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CORRIGENDUM: DYNAMICS OF A REACTION-DIFFUSION-ADVECTION MODEL FOR TWO COMPETING SPECIES

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Abstract. We provide a corrected proof of [4, Theorem 2.2], which preserves the validity of the theorem exactly under those assumptions as stated in the original paper.

1. **Corrigendum.** This Corrigendum concerns the proof of [4, Theorem 2.2]. In the original proof we used [1, Theorem 2.4] and [3, Proposition 3.2], which require more restrictive conditions than necessary. We provide here an elementary maximum principle argument which preserves the validity of Theorem 2.2, exactly under the assumptions as appeared in [4]. For the reader's convenience we recall the statement of Theorem 2.2 and give its complete proof.

The result concerns the unique positive solution $\theta_{\mu,\alpha}$ ($\mu > 0$, $\alpha \geq 0$) of (See [2] for existence and uniqueness results)

$$\begin{cases} \nabla \cdot (\mu \nabla \theta - \alpha \theta \nabla m) + \theta(m - \theta) = 0 & \text{in } \Omega, \\ \mu \frac{\partial \theta}{\partial n} - \alpha \theta \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative. Denote the set of local maximum points of m by $\mathfrak{M}$ and

$$\Sigma_0 = \{x \in \Omega : \nabla m = 0 \text{ and } x \notin \mathfrak{M}\},$$
$$\mathfrak{M}_+ = \{x \in \mathfrak{M} : m(x) > 0\}.$$

We recall the following non-degeneracy assumption on $m(x)$ contained in [4]:

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(M1) Every critical points of m are non-degenerate, and $\Delta m > 0$ on Σ_0. Moreover, $\frac{\partial m}{\partial n} < 0$ on $\partial\Omega$.

Theorem 2.2. *Assume* (M1). *There exist some positive constants* α_1, C, r, γ *and* $\delta^* < 1$ *such that for all* $\mu > 0$ *and* $\alpha \geq \alpha_1$,

$$\theta_{\mu,\alpha}(x) \leq \begin{cases} Ce^{\alpha\delta^*[m(x)-m(x_0)]/\mu} & \text{in } B_r(x_0), \text{ for any } x_0 \in \mathfrak{M}_+, \\ e^{-\gamma\alpha/\mu} & \text{in } \Omega \setminus \cup_{x_0\in\mathfrak{M}_+} B_r(x_0). \end{cases}$$

Proof of Theorem 2.2. Transform the equation by $w(x) = e^{-\alpha m(x)/\mu}\theta_{\mu,\alpha}$ which satisfies

$$\begin{cases} \mu\nabla\cdot(e^{\alpha m/\mu}\nabla w) + e^{\alpha m/\mu}w\left[m(x) - e^{\alpha m/\mu}w\right] = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

If α/μ is bounded, by applying the maximum principle, we have

$$\|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)} \leq \|e^{\alpha m/\mu}\|_{L^\infty(\Omega)}\|w\|_{L^\infty(\Omega)} \leq \|e^{\alpha m/\mu}\|_{L^\infty(\Omega)}\|me^{-\alpha m/\mu}\|_{L^\infty(\Omega)}. \quad (2)$$

Next we consider $\alpha/\mu \to \infty$. As a consequence of (M1), $\mathfrak{M}$ consists of finitely many points. Denote

$$\{m(x) : x \in \mathfrak{M}\} = \{m_1, m_2, ..., m_k\}, \quad m_1 < m_2 < ... < m_k;$$
$$\mathfrak{M}_i = \{x \in \mathfrak{M} : m(x) = m_i\}, \quad i = 1, \ldots, k.$$

By the non-degeneracy of critical points of m, there exist $r > 0$, $K > 0$ such that for any $z \in \mathfrak{M}$,

$$\begin{cases} \frac{1}{K}|z-x|^2 \leq m(z) - m(x) \leq K|z-x|^2 \\ \frac{1}{K}|z-x| \leq |\nabla m(x)| \leq K|z-x| \end{cases} \quad (3)$$

for all $x \in B_r(z)$. Set $m_0 = \min_{\overline{\Omega}} m$ and choose $0 < \eta < \min_{1\leq i\leq k}\{m_i - m_{i-1}, r^2/K\}$ such that

$$m_i - \eta \quad \text{are regular values of } m \textit{ as well as } m\big|_{\partial\Omega} \text{ for all } i. \quad (4)$$

Fix $0 < \delta_1 < 1$ and define recursively

$$\delta_{i+1} = \frac{\delta_i\eta}{m_{i+1} - m_i + \eta}, \quad i = 1, 2, ..., k-1. \quad (5)$$

Then we have

$$1 > \delta_1 > \delta_2 > \cdots > \delta_k \equiv \delta^* = \delta_1\prod_{i=1}^{k-1}\frac{\eta}{m_{i+1} - m_i + \eta} > 0.$$

Furthermore, by (3) and (M1) there exists a large constant K_1 independent of μ, α such that

$$\frac{\delta^*\alpha}{\mu}|\nabla m|^2 + \Delta m > 0 \quad \text{in } \overline{\Omega\setminus D},\ D = \cup_{z\in\mathfrak{M}}\overline{B_{\sqrt{\frac{\mu}{\alpha}}K_1}(z)}. \quad (6)$$

Define

$$\Omega_1 = \Omega, \quad \Omega_{i+1} = \{x \in \Omega : m(x) > m_i - \eta\} \setminus \cup_{z\in\mathfrak{M}_i}\overline{B_r(z)}.$$

By the choice of η as in (4) and the fact that $\frac{\partial m}{\partial n}\big|_{\partial\Omega} < 0$, the domains $\Omega_i, \Omega_i \setminus D$ are piecewise smooth. Moreover, $\Omega_{i+1} \subset \Omega_i$, since $\{x \in \Omega : m(x) > m_i - \eta\} \subset \Omega_i$. Define

$$M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)}, \quad d = KK_1^2, \quad \phi_i = Me^d e^{\alpha\delta_i(m(x)-m_i)/\mu}$$

and

$$N[\phi] := -\nabla\cdot(\mu\nabla\phi - \alpha\phi\nabla m) - \phi(m - \theta_{\mu,\alpha}).$$

Then we have

$$N[\phi_i] \geq \phi_i \left[\alpha(1-\delta_i)\left(\frac{\delta_i\alpha}{\mu}|\nabla m|^2 + \Delta m\right) - m\right] \geq 0 \tag{7}$$

in $\Omega_1 \setminus D = \Omega \setminus D$ for $i = 1, ..., k$ by (6) and by choosing $\alpha \geq \alpha_1$ large. Moreover, by (M1) we see that

$$\mu\frac{\partial\phi_i}{\partial n} - \alpha\phi_i\frac{\partial m}{\partial n} = \alpha(\delta_i - 1)\phi_i\frac{\partial m}{\partial n} > 0 \quad \text{on } \partial\Omega. \tag{8}$$

Note that in $D \cap \Omega_i$, $m(x) - m_i \geq -K(K_1\sqrt{\mu/\alpha})^2$. Hence for all i,

$$\phi_i(x) = Me^d e^{\delta_i\alpha(m(x)-m_i)/\mu} \geq Me^d e^{\delta_i\alpha(-KK_1^2\mu/\alpha)/\mu} \geq M \geq \theta_{\mu,\alpha} \quad \text{in } D \cap \Omega_i. \tag{9}$$

Now by (7), and the fact that $N[\theta_{\mu,\alpha}] = 0$,

$$N[\phi_i - \theta_{\mu,\alpha}] \geq 0 \quad \text{in } \Omega_i \setminus D, \text{ for } i = 1, 2, ..., k. \tag{10}$$

We shall show by induction that $\theta_{\mu,\alpha} \leq \phi_i$ in Ω_i, for $i = 1, ..., k$. Consider ϕ_1 on $\Omega_1 = \Omega$. By (9), it remains to prove that $\phi_1 \geq \theta_{\mu,\alpha}$ in $\Omega_1 \setminus D$. We already have a differential inequality given in (10). Therefore, we proceed to look at the boundary condition satisfied by $\phi_1 - \theta_{\mu,\alpha}$. Since $\Omega = \Omega_1$ and $\frac{\alpha}{\mu}$ is large, one may decompose $\partial(\Omega_1 \setminus D) = \partial D \cup \partial\Omega$. By (9),

$$\phi_1 - \theta_{\mu,\alpha} \geq 0 \quad \text{in } \partial(\Omega_1 \setminus D) \cap \partial D, \tag{11}$$

while

$$\mu\frac{\partial}{\partial n}(\phi_1 - \theta_{\mu,\alpha}) - \alpha(\phi_1 - \theta_{\mu,\alpha})\frac{\partial m}{\partial n} \geq 0 \quad \text{in } \partial(\Omega_1 \setminus D) \cap \partial\Omega. \tag{12}$$

FIGURE 1. Diagram illustrating the case when $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$.

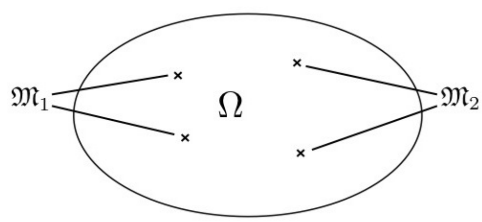
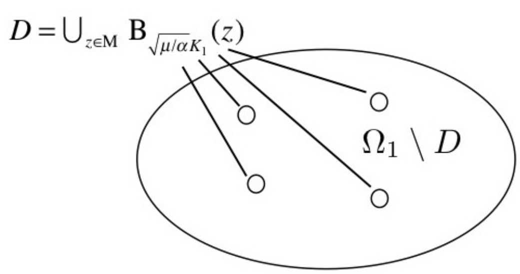


FIGURE 2. Diagram illustrating $\Omega_1 \setminus D$ when $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$.



Now ϕ_1 is a supersolution which is strictly positive on $\partial\Omega$ and that $\phi_1, \theta_{\mu,\alpha} \in C^2(\bar{\Omega})$. It is elementary that the maximum principle applies to yield that $\phi_1 \geq \theta_{\mu,\alpha}$

on $\Omega_1 \setminus D$. But for the sake of completeness, we include a proof here. Using the fact that $\phi > 0$ in $\bar{\Omega}$, we define $z_1 := \frac{\phi_1 - \theta_{\mu,\alpha}}{\phi_1}$, which satisfies

$$\Delta z_1 + \left(\frac{2e^{\alpha m/\mu}}{\mu \phi_1} \nabla (e^{-\alpha m/\mu} \phi_1) + \frac{\alpha}{\mu} \nabla m \right) \cdot \nabla z_1 - \frac{N[\phi_1]}{\mu \phi_1} z_1 \leq 0.$$

Since $z_1 \in C^2(\bar{\Omega})$, $\phi_1 > 0$ in $\bar{\Omega}$ and $N[\phi_1] \geq 0$ (by (7)), we easily deduce that $\inf_{\Omega_1 \setminus D} z_1$ is attained on $\partial(\Omega_1 \setminus D) = (\partial D) \cup (\partial \Omega)$.

Case (i). $\inf_{\Omega_1 \setminus D} z_1 = z_1(x_0)$ for some $x_0 \in \partial D$.

Then $\inf_{\Omega_1 \setminus D} z_1 = z_1(x_0) = \frac{\phi_1 - \theta_{\mu,\alpha}}{\phi_1}(x_0) \geq 0$ by (11).

Case (ii). $\inf_{\Omega_1 \setminus D} z_1 = z_1(x_0)$ for some $x_0 \in \partial \Omega$.

Since $\partial\Omega$ is smooth, and $\partial(\Omega_1 \setminus D) = \partial\Omega \cup \partial D$, the outer normal derivative $\frac{\partial}{\partial n}$ is well defined at x_0,

$$0 \leq -\frac{\partial z_1}{\partial n} = \left[\frac{1}{\mu \phi_1} \left(\mu \frac{\partial \phi_1}{\partial n} - \alpha \phi_1 \frac{\partial m}{\partial n} \right) \right] z_1(x_0).$$

Since the terms in the square bracket is strictly positive (by (8)), we deduce that $\inf_{\Omega_1 \setminus D} z_1 \geq 0$.

Therefore, in any case we have $\inf_{\Omega_1 \setminus D} \frac{\phi_1 - \theta_{\mu,\alpha}}{\phi_1} \geq 0$, and hence $\phi_1 \geq \theta_{\mu,\alpha}$ in $\Omega_1 \setminus D$. Combining with (9), we have proved that $\phi_1 \geq \theta_{\mu,\alpha}$ in Ω_1.

Next, suppose for induction that for some $1 \leq i \leq k-1$,

$$\phi_i \geq \theta_{\mu,\alpha} \quad \text{in } \Omega_i. \tag{13}$$

By (9), it remains to show that $\phi_{i+1} \geq \theta_{\mu,\alpha}$ in $\Omega_{i+1} \setminus D$. By (7), we have $N[\phi_{i+1} - \theta_{\mu,\alpha}] \geq 0$ in $\Omega_{i+1} \setminus D$. Again, ϕ_{i+1} satisfies a differential inequality given by (10). We turn to the boundary condition of $\phi_{i+1} - \theta_{\mu,\alpha}$. Firstly, by (8),

$$\mu \frac{\partial}{\partial n}(\phi_{i+1} - \theta_{\mu,\alpha}) - \alpha(\phi_{i+1} - \theta_{\mu,\alpha}) \frac{\partial m}{\partial n} \geq 0 \quad \text{in } \partial(\Omega_{i+1} \setminus D) \cap \partial\Omega. \tag{14}$$

(Note that by (4) and the fact that $\frac{\partial m}{\partial n}\big|_{\partial\Omega} < 0$, $\frac{\partial}{\partial n}(\phi_{i+1} - \theta_{\mu,\alpha})$ is well-defined by values in $\Omega_{i+1} \setminus D$ even at $x_0 \in \{y \in \partial\Omega : m(y) = m_i - \eta\}$. Here n denotes the unit outer normal of $\partial\Omega$ at x_0.) Secondly, observe that

$$\partial(\Omega_{i+1} \setminus D) = [\partial(\Omega_{i+1} \setminus D) \cap \partial\Omega] \cup [\partial(\Omega_{i+1} \setminus D) \cap \Omega],$$

and that

$$[\partial(\Omega_{i+1} \setminus D) \cap \Omega] \subset [\Omega_{i+1} \cap (\partial D)] \cup [(\partial\Omega_{i+1}) \cap \Omega].$$

We claim that $\phi_{i+1} - \theta_{\mu,\alpha} \geq 0$ in $\partial\Omega_{i+1} \cap \Omega$. By (9),

$$\phi_{i+1} - \theta_{\mu,\alpha} \geq 0 \quad \text{in } \Omega_{i+1} \cap (\partial D). \tag{15}$$

Whereas in $(\partial\Omega_{i+1}) \cap \Omega$, we have $m(x) \geq m_i - \eta$. We either have (i) $x \in \cup_{z \in \mathfrak{M}_i} \partial B_r(z)$; or (ii) $x \notin \cup_{z \in \mathfrak{M}_i} \partial B_r(z)$ and $m(x) = m_i - \eta$. But (i) is impossible, since on $\cup_{z \in \mathfrak{M}_i} \partial B_r(z)$,

$$m(x) \leq m_i - \frac{1}{K}|x - z|^2 = m_i - \frac{r^2}{K} < m_i - \eta.$$

So we must have (ii), i.e. $m(x) = m_i - \eta$. Consequently on $\partial\Omega_{i+1} \cap \Omega$,

$$\begin{aligned} \frac{\phi_{i+1}}{\phi_i} &= \exp\{\delta_{i+1}\alpha(m(x) - m_{i+1})/\mu - \delta_i \alpha(m(x) - m_i)/\mu\} \\ &= \exp\{\alpha[\delta_{i+1}(m_i - \eta - m_{i+1}) + \delta_i \eta]/\mu\} \\ &= 1 \quad \text{by (5)}. \end{aligned}$$

Hence $\phi_{i+1} = \phi_i$ on $\partial\Omega_{i+1} \cap \Omega$. Also, $(\partial\Omega_{i+1} \cap \Omega) \subset \Omega_i$, so by (13)

$$\phi_{i+1} - \theta_{\mu,\alpha} \geq \phi_i - \theta_{\mu,\alpha} \geq 0 \quad \text{on } \partial\Omega_{i+1} \cap \Omega. \tag{16}$$

Now let $z_{i+1} := \frac{\phi_{i+1} - \theta_{\mu,\alpha}}{\phi_{i+1}}$, then z_{i+1} satisfies

$$\Delta z_{i+1} + \left(\frac{2e^{\alpha m/\mu}}{\mu\phi_{i+1}} \nabla(e^{-\alpha m/\mu}\phi_{i+1}) + \frac{\alpha}{\mu}\nabla m \right) \cdot \nabla z_{i+1} - \frac{N[\phi_{i+1}]}{\mu\phi_{i+1}} z_{i+1} \leq 0 \quad \text{in } \Omega_{i+1} \setminus D.$$

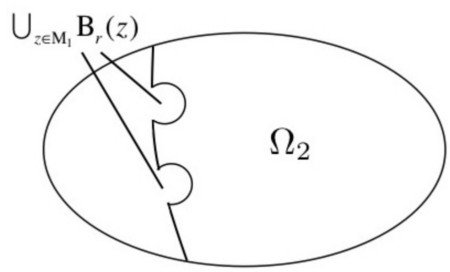
Since $z_{i+1} \in C^2(\bar{\Omega})$ and $\frac{N[\phi_{i+1}]}{\mu\phi_{i+1}} \geq 0$, we deduce that $\inf_{\Omega_{i+1}\setminus D} z_{i+1}$ is attained on $\partial(\Omega_{i+1} \setminus D)$.

Claim 1. $\inf_{\Omega_{i+1}\setminus D} z_{i+1} \geq 0$.

Suppose to the contrary that

$$\inf_{\Omega_{i+1}\setminus D} z_{i+1} = \inf_{\partial(\Omega_{i+1}\setminus D)} z_{i+1} < 0. \tag{17}$$

FIGURE 3. Diagram illustrating Ω_2 when $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$.



We decompose as before

$$\partial(\Omega_{i+1} \setminus D) = [\partial(\Omega_{i+1} \setminus D) \cap \partial\Omega] \cup [\partial(\Omega_{i+1} \setminus D) \cap \Omega].$$

Since by (15) and (16),

$$z_{i+1} \geq 0 \quad \text{in } [\partial(\Omega_{i+1} \setminus D)] \cap \Omega = \partial(\Omega_{i+1} \setminus D) \cap [\partial D \cup \{x \in \Omega : m(x) = m_i - \eta\}].$$

Hence necessarily $x_0 \in [\partial(\Omega_{i+1} \setminus D)] \cap (\partial\Omega)$. Moreover, x_0 is bounded away from $[\partial(\Omega_{i+1} \setminus D)] \cap \Omega$, and hence $\partial(\Omega_{i+1} \setminus D)$ contains a smooth neighborhood of x_0 in $\partial\Omega$. Hence the outer normal derivative $\frac{\partial}{\partial n}\left(z_{i+1}|_{\Omega_{i+1}\setminus D} \right)(x_0)$ is well-defined. Since the minimum of z_{i+1} is attaned at x_0,

$$0 \leq -\frac{\partial z_{i+1}}{\partial n}(x_0) = \left[\frac{1}{\mu\phi_{i+1}} \left(\mu\frac{\partial\phi_{i+1}}{\partial n} - \alpha\phi_{i+1}\frac{\partial m}{\partial n} \right) \right]\Bigg|_{x=x_0} z_{i+1}(x_0).$$

This contradicts the strict positivity of the square bracket term (by (8)) and the hypothesis that $z_{i+1}(x_0) = \inf_{\partial(\Omega_{i+1}\setminus D)} z_{i+1} < 0$. This contradiction establishes that $\inf_{\Omega_{i+1}\setminus D}(\phi_{i+1} - \theta_{\mu,\alpha}) \geq 0$. Combining with (9), we deduce that $\phi_{i+1} \geq \theta_{\mu,\alpha}$ in Ω_{i+1}.

By induction, $\phi_i \geq \theta_{\mu,\alpha}$ on Ω_i, $i = 1, ..., k$. Hence there exists $r_1 \in (0, r]$ such that

$$\text{for all } i, \quad \theta_{\mu,\alpha}(x) \leq Me^d e^{\delta^*\alpha(m(x)-m_i)/\mu} \quad \text{in } \cup_{z\in\mathfrak{M}_i} B_{r_1}(z), \tag{18}$$

$$\theta_{\mu,\alpha}(x) \leq Me^d e^{-\delta^*\alpha r_1^2/(\mu K)} \quad \text{in } \Omega \setminus \cup_{z\in\mathfrak{M}} B_{r_1}(z). \tag{19}$$

It remains to show that M is bounded independent of $\mu > 0$ and $\alpha \geq \alpha_1$. Firstly, there exists $R_0 > 0$ such that for each i and each $z \in \mathfrak{M}_i$, (by (3))

$$d - \frac{\delta^* \alpha (m(x) - m_i)}{\mu} < d - \frac{\delta^* \alpha |x-z|^2}{\mu K} < -\log 2 \quad \text{in } B_{r_1}(z) \setminus B_{\sqrt{\frac{\mu}{\alpha}} R_0}(z).$$

Secondly, since $\alpha/\mu \to \infty$, we may assume $d - \frac{\delta^* \alpha r_1^2}{\mu K} < -\log 2$. Hence, by (18) and (19),

$$\theta_{\mu,\alpha}(x) \leq \frac{M}{2} \qquad \text{in } \Omega \setminus \left(\cup_{z \in \mathfrak{M}} B_{\sqrt{\frac{\mu}{\alpha}} R_0}(z) \right)$$

and the maximum value $M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)}$ must be attained in $B_{\sqrt{\frac{\mu}{\alpha}} R_0}(z_{\mu,\alpha})$ for some $z_{\mu,\alpha} \in \mathfrak{M}$. Set $x = z_{\mu,\alpha} + \sqrt{\frac{\mu}{\alpha}} y$, then

$$\mu \left(\frac{\alpha}{\mu} \Delta_y \theta_{\mu,\alpha} \right) - \alpha \sqrt{\frac{\alpha}{\mu}} \nabla_x m \cdot \nabla_y \theta_{\mu,\alpha} + \theta_{\mu,\alpha}(m - \theta_{\mu,\alpha} - \alpha \Delta_x m) = 0.$$

Divide the above equation by α,

$$\Delta_y \theta_{\mu,\alpha} - \sqrt{\frac{\alpha}{\mu}} \nabla_x m \cdot \nabla_y \theta_{\mu,\alpha} + \left(\frac{m - \theta_{\mu,\alpha} - \alpha \Delta m}{\alpha} \right) \theta_{\mu,\alpha} = 0. \tag{20}$$

By applying the maximum principle to $\theta_{\mu,\alpha}$ and using $\frac{\partial m}{\partial n} \leq 0$, we have $M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)} \leq \|m\|_{L^\infty(\Omega)} + \alpha \|\Delta m\|_{L^\infty(\Omega)}$. Also, the middle term $\sqrt{\alpha/\mu} \nabla_x m(z_{\mu,\alpha} + \sqrt{\frac{\mu}{\alpha}} y)$ in the above equation is bounded by $2\|D^2 m\|_{L^\infty(\Omega)} \|y\|$. Hence the coefficients of (20) are bounded in $L^\infty(B_{4R_0}(0))$. By the Harnack Inequality (Theorem 8.20, [5]), there exists a constant $c = c(N, R_0) > 0$ (N being the dimension) such that

$$\theta_{\mu,\alpha}(x) \geq cM \quad \text{in } B_{\sqrt{\frac{\mu}{\alpha}} R_0}(z_{\mu,\alpha}).$$

Hence

$$c^2 M^2 \left(\frac{\mu}{\alpha} \right)^{N/2} R_0^N \mathrm{Vol}(B_1(0)) \leq \int_{B_{\sqrt{\frac{\mu}{\alpha}} R_0}(z_{\mu,\alpha})} \theta_{\mu,\alpha}^2 \leq \int_\Omega \theta_{\mu,\alpha}^2. \tag{21}$$

Moreover, by (18) and (19),

$$\int_\Omega \theta_{\mu,\alpha} m \leq \|m\|_{L^\infty(\Omega)} \int_\Omega \theta_{\mu,\alpha} \leq CM \left(\frac{\mu}{\alpha} \right)^{N/2} \mathrm{Vol}(B_1(0)). \tag{22}$$

Now integrating the equation of $\theta_{\mu,\alpha}$ to obtain

$$\int_\Omega \theta_{\mu,\alpha}^2 = \int_\Omega \theta_{\mu,\alpha} m. \tag{23}$$

Combining (21), (22) and (23) we infer that

$$c^2 M^2 \left(\frac{\mu}{\alpha} \right)^{N/2} R_0^N \leq CM \left(\frac{\mu}{\alpha} \right)^{N/2}.$$

This gives the boundedness of M as $\alpha/\mu \to \infty$ and proves the theorem in the case $\mathfrak{M} = \mathfrak{M}_+$, i.e. $m(x) > 0$ for all $x \in \mathfrak{M}$. If it is not the case, assume

$$m_1 < m_2 < \ldots < m_{l-1} \leq 0 < m_l < \ldots < m_k, \qquad \text{for some } l \geq 2.$$

Then (18) and (19) can be obtained as before. Next, define $\phi_0 = M e^d e^{\alpha(m(x) - \hat{\eta})/\mu}$ where $-\hat{\eta}$ is a regular value of both m and $m\big|_{\partial\Omega}$, chosen such that $\mathfrak{M} \cap [-\hat{\eta}, 0) = \emptyset$ and

$$0 < \hat{\eta} < \min \left\{ \eta, \frac{\delta_l m_l}{2 - \delta_l} \right\}. \tag{24}$$

Now consider $\Omega_0 = \{x \in \Omega : m < -\hat{\eta}\} \cup (\cup_{z\in\mathfrak{M}_0} B_r(z))$ where $\mathfrak{M}_0 := \{x \in \mathfrak{M} : m(x) = 0\}$ (possibly empty). Note that by similar considerations as before $\partial\Omega_0 \setminus \partial\Omega \subset \{x \in \Omega : m(x) = -\hat{\eta}\}$ and it is smooth as $-\hat{\eta}$ is a regular value of m. Since $m \le 0$ in Ω_0, it is easy to see that $N[\phi_0 - \theta_{\mu,\alpha}] \ge 0$ in Ω_0. Define

$$\mathcal{B}_0 u = \begin{cases} \mu\frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} & \text{on } \partial\Omega_0 \cap \partial\Omega, \\ u & \text{on } \partial\Omega_0 \setminus \partial\Omega. \end{cases}$$

Then $\mathcal{B}_0[\phi_0 - \theta_{\mu,\alpha}] = 0$ on $\partial\Omega_0 \cap \partial\Omega$ by simple calculation, and on $\partial\Omega_0 \cap \Omega \subset \{x \in \Omega : m(x) = -\hat{\eta}\} \cap \Omega_l$,

$$\begin{aligned} \phi_0 &= Me^d e^{\alpha(-\hat{\eta}-\hat{\eta})/\mu} \\ &> Me^d e^{\delta_l \alpha(-\hat{\eta}-m_l)/\mu} \qquad \text{by (24)} \\ &= \phi_l \ge \theta_{\mu,\alpha}. \end{aligned}$$

Therefore, by applying the maximum principle much as before, $\phi_0 - \theta_{\alpha,\mu} \ge 0$ in Ω_0. This completes the proof of the general case. □

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