0 \in \mathbb{R}$, and

$$\lim_{\epsilon \to 0} \lambda = \lambda_0 = \frac{\int_D [\underline{\alpha}|\nabla_x \Phi_0|^2 + (2\theta_{\underline{\alpha}} - m)|\Phi_0|^2 \, dx}{\int_D |\Phi_0|^2 \, dx}. \tag{4.15}$$

Finally, observe that by the definition of $\theta_{\underline{\alpha}}$ being the unique positive solution to (1.4), the principal eigenvalue of the related eigenvalue problem must be zero. Variational characterization then says

$$0 = \inf_{\Phi \in H^1(D) \setminus \{0\}} \frac{\int_D \underline{\alpha}|\nabla_x \Phi|^2 + (\theta_{\underline{\alpha}} - m)|\Phi|^2 \, dx}{\int_D |\Phi|^2 \, dx}.$$

Thus (4.15) implies

$$\lim_{\epsilon \to 0} \lambda = \lambda_0 \geq \frac{\int_D \theta_{\underline{\alpha}} |\Phi_0|^2 \, dx}{\int_D |\Phi_0|^2 \, dx} > 0.$$

This contradicts the fact that $\operatorname{Re} \lambda \leq 0$ for all $\epsilon$. □

# 5 Uniqueness Result

In this section we derive the uniqueness of positive steady states of (1.1) when $\epsilon$ is small. Our proof is based on a topological index argument, which makes use of the linear stability stability result proved in the previous section.

*Proof of Theorem 1.2.* By Theorem 1.1(b)(ii), there is $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the equation (1.1) has at least one positive steady state. Moreover, Proposition 4.1 asserts that there is $\epsilon_* \in (0, \epsilon^*]$ such that for all $\epsilon \in (0, \epsilon_*)$, every positive steady state $u_\epsilon$ of (1.1) is linearly stable. i.e.

$$\sigma(\epsilon^2 \partial_\alpha^2 + \alpha \Delta_x + (m - \hat{u}_\epsilon) - u_\epsilon \,\hat{\cdot}\, ; \mathcal{N}) \subset \{\mu \in \mathcal{C} : \operatorname{Re} \mu > 0\}. \tag{5.1}$$

In particular, all of such solutions are non-degenerate. It remains to show the uniqueness of positive steady states of (1.1). Combining the nondegeneracy of steady states with the apriori estimate of Corollary 2.4, we conclude that, for each $\epsilon > 0$ small, there are at most finitely many positive steady states of (1.1). By Lemma 2.5, we have

$$1 = \deg(A_0, X_1, 0) = \sum_u \mathrm{index}_X(A_0'(u), 0), \tag{5.2}$$

where the summation is taken over all positive steady states $u$ of (1.1). Since every positive steady state is linearly stable (by Proposition 4.1), Proposition C.1 then implies that $\mathrm{index}_X(A_0'(u), 0) = 1$. Therefore, (5.2) implies

$$\#\{\text{positive s.s. of (1.1)}\} = \sum_u \mathrm{index}_X(A_0'(u), 0) = \deg(A_0, X_1, 0) = 1.$$

This proves the theorem. □

# 6 Global dynamics when $m(x)$ is constant

In this section we consider the case when $m(x) \equiv m_0$ for some constant $m_0 \in \mathbb{R}$. We will show that for $m_0 \leq 0$, every non-negative solution of (1.1) converges to zero; while for $m_0 > 0$, every non-negative, non-trivial solution converges to the unique (homogeneous) positive steady state $m_0$. The latter is accomplished by the construction of a Lyapunov functional and application of LaSalle's Invariance Principle.

*Proof of Theorem 1.3.* If $m_0 \leq 0$, then $\mu_1 = -m_0 \geq 0$ (where $\mu_1$ is the p.e.v. of (1.5)) and Theorem 1.3(a) says that, for all $\epsilon > 0$, any non-negative solution $u$ of (1.1) converges to zero as $t \to \infty$, i.e. $\|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})} \to 0$ as $t \to \infty$.

Henceforth assume $m_0 > 0$. We make the following change of variables:

$$x' = |D|^{-\frac{1}{k}} x, \quad \alpha' = \frac{1}{\overline{\alpha} - \underline{\alpha}} \alpha, \quad t' = |D|^{-\frac{2}{k}} (\overline{\alpha} - \underline{\alpha}) t, \quad \epsilon' = \frac{|D|^{\frac{1}{k}}}{(\overline{\alpha} - \underline{\alpha})^{\frac{3}{2}}} \epsilon$$

where $x' \in D' := \{x' \in \mathbb{R}^k : |D|^{\frac{1}{n}} x' \in D\}$ with $|D'| = 1$ and $\alpha' \in (\underline{\alpha}', \overline{\alpha}')$ such that

$$\overline{\alpha}' = \frac{\overline{\alpha}}{\overline{\alpha} - \underline{\alpha}}, \quad \underline{\alpha}' = \frac{\underline{\alpha}}{\overline{\alpha} - \underline{\alpha}}, \quad \overline{\alpha}' - \underline{\alpha}' = 1.$$

Set $u'(x', \alpha', t') = \frac{\overline{\alpha} - \underline{\alpha}}{m_0} u(x, \alpha, t)$, then remove the $'$, it suffices to consider non-negative solutions to the following equation (recall that $\hat{u}(x,t) := \int_{\underline{\alpha}}^{\overline{\alpha}} u(x, \alpha, t)\, d\alpha$):

$$\begin{cases} \partial_t u = \epsilon^2 u_{\alpha\alpha} + \alpha \Delta_x u + M u (1 - \hat{u}) & \text{for } x \in D, \alpha \in (\underline{\alpha}, \overline{\alpha}), t > 0, \\ \partial_n u = 0 & \text{for } x \in \partial D, \alpha \in (\underline{\alpha}, \overline{\alpha}), t > 0, \\ \partial_\alpha u = 0 & \text{for } x \in D, \alpha = \underline{\alpha}, \overline{\alpha}, t > 0, \end{cases} \tag{6.1}$$

where

$$|D| = 1, \quad \overline{\alpha} - \underline{\alpha} = 1 \quad \textit{and} \quad M \text{ is some positive constant.} \tag{6.2}$$

In this case we observe that the constant function $U(x,\alpha) \equiv 1$ is a positive steady state of (6.1). Integrate (6.1) over $\Omega = D \times (\underline{\alpha}, \overline{\alpha})$, we get

$$\frac{d}{dt}\int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u \, d\alpha dx = M \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u(1-\hat{u})\, d\alpha dx = M \int_D \hat{u}(1-\hat{u})\, dx. \tag{6.3}$$

By the Hölder's inequality, we have $-\int_D \hat{u}^2\, dx \leq -(\int_D \hat{u}\, dx)^2$, so that

$$\frac{d}{dt}\iint_\Omega u\, d\alpha dx \leq M \left(\iint_\Omega u\, d\alpha dx\right)\left(1 - \iint_\Omega u\, d\alpha dx\right), \tag{6.4}$$

where $\iint_\Omega u\, d\alpha dx = \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u(x,\alpha,t)\, d\alpha dx$. By the above differential inequality, we can deduce the following.

**Claim 6.1.** *For each $K \geq 1$, define*

$$X_K := \{u_0 \in C(\overline{\Omega}) : u_0(x,\alpha) \geq 0 \quad \textit{and} \quad \int_D \int_{\underline{\alpha}}^{\overline{\alpha}} u_0(x,\alpha)\, dx d\alpha \leq K\}.$$

*Then for each $K > 1$, $X_K$ is a Banach space on which* (6.1) *defines a dynamical system.*

Next, observe that any non-negative solution $u$ of (6.1) is a subsolution of a linear parabolic equation if we discard the term $-\hat{u}u$. Hence, by parabolic local maximum principle [38, Theorem 7.36], we deduce that for some constant $C > 0$ independent of $\ell \in \mathbb{N}$, such that

$$\sup_{\Omega \times (\ell, \ell+1)} |u| \leq C\|u\|_{L^1(\Omega\times(\ell-1,\ell+2))} \quad \text{for all } \ell \in \mathbb{N}.$$

Hence we may apply parabolic $L^p$ estimates to deduce that for any $p > 1$, $\|u\|_{W^{2,1,p}(\Omega\times(\ell,\ell+1))}$ is uniformly bounded in $\ell \in \mathbb{N}$. If we denote for each $u_0 \in X_K$, the orbit of $u_0$ to be $\{u(\cdot,\cdot,t) \in X_K : t \geq 0\}$, where $u$ is the solution of (6.1) with initial condition $u_0$, then the following holds.

**Claim 6.2.** *For each $u_0 \in X_K$, the orbit of $u_0$ under the dynamical system generated by* (6.1) *has compact closure in $X_K$.*

Next, we construct a Lyapunov function.

**Claim 6.3.** *$V(u) := \iint_\Omega (u - 1 - \log u)\, dx d\alpha$ is a Lyapunov function.*

Let $u$ be a solution of (6.1), we compute

$$\begin{aligned}
\frac{d}{dt}V(u(\cdot,\cdot,t)) &= \iint_\Omega \partial_t u - \frac{\partial_t u}{u}\,dxd\alpha \\
&= \iint_\Omega \left[Mu(1-\hat{u}) - \frac{\epsilon^2\partial_\alpha^2 u + \alpha\Delta_x u}{u} - M(1-\hat{u})\right] d\alpha dx \\
&= M\iint_\Omega (u-1)(1-\hat{u})d\alpha dx - \iint_\Omega \frac{\epsilon^2|\partial_\alpha u|^2 + \alpha|\nabla_x u|^2}{u^2} d\alpha dx \\
&\le -M\int_D (1-\hat{u})^2\,dx.
\end{aligned}$$

This shows that $V$ is a Lyapunov function. Next, define

$$\dot{V}(u_0) := \frac{d}{dt}V(u(\cdot,\cdot,t)\Big|_{t=0}, \quad \text{and} \quad E_K := \{u_0 \in X_K : \dot{V}(u_0) = 0\} \tag{6.5}$$

where $u$ is the unique solution of (1.1) with initial condition $u_0 \ge 0$.

**Claim 6.4.** *For each $K > 1$, the maximal invariant set $\mathfrak{M}_K$ of $E_K$ is the singleton set containing only the constant function* 1.

By the proof of Claim 6.3,

$$E_K = \{w \in X_K : \int_{\underline{\alpha}}^{\overline{\alpha}} w(x,\alpha)\,d\alpha = 1 \text{ for all } x \in D\}.$$

Let $w_0 \in \mathfrak{M}$ and let $w$ be an entire solution of (1.1) such that $w\big|_{t=0} = w_0$. Making use of $w \in E_K$ for all $t$, we see that $w$ satisfies

$$\begin{cases} \partial_t w = \epsilon^2\partial_\alpha^2 w + \alpha\Delta_x w & \text{in } \Omega\times\mathbb{R} \\ \partial_n w = 0 & \text{on } \partial D \times (\underline{\alpha},\overline{\alpha})\times\mathbb{R}, \\ \partial_\alpha w = 0 & \text{on } D\times\{\underline{\alpha},\overline{\alpha}\}\times\mathbb{R}. \end{cases} \tag{6.6}$$

Let $(\gamma_k, \phi_k(x,\alpha))$ be the eigenpairs of the operator $\epsilon^2\partial_\alpha^2 + \alpha\Delta_x$ in $\Omega$ subject to Neumann boundary conditions. Then we arrange things so that $0 = \gamma_1 < \gamma_2 \le \gamma_3 \ldots$ and $\iint_\Omega \phi_k\phi_\ell\,dxd\alpha = \delta_{k\ell}$. In particular, the principal eigenfunction $\phi_1(x,\alpha) \equiv 1$ in $\Omega$. The fact that $w_0 \in \mathfrak{M} \subset E_K$ implies that $\iint_\Omega w_0\,dxd\alpha = 1$. Thus we can write $w_0(x,\alpha) = 1 + \sum_{k=2}^\infty c_k\phi_k$, and the solution $w$ of (6.6) as

$$w(x,\alpha,t) = 1 + \sum_{k=2}^\infty c_k e^{-\lambda_k t}\phi_k(x,\alpha). \tag{6.7}$$

We claim that $c_k = 0$ for all $k \ge 2$. For multiplying (6.7) by $\phi_k$ and integrating, we have

$$\iint_\Omega w(x,\alpha,t)\phi_k(x,\alpha)\,dxd\alpha = c_k e^{-\lambda_k t} \quad \text{for all } t \in \mathbb{R}. \tag{6.8}$$

But since $w(\cdot,\cdot,t) \in \mathfrak{M} \subset E_K$ for all $t \in \mathbb{R}$ and $w \geq 0$, we can estimate the left hand side of the above by

$$\left|\iint_\Omega w(x,\alpha,t)\phi_k(x,\alpha)\,dxd\alpha\right| \leq \sup_\Omega |\phi_k| \iint_\Omega w(x,\alpha,t)\,dxd\alpha = \sup_\Omega |\phi_k| \tag{6.9}$$

for all $t \in \mathbb{R}$. By (6.8) and (6.9), we deduce that $c_k = 0$ for each $k \geq 2$. This gives $w_0 \equiv 1$ in $\Omega$ and proves Claim 6.4. Finally, the global asymptotic stability of $U(x,\alpha) \equiv 1$ follows from the LaSalle's invariace principle.

**Theorem 6.5** ([27]). *Let $X$ be a Banach space and let $V : X \to X$ be a Lyapunov function on $X$ with respect to the dynamical system generated by* (1.1) *(so that $\frac{d}{dt}V(w(\cdot,\cdot,t)) \leq 0$ for each solution $w$ of* (1.1)*), and define $E = \{w \in X : \dot{V}(w) = 0\}$ where $\dot{V}$ is defined in* (6.5)*. Let $\mathfrak{M}$ be the maximal invariant set of $E$. If for some solution $w$ of* (1.1)*, $\{w(\cdot,\cdot,t) : t \geq 0\}$ is precompact in $X$, then $w(\cdot,\cdot,t) \to \mathfrak{M}$ as $t \to \infty$.*

See, e.g. [29] for the proof of Theorem 6.5. This completes the proof. □

# A Proofs of the Persistence Results

*Proof of Lemma 2.2.* Since $u \geq 0$ is a subsolution to the linear equation

$$u_t = \epsilon^2 \partial_\alpha^2 u + \alpha \Delta_x u + mu \quad \text{in } \Omega \times (0,\infty), \tag{A.1}$$

with homogeneous Neumann boundary condition on $\partial\Omega$, we see that any solution $u$ of (2.2) exists for all $t \geq 0$. For simplicity, we consider the special case when $\overline{\alpha} - \underline{\alpha} = 1$ and $|D| = 1$. The general case follows by minor modifications. First, we estimate $\|u(\cdot,\cdot,t)\|_{L^1(\Omega)}$.

**Claim A.1.** *There exists $C_2 > 0$ such that* $\limsup_{t\to\infty} \|u(\cdot,\cdot,t)\|_{L^1(\Omega)} \leq C_2$.

To see this claim, let $U(t) = \int_\Omega u(x,\alpha,t)\,dxd\alpha$. Integrate (2.2) over $\Omega$, and use the Cauchy-Schwartz inequality

$$U(t)^2 \leq \int_{\underline{\alpha}}^{\overline{\alpha}} u^2\,dxd\alpha,$$

to obtain the following differential inequality:

$$\partial_t U \leq (\sup_D m)U - U^2.$$

This proves Claim A.1. Next, by the homogeneous Neumann boundary condition of $u$ on $\partial\Omega$ and the smoothness of $\partial D$, we may extend $u$ by reflection on $\partial\Omega \times (0,\infty)$ so that it is a subsolution to an equation similar to (A.1) in a larger domain. This allows the use of the local maximum principle [38, Theoremm 7.36] to yield (2.3) from Claim A.1. Finally, the precompactness assertion follows from standard parabolic $L^p$ estimates applied in $\Omega \times [0,T]$. □

*Proof of Proposition 2.3.* Fix $s \in [0,1]$, and let $u(x,\alpha,t)$ be a solution of (2.2). Let $\Phi_1(x,\alpha)$ be a positive eigenfunction of (2.1) corresponding to $\mu_{1,\epsilon}$ such that $\sup_\Omega \Phi_1 = (\overline{\alpha} - \underline{\alpha})^{-1}$. By strong maximum principle, if $u_0$ is non-trivial, then $u(x,\alpha,t) > 0$ for all $(x,\alpha) \in \overline{\Omega}$ and all $t > 0$. Hence we may assume without loss that $u(x,\alpha,t) > 0$ for all $(x,\alpha) \in \overline{\Omega} = \overline{D \times (\underline{\alpha}, \overline{\alpha})}$ and $t \geq 0$.

First, we assume $\mu_{1,\epsilon} \geq 0$ and prove (a). For each $t \geq 0$, choose $k_t$ such that $\sup_\Omega(u(\cdot,\cdot,t) - k_t\Phi_1(\cdot)) = 0$.

**Claim A.2.** *If $u_0$ is non-negative and non-trivial, then $k_t$ is strictly decreasing in $t \geq 0$.*

Let $0 \leq t' < t''$. Observe that the function $w(x,\alpha,t) := u(x,\alpha,t) - k_{t'}\Phi_1(x,\alpha)$ satisfies $w(\cdot,\cdot,t') \leq 0$ in $\Omega$, the differential inequality

$$\partial_t w - \epsilon^2\partial_\alpha^2 w - \alpha\Delta_x w - mw = -su^2 - (1-s)u\hat{u} - \mu_{1,\epsilon}k_{t'}\Phi_1 < 0 \quad \text{(A.2)}$$

in $\overline{\Omega} \times [t', t'']$, and homogeneous Neumann conditon on $\partial\Omega \times [t', t'']$. By comparison, we have $u - k_{t'}\Phi_1 = w < 0$ in $\overline{\Omega} \times (t', t'']$, i.e. $k_{t''} < k_{t'}$. This proves Claim A.2.

By Claim A.2, $k_* := \lim_{t\to\infty} k_t = \inf_{t\geq 0} k_t \geq 0$ exists. It remains to show that $k_* = 0$. Suppose to the contrary that $k_* > 0$, then there is a sequence $(x_j, \alpha_j, t_j)$ such that

$$(x_j, \alpha_j) \to (x_0, \alpha_0) \in \overline{\Omega}, \quad t_j \to \infty, \quad u(x_j, \alpha_j, t_j + 1) \to k_*\Phi_1(x_0, \alpha_0) > 0. \quad \text{(A.3)}$$

We consider the sequence $\{u^{t_j}\}$ which is defined and shown to be compact in Lemma 2.2(b). Hence, passing to a subsequence if necessary, $u^{t_j}$ converges in $C(\bar{\Omega} \times [0,1])$ to a subsolution $u_*$ of (A.1) satisfying homogeneous Neumann boundary condition on $\partial\Omega \times [0,1]$. Moreover, the initial condition satisfies

$$u_*(x,\alpha,0) = \lim_{j\to\infty} u^{t_j}(x,\alpha,0) \leq \lim_{j\to\infty} k_{t_j}\Phi_1(x,\alpha) = k_*\Phi_1(x,\alpha).$$

Since $u_*(x,\alpha,0) \not\equiv 0$ (otherwise $u_*(x,\alpha,1) \equiv 0$ and contradicts (A.3)), the strict inequality in (A.2) holds. One can then repeat the argument used to prove Claim A.1 to show that $u_*(x,\alpha,1) < k_*\Phi_1(x,\alpha)$ in $\overline{\Omega}$ so that by (A.3),

$$k_*\Phi_1(x_0,\alpha_0) = \lim_{j\to\infty} u^{t_j}(x_j,\alpha_j,1) = u_*(x_0,\alpha_0,1) < k_*\Phi_1(x_0,\alpha_0)$$

which is a contradiction. This proves $k_* = 0$, which implies (a).

Next, we suppose $\mu_{1,\epsilon} < 0$ and show (b). Define

$$v(x,\alpha,t) = u(x,\alpha,t)/\Phi_1(x,\alpha),$$

then it satisfies

$$\begin{cases} \partial_t v - A_\epsilon v = v\left(-sv\Phi_1 - (1-s)\int_{\underline{\alpha}}^{\overline{\alpha}} v\Phi_1\, d\alpha - \mu_{1,\epsilon}\right) & \text{in } \Omega\times(0,\infty),\\ \partial_n v = 0 \qquad \text{on } \partial D\times(\underline{\alpha},\overline{\alpha})\times(0,\infty), \\ \partial_\alpha v = 0 \qquad \text{on } D\times\{\underline{\alpha},\overline{\alpha}\}\times(0,\infty), \\ v(x,\alpha,0) = v_0(x,\alpha) := u_0(x,\alpha)/\Phi_1(x,\alpha) \quad \text{in } \Omega. \end{cases} \tag{A.4}$$

where $A_\epsilon v = \epsilon^2\left(\partial_\alpha^2 v + 2\frac{\partial_\alpha \Phi_1}{\Phi_1}\partial_\alpha v\right) + \alpha\left(\Delta_x + 2\frac{\nabla_x\Phi_1}{\Phi_1}\nabla_x v\right)$. Define

$$V^*(t) = \sup_\Omega v(\cdot,\cdot,t), \qquad \text{and} \qquad V_*(t) = \inf_\Omega v(\cdot,\cdot,t).$$

Since $\limsup_{t\to\infty}\|u(\cdot,\cdot,t)\|_{C(\overline{\Omega})}$, and hence $\limsup_{t\to\infty}\|v(\cdot,\cdot,t)\|_{C(\overline{\Omega})}$, are bounded independent of initial conditions, we may apply the Harnack inequality for positive solutions of parabolic Neumann problems to obtain a constant $C>0$ independent of initial conditions such that

$$V^*(t) \le CV_*(t+1) \text{ for all } t\ge 1. \tag{A.5}$$

Therefore, it remains to show $\liminf\limits_{t\to\infty} V^*(t) \ge \delta_2$ for some $\delta_2$ independent of initial condition $v_0$. First, we observe from (A.4) that $\partial_t V^*(t) \le -\mu_{1,\epsilon}V^*(t)$. This implies

$$V^*(t_2) \le \exp(-\mu_{1,\epsilon}(t_2-t_1))V^*(t_1) \quad \text{ for all } t_2\ge t_1\ge 0. \tag{A.6}$$

If $\liminf_{t\to\infty} V^*(t) \ge -\mu_{1,\epsilon}/4$, then we are done. Suppose $\liminf_{t\to\infty} V^*(t) < |\mu_{1,\epsilon}|/4$, we claim then that $\limsup_{t\to\infty} V^*(t) \ge 2|\mu_{1,\epsilon}|/3$. Otherwise by the normalization $\sup_\Omega \Phi_1 = \max\{1, (\overline{\alpha}-\underline{\alpha})^{-1}\}$, we have

$$\begin{aligned} &-sv(x,\alpha,t)\Phi_1(x,\alpha) - (1-s)\int_{\underline{\alpha}}^{\overline{\alpha}} v(x,\alpha,t)\Phi_1(x,\alpha)\,dxd\alpha - \mu_{1,\epsilon} \\ &\ge -V^*(t) + |\mu_{1,\epsilon}| \ge |\mu_{1,\epsilon}|/3 \end{aligned} \tag{A.7}$$

for all $t\gg 1$, and (A.4) implies $\partial_t V_*(t) \ge (|\mu_{1,\epsilon}|/3)V_*(t)$ for all $t\gg 1$, and thus $\lim_{t\to+\infty} V_*(t) = +\infty$. This contradicts the last inequality of (A.7). Henceforth assume

$$\liminf_{t\to\infty} V^*(t) < |\mu_{1,\epsilon}|/4 \quad \text{ and } \quad \limsup_{t\to\infty} V^*(t) \ge 2|\mu_{1,\epsilon}|/3. \tag{A.8}$$

**Claim A.3.** *Suppose $V^*(t_0) = |\mu_{1,\epsilon}|/4$. Let*

$$T = \inf\{t>0 : V^*(t_0+t) \ge |\mu_{1,\epsilon}|/2\}.$$

*Then $T \le 1 + C'/|\mu_{1,\epsilon}|$ for some $C'>0$ independent of initial conditions.*

By (A.8), $T < \infty$. If $T \leq 1$ then we are done. Therefore we may assume $T > 1$. When $t \in [0, T]$, (since $0 \leq \Phi_1 \leq \max\{1, (\overline{\alpha} - \underline{\alpha})^{-1}\}$)

$$\begin{aligned}
&- sv(x, \alpha, t_0 + t)\Phi(x, \alpha) - (1 - s)\int_{\underline{\alpha}}^{\overline{\alpha}} v(x, \alpha, t_0 + t)\Phi_1(x, \alpha)\, dxd\alpha - \mu_{1,\epsilon} \\
&\geq -V^*(t_0 + t) + |\mu_{1,\epsilon}| \geq |\mu_{1,\epsilon}|/2,
\end{aligned}$$

so (A.4) implies $\partial_t V_*(t) \geq (|\mu_{1,\epsilon}|/2)V_*(t)$ for $t \in [t_0, t_0 + T]$. Thus

$$V_*(t_0 + T) \geq \exp(|\mu_{1,\epsilon}|(T - 1)/2)V_*(t_0 + 1). \tag{A.9}$$

Combining with the Harnack's inequality (A.5), we have

$$\begin{aligned}
\frac{|\mu_{1,\epsilon}|}{2} &= V^*(t_0 + T) \geq V_*(t_0 + T) \geq \exp(|\mu_{1,\epsilon}|(T - 1)/2)V_*(t_0 + 1) \\
&\geq \frac{1}{C}\exp(|\mu_{1,\epsilon}|(T - 1)/2)V^*(t_0) = \frac{1}{C}\exp(|\mu_{1,\epsilon}|(T - 1)/2)\left(\frac{|\mu_{1,\epsilon}|}{4}\right)
\end{aligned}$$

Hence

$$T \leq 1 + \frac{2}{|\mu_{1,\epsilon}|}\log(2C) \leq 1 + \frac{C'}{|\mu_{1,\epsilon}|},$$

where $C'$ is independent of initial conditions. This proves Claim A.3. Now, take $t_1 \in [t_0, t_0 + T]$ and $t_2 = t_0 + T$ in (A.6), we deduce that

$$\min_{t\in[0,T]} V^*(t_0 + t) \geq \exp(-|\mu_{1,\epsilon}|T)V^*(t_0 + T) = \exp(-|\mu_{1,\epsilon}| - C')\left|\frac{\mu_{1,\epsilon}}{2}\right|.$$

Hence $\liminf_{t\to\infty} V^*(t) \geq \min\{|\mu_{1,\epsilon}|/4, \exp(-|\mu_{1,\epsilon}| - C')\,|\mu_{1,\epsilon}|/2\}$. In view of (A.5), this proves the assertion (b). □

# B Proof of Estimates

In this section we prove the technical estimates in Proposition 3.3. Parts (i), (ii) will be proved in Subsection B.1. Part (iii), which is a Poincaré inequality, will be proved in Subsection B.2 via establishing a bound for the Poincarè's constant in terms of $\epsilon$.

## B.1 Estimates of $\varphi$

In this subsection, we will first prove a rough estimate of $\|\varphi\|_{L^\infty(\Omega)}$ (Lemma B.1). The proof is based on bootstrapping standard elliptic $L^p$ estimates. Then we will show exponential decay away from the boundary subset $D\times\{\underline{\alpha}\}$ by constructing an upper solution (Lemma B.2). These two equation shows the concentration of $\varphi$ away from the the boundary subset $D \times \{\underline{\alpha}\}$.

**Lemma B.1.** *Fix $M > 0$, and let $(\lambda, \varphi)$ be an eigenpair of* (1.6) *such that $|\lambda| \leq M$ and $\varphi$ is normalized by* (3.3). *Then there exists $C_1, N_1$ independent of $\epsilon$ such that*

$$\|\varphi\|_{L^\infty(\Omega)} \leq C_1 \epsilon^{-N_1}.$$

*Proof.* Extending $\varphi$ by reflection with respect to the boundary portions $D \times \{\underline{\alpha}, \overline{\alpha}\}$, we may assume $\varphi$ is defined in $D \times (\underline{\alpha} - 1, \overline{\alpha} + 1)$. Fix $\alpha_0 \in [\underline{\alpha}, \overline{\alpha}]$ and $\epsilon \in (0, 1/2]$, and let $\Phi(x, \tau) = \varphi(x, \alpha_0 + \epsilon\tau)$ for $|\tau| \leq 2$, then $\Phi$ satisfies

$$\Phi_{\tau\tau} + \alpha(\tau)\Delta_x \Phi + (m(x) - \hat{u}_\epsilon(x) + \lambda)\Phi = \hat{\varphi}(x) u_\epsilon(x, \alpha_0 + \epsilon\tau) \tag{B.1}$$

then $L^p$ estimates yields, for each $q > 1$,

$$\|\Phi\|_{W^{2,q}(D\times(-1,1))} \leq C \left[\|\Phi\|_{L^1(D\times(-2,2))} + \|\hat{\varphi}(x) u_\epsilon(x, \alpha_0 + \epsilon\tau)\|_{L^q(D\times(-2,2))}\right]. \tag{B.2}$$

Since $\|\Phi\|_{L^1(D\times(-2,2))} \leq \epsilon^{-1}\|\varphi\|_{L^1(\Omega)} \leq C\epsilon^{-1}$ and $\|u_\epsilon\|_{L^\infty(\Omega)} \leq \epsilon^{-2/3}$ (by Lemma 3.2(i)), the inequality (B.2) becomes

$$\|\Phi\|_{W^{2,q}(D\times(-1,1))} \leq C \left[\epsilon^{-1} + \epsilon^{-2/3}\|\hat{\varphi}\|_{L^q(D)}\right]. \tag{B.3}$$

Take $q = 2$, then by (3.3) we have ($C$ may change from line to line but remains independent of $\epsilon \in (0, 1/2]$)

$$\|\Phi\|_{W^{2,2}(D\times(-1,1))} \leq C(\epsilon^{-1} + \epsilon^{-2/3}\|\hat{\varphi}\|_{L^2(D)}) \leq C\epsilon^{-1}.$$

Recall that $k$ is the spatial dimension of $D$ and that $k' := k + 1$ is the dimension of $\Omega$. If $k' < 4$, then we are done. For, suppose $\|\varphi\|_{L^\infty(\Omega)} = |\varphi(x_\epsilon, \alpha_\epsilon)|$, the preceeding arguments with $\alpha_0 = \alpha_\epsilon$ show that

$$|\varphi(x_\epsilon, \alpha_\epsilon)| \leq \|\Phi\|_{L^\infty(D\times(-1,1))} \leq \|\Phi\|_{W^{2,2}(D\times(-1,1)} \leq C\epsilon^{-1}.$$

Suppose now $k' \geq 4$, then for $q_1 = \frac{2k'}{k'-4}$

$$\|\varphi\|_{L^{q_1}(D\times(\alpha_0-\epsilon,\alpha_0+\epsilon))} \leq \|\Phi\|_{L^{q_1}(D\times(-1,1))} \leq C\|\Phi\|_{W^{2,2}(D\times(-1,1))} \leq C\epsilon^{-1}. \tag{B.4}$$

(Here we used the fact that $\epsilon^{1/q_1} \leq 1$ for $\epsilon \in (0, 1/2]$.) Partition $[\underline{\alpha}, \overline{\alpha}]$ into subintervals of length $2\epsilon$, then (B.4) implies

$$\|\varphi\|_{L^{q_1}(\Omega)} \leq C\epsilon^{-2}.$$

In particular, $\|\hat{\varphi}\|_{L^{q_1}(D)} \leq C\|\varphi\|_{L^{q_1}(\Omega)} \leq C\epsilon^{-2}$. We may then take $q = q_1$ in (B.3) to yield

$$\|\Phi\|_{W^{2,q_1}(D\times(-1,1))} \leq C\epsilon^{-8/3}.$$

Again, if $k' < 2q_1$ (i.e. $k' < 8$), then $L^\infty \subset W^{2,q_1}$ and we are done. Otherwise we continue by bootstrapping. As the process must terminate within $[k'/4]+1$ steps, the lemma is proved. □

Next, define

$$w(x,s) = \frac{\varphi(x, \underline{\alpha} + \epsilon^{2/3}s)}{\psi_\epsilon(x, \underline{\alpha} + \epsilon^{2/3}s)} \quad \text{for } x \in D, s \in (0, s_\epsilon), \tag{B.5}$$

where $s_\epsilon = (\overline{\alpha} - \underline{\alpha})/\epsilon^{2/3}$, then for $\alpha = \underline{\alpha} + \epsilon^{2/3}s$, then equation (1.6) becomes

$$\begin{cases} -\partial_s(\psi_\epsilon^2 \partial_s w) - \frac{\alpha}{\epsilon^{2/3}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x w) + \psi_\epsilon^2 \left( \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s)}{\epsilon^{2/3}} - \epsilon^{4/3} \frac{\partial_\alpha^2 \psi_\epsilon}{\psi_\epsilon} - \frac{\lambda}{\epsilon^{2/3}} \right) w \\ \qquad = -\psi_\epsilon \hat{\varphi} u_\epsilon \quad \text{in } D \times (0, s_\epsilon), \\ \partial_n w = 0 \quad \text{on } \partial D \times (0, s_\epsilon), \\ \partial_s w = -\epsilon^{2/3} \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} w \quad \text{on } D \times \{0, s_\epsilon\}. \end{cases} \tag{B.6}$$

And the function $W = |w|^2 = w\overline{w}$ (where $\overline{w}$ denote the complex conjugate of $w$) satisfies (for $\alpha = \underline{\alpha} + \epsilon^{2/3}s$)

$$\begin{cases} -\partial_s(\psi_\epsilon^2 \partial_s W) - \frac{\alpha}{\epsilon^{2/3}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x W) + 2\psi_\epsilon^2 \left( \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s)}{\epsilon^{2/3}} - \epsilon^{4/3} \frac{\partial_\alpha^2 \psi_\epsilon}{\psi_\epsilon} - \frac{\mathrm{Re}\,\lambda}{\epsilon^{2/3}} \right) W \\ \qquad \le -2\mathrm{Re}\,(\bar{\varphi}\hat{\varphi}) u_\epsilon \le 2|\varphi||\hat{\varphi}| u \quad \text{in } D \times (0, s_\epsilon), \\ \partial_n W = 0 \quad \text{on } \partial D \times (0, s_\epsilon), \\ \partial_s W = -2\epsilon^{2/3} \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} W \quad \text{on } D \times \{0, s_\epsilon\}. \end{cases} \tag{B.7}$$

For some $s_0 > 0$ (independent of $\epsilon$) to be specified later, we are going to apply comparison principle to estimate $W$ (and thus $|\varphi|$), in the domain $D \times (s_0, s_\epsilon)$ with Dirichlet boundary condition on $D \times \{s_0\}$ and Neumann boundary condition on the rest of the boundary portions. A sufficient condition for the comparison principle to hold is the existence of a strictly positive supersolution, which we are constructing next.

**Lemma B.2.** *Fix $\beta > 0$, define $W^*(x,s) = \exp(-\beta s) + \exp(\beta(s - 3s_\epsilon/2))$, then for some $s_0 > 0$ independent of $\epsilon$ we have, for all $\epsilon$ sufficiently small,*

$$\begin{cases} -\partial_s(\psi_\epsilon^2 \partial_s W^*) - \frac{\alpha}{\epsilon^{2/3}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x W^*) + 2\psi_\epsilon^2 \left( \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s)}{\epsilon^{2/3}} + \epsilon^{4/3} \left| \frac{\partial_\alpha^2 \psi_\epsilon}{\psi_\epsilon} \right| \right) W^* \\ \qquad \ge W^* \quad \textit{in } D \times (s_0, s_\epsilon), \\ \partial_n W^* = 0 \quad \textit{on } \partial D \times (0, s_\epsilon), \\ \partial_s W^* \ge -2\epsilon^{2/3} \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} W^* \quad \textit{on } D \times \{s_\epsilon\}, \\ W^* > 0 \quad \textit{in } \overline{D} \times [0, s_\epsilon], \end{cases} \tag{B.8}$$

*where $\alpha = \underline{\alpha} + \epsilon^{2/3}s$ and $s_\epsilon = \epsilon^{-2/3}(\overline{\alpha} - \underline{\alpha})$.*

*Proof.* By Lemma 3.1 (ii) and (iii), there are positive constants $a_0', a_2$ independent of $\epsilon$ and $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ such that $\epsilon^{-2/3}\sigma_\epsilon(\underline{\alpha}) \ge -a_0'$ and $\frac{\partial}{\partial \alpha}\sigma_\epsilon(\alpha) \ge a_2 > 0$. Therefore

$$\epsilon^{-2/3}\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s) = \epsilon^{-2/3}[\sigma_\epsilon(\underline{\alpha}) + \sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3}s) - \sigma_\epsilon(\underline{\alpha})] \ge -a_0' + a_2 s,$$

and, for each given $\beta > 0$, we can choose $s_0 > 0$ independent of $\epsilon$ such that

$$-\beta^2 - \beta + 2(-a_0' + a_2 s_0 - 1) \geq \left(\inf_{0<\epsilon<1} \inf_{\Omega} \psi_\epsilon\right)^{-2}. \tag{B.9}$$

Note that the right hand side of (B.9) is a positive constant, by Lemma 3.1(iv). Setting $W^*(x,s) = \exp(-\beta s) + \exp(\beta(s - 3s_\epsilon/2))$, we can compute

$$\begin{aligned}
&- \partial_s(\psi_\epsilon^2 \partial_s W^*) - \frac{\alpha}{\epsilon^{2/3}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x W^*) + 2\psi_\epsilon^2 \left( \frac{\sigma_\epsilon(\underline{\alpha} + \epsilon^{2/3} s)}{\epsilon^{2/3}} + \epsilon^{4/3} \left| \frac{\partial_\alpha^2 \psi_\epsilon}{\psi_\epsilon} \right| \right) W^* \\
&\geq \psi_\epsilon^2 \left[ -(\beta^2 + o(1)\beta) + 0 + 2(-a_0' + a_2 s + o(1)) \right] W^* \\
&\geq \psi_\epsilon^2 \left( \inf_{0<\epsilon<1} \inf_{\Omega} \psi_\epsilon \right)^{-2} W^* \geq W^*
\end{aligned}$$

for $0 < \epsilon < 1$, $\alpha = \underline{\alpha} + \epsilon^{2/3} s$ and $s \geq s_0$, where we have used (B.9) in the second last inequality. It is easy to see that $\frac{\partial}{\partial n} W^* = 0$ on $\partial D \times (0, s_\epsilon)$ and that $W^* > 0$ in $\overline{D} \times [0, s_\epsilon]$. It remains to compute, on $D \times \{s_\epsilon\}$,

$$\begin{aligned}
\partial_s W^* + \epsilon^{2/3} \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} W^* &= \beta[-\exp(-\beta s_\epsilon) + \exp(-\beta s_\epsilon/2)] + \epsilon^{2/3} \left| \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} \right| W^* \\
&= W^* \left[ \beta \frac{-\exp(-\beta s_\epsilon) + \exp(-\beta s_\epsilon/2)}{\exp(-\beta s_\epsilon) + \exp(-\beta s_\epsilon/2)} + \epsilon^{2/3} \left| \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} \right| \right] \geq 0
\end{aligned}$$

for $\epsilon \ll 1$ (and $s_\epsilon = \epsilon^{-2/3}(\overline{\alpha} - \underline{\alpha}) \gg 1$)m the expression in square bracket converges to $\beta > 0$. This proves the lemma. □

**Proposition B.3.** *For a given $M > 0$, let $(\lambda, \varphi)$ be an eigenpair of* (1.6) *such that* $\operatorname{Re}\lambda \leq 0$, $|\lambda| \leq M$ *and $\varphi$ satisfies* (3.3). *Then the following statements hold.*

(i) *There exists $C_1$ and $N_2$ such that*

$$|\varphi(x,\alpha)| \leq C_1 \epsilon^{-N_2} \exp(-\epsilon^{-2/3}(\alpha - \underline{\alpha})).$$

(ii) *For each $q \in (0, 2/3)$, we have*

$$\|\varphi\|_{L^\infty(D \times (\underline{\alpha} + \epsilon^q, \overline{\alpha}))} \leq o(\epsilon^p) \quad \textit{for all } p \geq 1.$$

(iii) *For each $q \in (0, 2/3)$,*

$$\left\| \int_{\underline{\alpha}}^{\overline{\alpha}} \alpha \varphi(x,\alpha)\, d\alpha - \underline{\alpha} \hat{\varphi}(x) \right\|_{L^2(D)} \leq C\epsilon^q.$$

(iv) *For each* $q \in (0, 2/3)$, *there exists* $r_\epsilon \in \mathbb{R}$ *such that*

$$\left(\int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp(x,\alpha)|\,d\alpha\right)^2 \le 2\epsilon^q \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp(x,\alpha)|^2\,d\alpha + r_\epsilon \qquad \text{for } x \in D,$$

*where* $r_\epsilon$ *is independent of* $x$ *and that* $r_\epsilon = o(\epsilon^p)$ *for all* $p > 1$. *Here* $\varphi^\perp(x,\alpha)$ *is given by* (3.4).

*Proof of Proposition B.3.* Fix a constant $M > 0$. Suppose $(\lambda, \varphi)$ be given, such that $\operatorname{Re}\lambda \le 0$, $|\lambda| \le M$ and $\varphi$ satisfies (3.3). Then recall $\hat{\varphi}(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi(x,\alpha)\,d\alpha$ and the function $W(x,s)$ satisfying (B.7). By Lemma B.1, there exists $C_2, N_1$ such that

$$\|\varphi\|_{L^\infty(\Omega)} + \|\hat{\varphi}\|_{L^\infty(D)} + \|W\|_{L^\infty(D\times(0,s_\epsilon))} \le C_2\epsilon^{-2N_1}, \tag{B.10}$$

and by Lemma 3.2(i), for each $\beta > 0$,

$$0 < u_\epsilon(x, \underline{\alpha} + \epsilon^{2/3}s) \le C_3\epsilon^{-2/3}\exp(-\beta s) \qquad \text{for all } x \in D, s \in (0, s_\epsilon), \tag{B.11}$$

for some $C_3 > 0$ independent of $\epsilon$. Let $s_0 > 0$ be specified by Lemma B.2 and let

$$W^{**}(x,s) := 2(C_2\epsilon^{-2N_1})^2(C_3\epsilon^{-2/3})\exp(\beta s_0)W^*(x,s).$$

(Recall that $W^*(x,s) = \exp(-\beta s) + \exp(\beta(s - 3s_\epsilon/2))$.) For each $\beta > 0$, there exists (by Lemma B.2) some $s_0 > 0$ such that $W^{**}$ satisfies

$$\begin{cases} -\partial_s(\psi_\epsilon^2\partial_s W^{**}) - \frac{\alpha}{\epsilon^{2/3}}\nabla_x\cdot(\psi_\epsilon^2\nabla_x W^{**}) + 2\psi_\epsilon^2\left(\frac{\sigma_\epsilon(\underline{\alpha}+\epsilon^{2/3}s)}{\epsilon^{2/3}} + \epsilon^{4/3}\left|\frac{\partial_\alpha^2\psi_\epsilon}{\psi_\epsilon}\right|\right)W^{**} & \\ \qquad \ge 2(C_2\epsilon^{-2N_1})^2(C_3\epsilon^{-2/3}\exp(-\beta s)) \ge 2|\varphi||\hat{\varphi}|u_\epsilon & \text{in } D\times(s_0,s_\epsilon), \\ \partial_n W^{**} = 0 & \text{on } \partial D\times(0,s_\epsilon), \\ \partial_s W^{**} \ge -2\epsilon^{2/3}\frac{\partial_\alpha\psi_\epsilon}{\psi_\epsilon}W^{**} & \text{on } D\times\{s_\epsilon\} \\ W^{**} \ge W & \text{on } D\times\{s_0\}, \\ W^* > 0 & \text{in } \overline{D}\times[s_0,s_\epsilon], \end{cases} \tag{B.12}$$

i.e. $W^{**}$ is a positive strict supersolution of (B.7). Hence we may apply the maximum principle to $\frac{W^{**}-W}{W^{**}}$ (see, e.g. [6, p. 48]) to conclude that (recall that $W = |w|^2$ and $w$ is defined in (B.5))

$$W(x,s) \le W^{**}(x,s) \le C\epsilon^{-2N_1-2/3}\exp(-\beta s/2) \qquad \text{for } x \in D, s \in (s_0,s_\epsilon).$$

Combining with (B.10), and that $W = |w|^2$ ($w$ defined in (B.5))

$$\begin{aligned} |\varphi(x,\underline{\alpha}+\epsilon^{2/3}s)|^2 &\le |\psi_\epsilon(x,\underline{\alpha}+\epsilon^{2/3}s)|^2 W(x,s) \\ &\le C\|\psi_\epsilon\|_{L^\infty(\Omega)}\epsilon^{-2N_1-2/3}\exp(-\beta s/2) \\ &\le C\epsilon^{-4N_1-2/3}\exp(-\beta s/2) \end{aligned}$$

where we used boundedness of $\|\psi_\epsilon\|_{L^\infty}$ (Lemma 3.1(iv)). Hence (i) is proved.

Assertion (ii) is a direct consequence of (i).

For assertion (iii), fix $q \in (0, 2/3)$, then

$$\left| \int_{\underline{\alpha}}^{\overline{\alpha}} \alpha\varphi(x,\alpha)\, d\alpha - \underline{\alpha}\hat{\varphi}(x) \right| \leq \int_{\underline{\alpha}}^{\overline{\alpha}} (\alpha - \underline{\alpha})|\varphi|\, d\alpha$$
$$\leq \epsilon^q \int_{\underline{\alpha}}^{\underline{\alpha}+\epsilon^q} |\varphi|\, d\alpha + C \int_{\underline{\alpha}+\epsilon^q}^{\overline{\alpha}} |\varphi|\, d\alpha$$

Combining with the facts that

$$\left\| \int_{\underline{\alpha}}^{\underline{\alpha}+\epsilon^q} |\varphi|\, d\alpha \right\|_{L^2(D)} \leq 1, \quad \text{and} \quad \left\| \int_{\underline{\alpha}+\epsilon^q}^{\overline{\alpha}} |\varphi|\, d\alpha \right\|_{L^2(D)} \leq C\epsilon^q,$$

which follows from (3.3) and assertion (ii), we deduce (iii).

For assertion (iv), first apply Cauchy-Schwartz inequality to yield

$$\left( \int_{\underline{\alpha}}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha \right)^2 \leq 2 \left( \int_{\underline{\alpha}}^{\underline{\alpha}+\epsilon^q} |\varphi^\perp|\, d\alpha \right)^2 + 2 \left( \int_{\underline{\alpha}+\epsilon^q}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha \right)^2$$
$$\leq 2\epsilon^q \int_{\underline{\alpha}}^{\underline{\alpha}+\epsilon^q} |\varphi^\perp|^2\, d\alpha + k_\epsilon(x),$$

where

$$k_\epsilon(x) = 2 \left( \int_{\underline{\alpha}+\epsilon^q}^{\overline{\alpha}} |\varphi^\perp|\, d\alpha \right)^2. \tag{B.13}$$

It remains to show that $\|k_\epsilon\|_{L^\infty(D)} \leq o(\epsilon^p)$ for all $p > 1$.

From the facts $\|u_\epsilon\|_{L^\infty(\Omega)} \leq C\epsilon^{-2/3}$ and (by Cauchy-Schwartz and Lemma 3.2(iii))

$$\liminf_{\epsilon\to 0} \|u_\epsilon\|_{L^2(\Omega)} \geq \liminf_{\epsilon\to\infty} (\overline{\alpha} - \underline{\alpha})^{-1/2} \|\hat{u}\|_{L^2(D)} > 0,$$

and assertion (i) of the lemma, we deduce that for some $N_3 > 0$,

$$|a_\epsilon| = \left| \frac{\iint_\Omega u_\epsilon \varphi\, d\alpha dx}{\iint_\Omega u_\epsilon^2\, d\alpha dx} \right| \leq C\epsilon^{-N_3}, \tag{B.14}$$

where $a_\epsilon$ is given by (3.4). Observe from $\varphi^\perp = \varphi - a_\epsilon u_\epsilon$ that

$$\left\|\varphi^\perp\right\|_{L^\infty(D\times(\underline{\alpha}+\epsilon^q,\overline{\alpha}))} \leq \|\varphi\|_{L^\infty(D\times(\underline{\alpha}+\epsilon^q,\overline{\alpha}))} + \|a_\epsilon u_\epsilon\|_{L^\infty(D\times(\underline{\alpha}+\epsilon^q,\overline{\alpha}))}.$$

By assertion (ii), (B.14) and Lemma 3.2(i), we deduce that

$$\|\varphi^\perp\|_{L^\infty(D\times(\underline{\alpha}+\epsilon^q,\overline{\alpha})} \leq o(\epsilon^p) \quad \text{for all } p > 1.$$

Hence, by (B.13), $\|k_\epsilon\|_{L^\infty(D)} \leq o(\epsilon^p)$ for all $p > 1$. This proves (iv). □

## B.2 A Poincare-type inequality

Let $u_\epsilon$ be a positive steady state solution of (1.1), and recall that $\hat{u}_\epsilon(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon(x,\alpha)\,d\alpha$. Consider the elliptic eigenvalue problem in $\Omega = D \times (\underline{\alpha}, \overline{\alpha})$.

$$\begin{cases} \epsilon^2 \partial_\alpha^2 \phi + \alpha \Delta_x \phi + (m(x) - \hat{u}_\epsilon(x))\phi + \mu_2 \phi = 0 & \text{for } x \in D,\ \underline{\alpha} < \alpha < \overline{\alpha}, \\ \partial_n \phi = 0 & \text{for } x \in \partial D,\ \underline{\alpha} < \alpha < \overline{\alpha}, \\ \partial_s \phi = 0 & \text{for } x \in D,\ \alpha = \overline{\alpha} \text{ or } \underline{\alpha}. \end{cases} \tag{B.15}$$

It can be readily seen that the principal eigenvalue of (B.15) is zero, and $u_\epsilon > 0$ is a principal eigenfunction. Since problem (B.15) is self-adjoint, we deduce that the eigenvalues of (B.15) $\{\mu_j\}_{j\geq 1}$, can be arranged so that (counting multiplicities),

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots$$

**Proposition B.4.** *There exists a constant $c_0 > 0$ independent of $\epsilon$ such that the first positive eigenvalue $\mu_2$ of (B.15) satisfies $\mu_2 \geq c_0 \epsilon^{2/3}$.*

The first positive eigenvalue $\mu_2$ characterizes the optimal constant for a Poincare-type inequality.

**Corollary B.5.** *Let $(\lambda, \varphi)$ be an eigenpair of (1.6). Suppose $Re\,\lambda \leq 0$, then*

$$\iint_\Omega |\varphi^\perp|^2 \,d\alpha dx \leq C\epsilon^{-4/3} \int_D |\hat{\varphi}|^2\,dx$$

*for some constant $C > 0$ independent of $\epsilon$, where*

$$\varphi^\perp = \varphi - \left( \frac{\iint_\Omega \varphi u_\epsilon \,d\alpha dx}{\iint_\Omega u_\epsilon^2 \,d\alpha dx} \right) u_\epsilon \quad \textit{and} \quad \hat{\varphi}(x) = \int_{\underline{\alpha}}^{\overline{\alpha}} \varphi(x,\alpha)\,d\alpha.$$

*Proof of Corollary B.5.* By the Poincare's inequality applied to (1.6), we have for $\lambda \in \mathbb{C}$ with non-positive real part,

$$\iint_\Omega |\varphi^\perp|^2\,d\alpha dx \leq \frac{C}{|\lambda - \mu_2|} \iint_\Omega u_\epsilon^2(x,\alpha) |\hat{\varphi}(x)|^2\,d\alpha dx. \tag{B.16}$$

Since $\mu_2 > 0$ and $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}\,\lambda \leq 0$, we have $|\lambda - \mu_2| \geq \mu_2 \geq c_0 \epsilon^{2/3}$. Also, for some $C$ independent of $\epsilon$ and $x$,

$$\int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon^2(x,\alpha)\,d\alpha \leq \|u_\epsilon\|_{L^\infty(\Omega)} \hat{u}(x) \leq C\epsilon^{-2/3} \quad \text{for } x \in D$$

by Lemma 3.2(i). Hence, (B.16) gives

$$\iint_\Omega |\varphi^\perp|^2\,d\alpha dx \leq C\epsilon^{-2/3} \int_D \left[ \int_{\underline{\alpha}}^{\overline{\alpha}} u_\epsilon^2\,d\alpha \right] |\hat{\varphi}(x)|^2\,dx \leq C\epsilon^{-4/3} \int_D |\hat{\varphi}|^2\,dx.$$

This proves the corollary. □

Since the proof of Proposition B.4 is fairly technical, we briefly outline it here. We suppose for contradiction that $\mu_1 = o(\epsilon^{2/3}$. In this case,if we normalize so that

$$\int_0^{s_\epsilon} \int_D \phi^2(x, \underline{\alpha} + \epsilon^{2/3}s)\, dxds = 1, \tag{B.17}$$

then energy estimate implies

$$\phi(x, \underline{\alpha} + \epsilon^{2/3}s) \to 0 \quad \text{in } C_{loc}(\bar{D} \times [0, \infty)). \tag{B.18}$$

However, Lemmas B.6 and B.7 together shows that

$$\phi(x, \underline{\alpha} + \epsilon^{2/3}s) \le C \exp(-s/4). \tag{B.19}$$

It remains to observe that (B.18) and (B.19) is in contradiction with (B.17).

**Lemma B.6.** *Fix $\beta > 0$ and let $\phi$ be the eigenfunction of* (B.15) *corresponding to the eigenvalue $\mu_2$. Suppose $\mu_2 \le \epsilon^{2/3}$, then there exists $C, s_0 > 0$ independent of $\epsilon$ such that for any $s' \in [s_0, s_0 + 3]$,*

$$|\phi(x, \underline{\alpha} + \epsilon^{2/3}s)| \le C \left[\sup_{x\in D} |\phi(x, \underline{\alpha} + \epsilon^{2/3}s')|\right] \exp(-\beta s/4)$$

*for $x \in D, s \in [s', s_\epsilon)$, where $s_\epsilon = \epsilon^{-2/3}(\overline{\alpha} - \underline{\alpha})$.*

*Proof.* Define $\Phi_\epsilon(x, s) := \left|\frac{\phi(x,\underline{\alpha}+\epsilon^{2/3}s)}{\psi_\epsilon(x,\underline{\alpha}+\epsilon^{2/3}s)}\right|^2$. Then from the calculations in Appendix B, we have

$$\begin{cases} -\partial_s(\psi_\epsilon^2 \partial_s \Phi_\epsilon) - \frac{\alpha}{\epsilon^{2/3}} \nabla_x \cdot (\psi_\epsilon^2 \nabla_x \Phi_\epsilon) + 2\psi_\epsilon^2 \left(\frac{\sigma_\epsilon(\underline{\alpha}+\epsilon^{2/3}s)}{\epsilon^{2/3}} - \epsilon^{4/3} \frac{\partial_\alpha^2 \psi_\epsilon}{\psi_\epsilon}\right) \Phi_\epsilon \\ \qquad = \frac{\mu_2}{\epsilon^{2/3}} \Phi_\epsilon \quad \text{in } D \times (0, s_\epsilon), \\ \partial_n \Phi_\epsilon = 0 \qquad \text{on } \partial D \times (0, s_\epsilon), \\ \partial_s \Phi_\epsilon = -2\epsilon^{2/3} \frac{\partial_\alpha \psi_\epsilon}{\psi_\epsilon} \Phi_\epsilon \qquad \text{on } D \times \{0, s_\epsilon\} \end{cases} \tag{B.20}$$

Next, we notice that by Lemma B.2 and the fact that $0 \le \mu_2/\epsilon^{2/3} \le 1$, for any $s' \in [s_0, s_0 + 3]$, (here $W^*(x, s)$ is defined in Lemma B.2 and $s_0$ in Lemma B.2)

$$\Phi^*(x, s) = \left[\frac{\sup_{x\in D} \Phi_\epsilon(x, s')}{\inf_{x\in D} W^*(x, s')}\right] W^*(x, s) \tag{B.21}$$

is a strict positive supersolution to (B.20) in $D \times [s', s_\epsilon]$, where Dirichlet boundary condition is imposed on $D \times \{s'\}$. By comparison principle, we have (for each $s' \in [s_0, s_0 + 3]$,)

$$\Phi_\epsilon(x, s) \le C \left[\sup_{x\in D} \Phi_\epsilon(x, s')\right] W^*(x, s) \le C \left[\sup_{x\in D} \Phi_\epsilon(x, s')\right] 2 \exp(-\beta s/2)$$

for $x \in D$ and $s \in [s', s_\epsilon]$. Here we have used the fact that the denominator in (B.21) is bounded from below independent of $\epsilon > 0$:

$$\inf_{x \in D} W^*(x, s') = \exp(-\beta s') + \exp(\beta(s' - 3s_\epsilon/2)) \geq \exp(-\beta(s_0 + 3)),$$

and $s' \in [s_0, s_0 + 3]$. Since $\psi_\epsilon$ is uniformly bounded from above and below by positive constants independent of $\epsilon$ (Lemma 3.1(iv)), we have

$$|\phi(x, \underline{\alpha} + \epsilon^{2/3} s)| \leq C \left[\sup_{x \in D} |\phi(x, \underline{\alpha} + \epsilon^{2/3} s')|\right] \exp(-\beta s/4)$$

for $x \in D$ and $s \in [s', s_\epsilon]$. □

**Lemma B.7.** *Suppose $\mu_2 \leq \epsilon^{2/3}$. There exists $C_5 > 0$ such that for each $\epsilon$ sufficiently small, there exists $s'_\epsilon \in [s_0 + 1, s_0 + 2]$ such that*

$$\sup_{x \in D} |\phi(x, \underline{\alpha} + \epsilon^{2/3} s'_\epsilon)| \leq C_5 \epsilon^{-1/3} \|\phi\|_{L^2(\Omega)},$$

*where $s_0$ (given in Lemma B.6) and $C_5$ are both independent of $\epsilon$.*

*Proof.* First we claim that

**Claim B.8.** *For all $\epsilon$, there exists $s'_\epsilon \in [s_0 + 1, s_0 + 2]$ such that*

$$\epsilon^{-1/3} \int_{-\epsilon^{1/3}}^{\epsilon^{1/3}} \int_D |\phi(x, \underline{\alpha} + \epsilon^{2/3}(s'_\epsilon + t))|^2 \, dx dt \leq 2\epsilon^{-2/3} \|\phi\|^2_{L^2(\Omega)}.$$

The claim follows from the fact that

$$\begin{aligned}
&\int_{s_0+1}^{s_0+2} \left[\epsilon^{-1/3} \int_{-\epsilon^{1/3}}^{\epsilon^{1/3}} \int_D |\phi(x, \underline{\alpha} + \epsilon^{2/3}(s' + t))|^2 \, dx dt\right] ds' \\
&= \epsilon^{-1/3} \int_{-\epsilon^{1/3}}^{\epsilon^{1/3}} \int_{s_0+1}^{s_0+2} \int_D |\phi(x, \underline{\alpha} + \epsilon^{2/3}(s' + t))|^2 \, dx ds' dt \\
&\leq \epsilon^{-1/3} \int_{-\epsilon^{1/3}}^{\epsilon^{1/3}} \left[\int_0^{s_\epsilon} \int_D |\phi(x, \underline{\alpha} + \epsilon^{2/3} s)|^2 \, dx ds\right] dt \\
&= 2\epsilon^{-2/3} \iint_\Omega |\phi|^2 \, dx d\alpha.
\end{aligned}$$

Next, let $\Psi_\epsilon(x, \tau) := \phi(x, \underline{\alpha} + \epsilon^{2/3} s'_\epsilon + \epsilon\tau)$ for $x \in D$ and $\tau \in [-1, 1]$. Then

$$\begin{cases} \Psi_{\epsilon,\tau\tau} + \alpha \Delta_x \Psi_\epsilon + (m - \hat{u}_\epsilon + \mu_2)\Psi_\epsilon = 0 & \text{in } D \times (-1, 1), \\ \frac{\partial}{\partial n} \Psi_\epsilon = 0 & \text{on } \partial D \times (-1, 1). \end{cases} \tag{B.22}$$

Hence elliptic $L^p$ estimate gives

$$\sup_{x \in D} |\phi(x, \underline{\alpha} + \epsilon^{2/3} s'_\epsilon)| = \sup_{x \in D} |\Psi_\epsilon(x, 0)| \leq C \left[\|\Psi_\epsilon\|_{L^2(D \times (-1,1))}\right]$$

Since

$$\|\Psi_\epsilon\|^2_{L^2(D\times(-1,1))} = \epsilon^{-1/3}\int_{-\epsilon^{1/3}}^{\epsilon^{1/3}}\int_D |\phi(x,\underline{\alpha}+\epsilon^{2/3}(s'_\epsilon+t))|^2\,dxdt,$$

the lemma follows from Claim B.8. □

We are now in position to prove Proposition B.4.

*Proof of Proposition B.4.* Let $\phi$ be the eigenfunction of (B.15) corresponding to the eigenvalue $\mu_2$. Set

$$\tilde{u}(x,s) := \epsilon^{2/3}u_\epsilon(x,\underline{\alpha}+\epsilon^{2/3}s), \quad \text{and} \quad \tilde{\phi}(x,s) = \frac{\phi(x,\underline{\alpha}+\epsilon^{2/3}s)}{\tilde{u}(x,s)},$$

then we have

$$(\tilde{u}^2\tilde{\phi}_s)_s + \frac{\alpha}{\epsilon^{2/3}}\nabla_x\cdot(\tilde{u}^2\nabla_x\tilde{\phi}) + \frac{\mu_2}{\epsilon^{2/3}}\tilde{u}^2\tilde{\phi} = 0 \quad \text{on } D\times(0,s_\epsilon), \tag{B.23}$$

and with homogeneous Neumann boundary condition on $\partial(D\times(0,s_\epsilon))$, where $s_\epsilon = (\overline{\alpha}-\underline{\alpha})/\epsilon^{2/3}$. We normalize the eigenfunction $\tilde{\phi}(x,s)$ by

$$\int_0^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}^2\,dxds = 1 \quad\Longleftrightarrow\quad \epsilon^{-2/3}\iint_\Omega \phi^2\,d\alpha dx = 1. \tag{B.24}$$

Suppose for contradiction that $\mu_2 = o(\epsilon^{2/3})$. Multiply by $\tilde{\phi}$ and integrate by parts, we have by (B.24),

$$\int_0^{s_\epsilon}\int_D \tilde{u}^2\left[|\tilde{\phi}_s|^2 + \frac{\alpha}{\epsilon^{2/3}}|\nabla_x\tilde{\phi}|^2\right] = \frac{\mu_2}{\epsilon^{2/3}}\int_0^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}^2\,dxds = o(1). \tag{B.25}$$

**Claim B.9.** *For each $M>0$, $\tilde{\phi}\to 0$ in $L^2(D\times(0,M))$.*

By Lemma 3.2(ii), for each $M>0$, there exists $c_M>0$ such that $\tilde{u}\geq c_M$ in $D\times(0,M)$. By (B.24) and (B.25), passing to a sequence if necessary, there is a constant $a_0$ such that for each $M>0$, $\tilde{\phi}\to a_0$ weakly in $H^1(D\times(0,M))$ and strongly in $L^2(D\times(0,M))$. It remains to show that $a_0=0$.

Suppose to the contrary that $a_0\neq 0$. Since constant function and $\tilde{\phi}$ are eigenfunctions of (B.23) corresponding to distinct eigenvalues, we must have

$$\int_0^{s_\epsilon}\!\!\int_D \tilde{u}^2\tilde{\phi}\,dxds = 0. \tag{B.26}$$

By Lemma 3.2(i), we can choose $M\geq 1$ independent of $\epsilon$ so that

$$\int_M^{s_\epsilon}\tilde{u}^2\,ds \leq \frac{|a_0|^2}{2}\left(\int_D |\theta_{\underline{\alpha}}(x)|^2\,dx\right)^2\left(\int_0^1 |\eta^*(s)|^2\,ds\right)^2, \tag{B.27}$$

where $\theta_{\underline{\alpha}}(x)$ and $\eta^*(s)$ are as given in Lemma 3.2(ii). So that

$$\begin{aligned}|a_0|\int_D |\theta_{\underline{\alpha}}(x)|^2\,dx\int_0^M |\eta^*(s)|^2\,ds &= \lim_{\epsilon\to 0}\left|\int_0^M\int_D \tilde{u}^2\tilde{\phi}\,dxds\right| \\ &= \lim_{\epsilon\to 0}\left|\int_M^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}\,dxds\right| \\ &\le \lim_{\epsilon\to 0}\left[\int_M^{s_\epsilon}\int_D \tilde{u}^2\,dxds\right]^{\frac{1}{2}}\left[\int_M^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}^2\,dxds\right]^{\frac{1}{2}} \\ &\le \lim_{\epsilon\to 0}\left[\int_M^{s_\epsilon}\int_D \tilde{u}^2\,dxds\right]^{\frac{1}{2}}\end{aligned}$$

where we used Lemma 3.2(ii) and the fact that $\tilde{\phi}\to a_0$ for the first equality; (B.26) for the second equality; and the normalization (B.24) for the last inequality. In view of (B.27), this contradicts the assumption that $a_0\neq 0$. Hence Claim B.9 is proved.

By Lemmas B.6 and B.7, and the normalization (B.24),

$$\begin{aligned}\tilde{u}(x,s)\tilde{\phi}(x,s) = \phi(x,\underline{\alpha}+\epsilon^{2/3}s) &\le C\left[\sup_{x\in D}|\phi(x,\underline{\alpha}+\epsilon^{2/3}s'_\epsilon)|\right]\exp(-\beta s/4) \\ &\le C'\epsilon^{-1/3}\|\phi\|_{L^2(\Omega)}\exp(-\beta s/4) \\ &= C'\exp(-\beta s/4)\end{aligned}$$

for $x\in D$ and $s\ge s_0+2$ (since $s'_\epsilon\in[s_0+1,s_0+2]$). Therefore, there exists $M\ge s_0+2$ independent of $\epsilon$ such that

$$\int_M^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}^2\,dxds \le \frac{1}{2}. \tag{B.28}$$

However, by (B.24) and (B.28),

$$\int_0^M\int_D \tilde{u}^2\tilde{\phi}^2\,dxds = 1-\int_M^{s_\epsilon}\int_D \tilde{u}^2\tilde{\phi}^2\,dxds \ge \frac{1}{2}.$$

Recalling that $\tilde{u}$ is bounded in $L^\infty(D\times(0,s_\epsilon))$ (Lemma 3.2), this contradicts $\tilde{\phi}\to 0$ in $L^2(D\times(0,M))$ (Claim B.9) and completes the proof. □

# C Fixed-point index for linearly stable operators

The result of this section is known among nonlinear functional analysts. Since the author cannot locate a proof in the literature, a proof is given here for completeness.

Let $X=C(\overline{\Omega})$, where $\Omega=D\times(\underline{\alpha},\overline{\alpha})$ such that $D$ is a bounded smooth domain in $\mathbb{R}^k$. Suppose $H:X\to X$ is a bounded linear operator such that

$$\sigma(-\epsilon^2\partial_\alpha^2-\alpha\Delta+I+H;\mathcal{N})\subseteq\{\mu\in\mathbb{C}:\operatorname{Re}\mu>0\}, \tag{C.1}$$

i.e. for any given $f \in X$ and $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu \le 0$, the following problem has a unique solution $u \in X$:

$$\begin{cases} \epsilon^2 \partial_\alpha^2 u + \alpha \Delta u - u - H(u) + \mu u = f \quad \text{in } \Omega, \\ \partial_n u = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \quad \text{and} \quad \partial_\alpha u = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}. \end{cases}$$

**Proposition C.1.** $\operatorname{index}_X((-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I)^{-1} \circ H, 0) = 1$*, where* $(-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I)^{-1}$ *is the inverse of* $-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I$ *in* $X$ *subject to Neumann boundary condition.*

*Proof.* Since $-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I$ is sectorial and $H$ is a bounded linear operator, there is some $\phi_0 \in (0, \pi/2)$ such that (using also (C.1))

$$\sigma(-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I + H; \mathcal{N}) \subset \{\mu \in \mathbb{C} : \operatorname{Re}\mu > 0 \quad \text{and } |\arg \mu| < \phi_0\}.$$

By closedness of the spectrum, there is some $\delta_1 > 0$ such that for all $0 \le s \le \delta_1$,

$$\sigma(-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I + (1-s)H; \mathcal{N}) \subset \{\mu \in \mathbb{C} : \operatorname{Re}\mu > \delta_1\}.$$

Choose also $\ell_0 > 0$ large such that for any $s \in [\delta_1, 1]$,

$$\sigma(-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I + (1-s)H; \mathcal{N}) \subset \{\mu \in \mathbb{C} : \operatorname{Re}\mu > -\delta_1\ell_0\}.$$

Then we see that for all $s \in [\delta_1, 1]$, (and thus all $s \in [0, 1]$)

$$\sigma(-\epsilon^2\partial_\alpha^2 - \alpha\Delta + (1 + s\ell_0)I + (1-s)H; \mathcal{N}) \subset \{\mu \in \mathbb{C} : \operatorname{Re}\mu > 0\}.$$

In particular, $L_s = -\epsilon^2\partial_\alpha^2 - \alpha\Delta + (1+s\ell_0)I + (1-s)H$, subject to Neumann boundary condition, is invertible for all $s \in [0, 1]$, so that $\operatorname{index}_X((-\epsilon^2\partial_\alpha^2 - \Delta + I)^{-1} \circ (s\ell_0 I + (1-s)H), 0)$ is independent of $s \in [0, 1]$. Hence,

$$\operatorname{index}_X((-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I)^{-1} \circ H, 0) = \operatorname{index}_X((-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I)^{-1} \circ (\ell_0 I), 0).$$

It remains to see that the last expression is equal to 1. For this purpose, we observe that for any $\tilde{s} \in [0, \ell_0]$, the problem

$$\begin{cases} -\epsilon^2 \partial_\alpha^2 u - \alpha \Delta u + u + \tilde{s} u = f \quad \text{in } \Omega, \\ \partial_n u = 0 \quad \text{on } \partial D \times (\underline{\alpha}, \overline{\alpha}), \quad \text{and} \quad \partial_\alpha u = 0 \quad \text{on } D \times \{\underline{\alpha}, \overline{\alpha}\}, \end{cases}$$

is always uniquely solvable. Hence

$$\operatorname{index}_X((-\epsilon^2\partial_\alpha^2 - \alpha\Delta + I)^{-1} \circ (\ell_0 I), 0) = \operatorname{index}_X(0, 0) = 1.$$

This completes the proof. □

**Remark 2.** *The same argument works for general weakly coupled elliptic systems with Dirichlet or Neumann boundary conditions, e.g. the predator-prey model linearized at a given positive steady state. See, e.g. [25, Lemma 2.6]*

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