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UNIQUENESS AND COMPLETE DYNAMICS IN HETEROGENEOUS COMPETITION-DIFFUSION SYSTEMS*

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Abstract. In this paper we study the interactions between diffusion and heterogeneity of the environment in the classical diffusive Lotka–Volterra competition systems. In the weak competition case, we establish the uniqueness, hence the global asymptotic stability, of coexistence steady states under various circumstances, and thereby we obtain a complete understanding of the change in dynamics when one of the interspecific competition coefficients is small.

Key words. spatial heterogeneity, reaction-diffusion, Lotka–Volterra, mathematical ecology

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1. Introduction.

Spatial characteristics of the environment play an important role in ecology and evolution. Using a competition-diffusion model, we shall illustrate the significant changes in dynamics caused by the introduction of spatial heterogeneity. More precisely, we study in this paper the following two-species Lotka–Volterra competition-diffusion model:

$$(1)\qquad \begin{cases} U_t = d_1\Delta U + U(m(x) - U - cV) & \text{in } \Omega\times\mathbf{R}^+,\\ V_t = d_2\Delta V + V(m(x) - bU - V) & \text{in } \Omega\times\mathbf{R}^+,\\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega\times\mathbf{R}^+,\\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x) & \text{in } \Omega, \end{cases}$$

where $U(x,t)$ and $V(x,t)$ represent the population densities of two competing species and are therefore assumed to be nonnegative, with corresponding migration rates d_1 and d_2. For simplicity we assume that both U_0 and V_0 are nonnegative and not identically zero. The function $m(x)$ represents their common (spatially inhomogeneous) intrinsic growth rate or carrying capacity, and b and c are interspecific competition coefficients. The habitat Ω is a bounded region in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. The zero Neumann (no-flux) boundary condition means that no individual crosses the boundary of the habitat; $\partial_\nu = \nu\cdot\nabla$, where ν denotes the outward unit normal vector on $\partial\Omega$. We shall assume that d_1, d_2 are positive constants; b, c are nonnegative constants; and the carrying capacity satisfies

(M) $m(x) \in C(\bar\Omega)$ is nonconstant and $m(x) > 0$ on $\bar\Omega$.

The model (1) has attracted considerable interest in the past two decades; see [CC, DHMP, HLBV, HLM1, HLM2, LWW, Lo, L1, L2, SK] and references therein.

Let θ_d be the unique positive solution of

$$(2)\qquad \begin{cases} d\Delta\theta + [m(x) - \theta]\theta = 0 & \text{in } \Omega,\\ \partial_\nu\theta = 0 & \text{on } \partial\Omega. \end{cases}$$

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(See, e.g., [CC] for the proof of existence and uniqueness results for (2).) System (1) has a trivial steady state $(0,0)$ and two semitrivial steady states $(\theta_{d_1},0)$ and $(0,\theta_{d_2})$. If a steady state (U,V) satisfying $U \geq 0$ and $V \geq 0$ is neither a trivial nor a semitrivial steady state, then by the maximum principle we must have $U > 0$ and $V > 0$ in $\bar{\Omega}$. In this case, we call (U,V) a *coexistence steady state.*

As (1) generates a monotone dynamical system [He, Hi, HiS] which preserves the order

$$(U_1,V_1) \preceq (U_2,V_2) \quad \text{if} \quad U_1 \leq U_2 \text{ and } V_1 \geq V_2 \text{ in } \Omega,$$

it is well known that, to a large extent, the dynamics of (1) is determined by its steady states and their stability properties.

To motivate our discussion, we first consider the special case $b = c = 1$ with homogeneous intrinsic growth rate $m(x) \equiv \bar{m}$. In this case, it is easy to see that, for any $d_1, d_2 > 0$, (1) has a compact global attractor consisting of a continuum of steady states $\{((1-t)\bar{m}, t\bar{m}) : t \in [0,1]\}$ connecting the two semitrivial steady states.

If we incorporate spatial inhomogeneity into the model, then the difference in diffusion rates takes effect, as the following well-known result in [DHMP] illustrates. Note that the positivity assumption on $m(x)$ can be relaxed here.

THEOREM 1.1 (see [DHMP]). *Suppose that $m(x)$ is nonconstant, $\int_\Omega m \geq 0$, and $b = c = 1$. Then the semitrivial steady state $(\theta_{d_1},0)$ of (1) is globally asymptotically stable when $d_1 < d_2$; i.e., every solution (U,V) of (1) converges to $(\theta_{d_1},0)$ as $t \to \infty$, regardless of initial conditions.*

This suggests that, in a spatially heterogeneous but temporally constant environment, a slower diffuser is competitively superior to its faster-moving counterpart.

To understand this phenomenon from a different angle, we turn to the weak competition case $(0 < b, c < 1)$. This approach first started with [L1], where among other things the local stability of the semitrivial steady state $(\theta_{d_1},0)$ is completely determined. To describe the result more precisely, we define as in [L1]

$$\begin{aligned} &\Sigma_b = \{(d_1,d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : (\theta_{d_1},0) \text{ is linearly stable}\}, \\ &\bar{\Sigma}_b = \{(d_1,d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : (\theta_{d_1},0) \text{ is linearly neutrally stable}\}, \end{aligned} \tag{3}$$

where $\mathbf{R}^+ = (0,\infty)$.

Remark 1.2. (i) We will state the precise definition of linear stability and linear neutral stability and prove that $\bar{\Sigma}_b$ is in fact the closure of Σ_b in $\mathbf{R}^+ \times \mathbf{R}^+$. (See (14) and (15) below.)

(ii) It can be proved [L1] that the linear stability of $(\theta_{d_1},0)$ does not depend on c (see Proposition 2.8 below), and hence Σ_b is well defined.

THEOREM 1.3 (see [L1]). *Let $b^* := \inf_{d>0} \int_\Omega m / \int_\Omega \theta_d$; then the following hold:*

(i) *if $b \in (0,b^*]$, then $(\theta_{d_1},0)$ is unstable for all $d_1,d_2 > 0$ (i.e., $\Sigma_b = \emptyset$);*

(ii) *if $b \in (b^*,1)$, then $(\theta_{d_1},0)$ is linearly stable if $(d_1,d_2) \in \Sigma_b$ and unstable if $(d_1,d_2) \notin \bar{\Sigma}_b$.*

An outstanding problem regarding the dynamics of (1) is the following.

CONJECTURE 1.4. *For all $b \in (b^*,1)$ and all $c \in [0,1]$,*

(i) *$(\theta_{d_1},0)$ is globally asymptotically stable if $(d_1,d_2) \in \Sigma_b$;*

(ii) *there exists a coexistence steady state which is globally asymptotically stable if $(d_1,d_2) \notin \bar{\Sigma}_b$ and $d_1 \leq d_2$.*

Our first main result establishes this conjecture when c is small.

THEOREM 1.5. *There exists $\bar{c} > 0$ such that for all $c \in (0,\bar{c})$ and for all $b \in [0,1]$,*

(i) *$(\theta_{d_1},0)$ is globally asymptotically stable for all (d_1,d_2) in $\bar{\Sigma}_b$, and*

(ii) *for any* $d_1 \leq d_2$ *such that* $(d_1, d_2) \notin \bar{\Sigma}_b$*,* (1) *has a unique coexistence steady state which is globally asymptotically stable.*

Remark 1.6. For c sufficiently small, the first part of Conjecture 1.4 is proved in [L1, Theorem 1.9], with the smallness of c depending on $b \in (b^*, 1)$. Here $\bar{c}$ is independent of $b \in [0, 1]$.

Since the structure of Σ_b can be quite nontrivial (e.g., it may have multiple connected components [LiL]), Theorem 1.5 suggests that the conclusions do not rely on the specific structure of Σ_b.

Concerning the second part of Conjecture 1.4, the following was shown previously in the case where the diffusion rates are both very small or both very large.

THEOREM 1.7 (see [HLM2]). *Given* $b, c \in (0, 1)$*, for all* d_1, d_2 *sufficiently small,* (1) *has a unique coexistence steady state which is globally asymptotically stable.*

THEOREM 1.8 (see [L3]). *Given* $b, c \in (0, 1)$*, for all* d_1, d_2 *sufficiently large,* (1) *has a unique coexistence steady state which is globally asymptotically stable.*

Our next result says that there exists a strip in the d_1-d_2 plane which connects the two above-mentioned regions. This gives a clearer picture of the dynamics of (1).

THEOREM 1.9. *For any* $b, c \in (0, 1)$*, there exists* $\delta_1 > 0$ *such that whenever* $|d_1 - d_2| < \delta_1$*,* (1) *has a unique coexistence steady state* (U^*, V^*) *which is globally asymptotically stable. Moreover,* $(U^*(x), V^*(x)) \to (\frac{1-c}{1-bc}\theta_d(x), \frac{1-b}{1-bc}\theta_d(x))$ *uniformly in* $\bar{\Omega}$ *as* $d_1, d_2 \to d > 0$.

Finally, we consider the case when $b \nearrow 1$. We have the following description of Σ_b as $b \nearrow 1$.

PROPOSITION 1.10. *For all* $b \in (0, 1)$*,* Σ_b *is increasing in* b *and* $\Sigma_b \subset \bar{\Sigma}_b \subset \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : d_1 < d_2\}$*. Moreover, for any* $\epsilon > 0$*,*

$$\{(d_1, d_2) : d_1 \in [\epsilon, 1/\epsilon] \text{ and } d_2 \geq d_1 + \epsilon\} \subset \Sigma_b$$

for all b *sufficiently close to* 1*. In particular,* $\Sigma_b \nearrow \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : d_1 < d_2\}$ *as* b *tends to* 1.

Remark 1.11. In fact, it is also proved in [L1] that for each $b \in (0, 1)$, if d_1 is sufficiently small, then $(\theta_{d_1}, 0)$ becomes unstable and (1) has at least one coexistence steady state which is locally asymptotically stable. (See Claim 2.13 below.) Therefore, in general Σ_b is bounded away from the d_2-axis $\{(d_1, d_2) : d_1 = 0\}$.

Define

$$\Gamma_{b,c} = \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : (\theta_{d_1}, 0) \text{ is globally asymptotically stable}\}.$$

It is obvious that $\Gamma_{b,c} \subset \bar{\Sigma}_b$ for any $b, c > 0$. By Theorem 1.5 (or Theorem 1.9 in [L1]), when c is small, $\Gamma_{b,c} = \bar{\Sigma}_b$. The following theorem shows that for all $c \in (0, 1)$,

$$\Gamma_{b,c} \nearrow \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : d_1 < d_2\} \qquad \text{when } b \nearrow 1.$$

THEOREM 1.12. *For any* $\epsilon > 0$*, there exists* $\delta_2 > 0$ *such that for all* $d_1 \in [\epsilon, 1/\epsilon]$*,* $d_2 \in [d_1 + \epsilon, \infty)$*,* $b \in (1 - \delta_2, 1]$*, and all* $c \in [0, 1 + \delta_2)$*,* $(\theta_{d_1}, 0)$ *is globally asymptotically stable.*

Remark 1.13. A version of this result for fixed $d_1 < d_2$ was mentioned in passing in [HLM1]. Also, it would be interesting to inquire whether (i) $\Gamma_{b,c}$ is nonempty for all $b \in (b^*, 1)$ and $c \in (0, 1)$; (ii) there exists some b, c in $(0, 1)$ such that $\Gamma_{b,c}$ is a proper subset of $\bar{\Sigma}_b$. The latter is equivalent to the existence of multiple coexistence steady states for (1).

By Theorem 1.12, there exists some $b < c$ so that for some $d_1 < d_2$, $(\theta_{d_1}, 0)$ is globally asymptotically stable. In other words, with a suitably smaller diffusion rate, a species with weaker competition abilities may still dominate.

Finally, let us fix $d_2 > 0$ and $b = c \in (b^*, 1)$ close to 1 and observe the change in dynamics when we decrease d_1 from d_2 to 0. (See Figure 1.) Initially, when $d_1 \sim d_2$, i.e., when the diffusion rates are similar, coexistence is guaranteed (for any initial values) by Theorem 1.9. If we decrease d_1, then we enter the shaded region, where for some intermediate range of $0 < d_1 < d_2$, U dominates V (Theorem 1.12). However, if we further decrease d_1, then we have coexistence again (Remark 1.11). Although the two species can still be regarded as equal in competition abilities (as $b = c$), the slower diffuser U is no longer being favored. This is in stark contrast to the case $b = c = 1$ (see [DHMP]), when the slower diffuser always prevails. This suggests that in the context of weak competition, a diffusion rate that is too slow might not be advantageous to a species. It also suggests that in this context a better strategy for winning against a certain species is to adopt a slower, yet somewhat comparable diffusion rate.

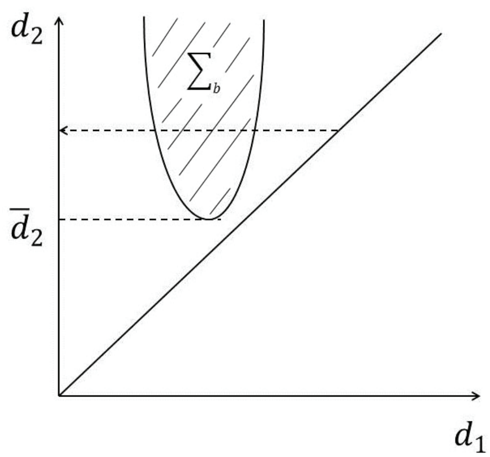

Fig. 1. *Intermediate slower diffuser wins.*

The rest of this paper is organized as follows. In section 2 we will give a characterization of Σ_b and $\bar{\Sigma}_b$, and prove Proposition 1.10. Then we will prove Theorem 1.9 in section 3. Finally, section 4 is devoted to the proofs of Theorems 1.12 and 1.5.

2. Preliminaries. We now define the notion of linear stability of a given steady state (U, V). Linearizing the steady state problem of (1) at (U, V), we have

$$\begin{cases} d_1\Delta\Phi + (m - 2U - cV)\Phi - cU\Psi + \lambda\Phi = 0 & \text{in } \Omega, \\ d_2\Delta\Psi + (m - bU - 2V)\Psi - bV\Phi + \lambda\Psi = 0 & \text{in } \Omega, \\ \partial_\nu\Phi = \partial_\nu\Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

If (U, V) is a coexistence steady state, then according to the Krein–Rutman theorem [KR, Sm], (4) has a principal eigenvalue $\lambda_1 \in \mathbf{R}$; i.e., λ_1 is simple and has the least real part among all eigenvalues. Moreover, we may choose the corresponding eigenfunction (Φ_1, Ψ_1) to satisfy $\Phi_1 > 0 > \Psi_1$ in $\bar{\Omega}$. If (U, V) is a trivial or semitrivial steady state, then the sign of the principal eigenvalue can be determined according to Corollary 2.10 below. In the following, we call a steady state (U, V) of (1) *linearly* *stable* (resp., *linearly unstable*) if the principal eigenvalue λ_1 of (4) is positive (resp., negative). And we call a steady state *linearly neutrally stable* if λ_1 is nonnegative. It is well known that *if a steady state of* (1) *is linearly stable (resp., linearly unstable), then it is asymptotically stable (resp., unstable).* (See, e.g., Theorem 7.6.2 in [Sm], or Proposition 2.15 below. Here the notions of *stability and asymptotic stability* are defined in the standard dynamical systems sense with the $C(\bar{\Omega}) \times C(\bar{\Omega})$ topology.) However, *linearly neutral stability is in general not enough to imply asymptotic stability, or even stability.*

Next, we collect some facts concerning the elliptic eigenvalue problem with an indefinite weight.

DEFINITION 2.1. *Given a positive constant d and a function $h \in L^\infty(\Omega)$, we define $\mu_k(d,h)$ to be the kth eigenvalue (counting multiplicities) of*

$$\begin{cases} d\Delta\psi + h\psi + \mu\psi = 0 & \text{in } \Omega, \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

In particular, we call $\mu_1(d,h)$ the first eigenvalue of (5).

The following eigenvalue comparison result is standard. (See, e.g., p. 95 in [CC] or p. 69 in [N].)

PROPOSITION 2.2. *If $h_1(x) \le h_2(x)$ in Ω, then $\mu_1(d,h_1) \ge \mu_1(d,h_2)$ with equality holds if and only if $h_1 = h_2$ a.e. in Ω. Assume in addition that h is nonconstant; then $\mu_1(d_1,h) < \mu_1(d_2,h)$ if $d_1 < d_2$.*

We also collect some useful facts.

LEMMA 2.3.

(a) *$\mu_1(d,h)$ depends smoothly on $d > 0$ and continuously on $h \in L^\infty(\Omega)$.*

(b) *$d \mapsto \theta_d$ is continuous from $\boldsymbol{R}^+$ to $W^{2,p}(\Omega) \cap C^{1,\gamma}(\bar{\Omega})$.*

(c) *$\mu_1(d_2, m - b\theta_{d_1})$ depends continuously on d_1, d_2.*

Proof. Part (a) is classical. (See, e.g., p. 418 in [CoH] and p. 93 in [CC].) Part (b) can be proved by an application of the implicit function theorem. (See Proposition 3.6 in [CC] and remarks there.) Part (c) follows readily from (a) and (b). ☐

PROPOSITION 2.4. *Let θ_d be the positive solution to* (2)*; then $\|\theta_d\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}$. Moreover, if (U,V) is any coexistence steady state of* (1)*, then $\|U\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}$ and $\|V\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}$.*

Proof. Let $w := \theta_d - \|m\|_{L^\infty(\Omega)}$. Then w satisfies

$$d\Delta w + w(m - \theta_d - \|m\|_{L^\infty(\Omega)}) \ge 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu w = 0 \text{ on } \partial\Omega.$$

Notice that $m - \theta_d - \|m\|_{L^\infty(\Omega)} < 0$, so the strong maximum principle (Theorem 9.6 of [GT]) applies. Since w is nonconstant, we see that w cannot attain a nonnegative maximum in Ω. It also cannot attain a nonnegative maximum on $\partial\Omega$, by the Hopf boundary point lemma. Hence $\theta_d - \|m\|_{L^\infty(\Omega)} = w < 0$ in $\bar{\Omega}$. Similar arguments yield $\|U\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}$ and $\|V\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}$ for any steady state (U,V) of (1). ☐

For $d > 0$ and $h \in L^\infty(\Omega)$ with $h > 0$ in a set of positive measure in Ω, define $\vartheta(d,h)$ to be the unique positive solution (if it exists) of

$$d\Delta\vartheta + \vartheta(h - \vartheta) = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu\vartheta = 0 \text{ on } \partial\Omega. \tag{6}$$

It is well known that

$$\vartheta(d,h) > 0 \ \text{ exists if and only if } \ \mu_1(d,h) < 0. \tag{7}$$

(See, e.g., Proposition 3.2 in [CC] or section 4.1 in [N].) We have the following useful result.

PROPOSITION 2.5. *Let $\{h_k\}$ be a sequence of functions in $C(\bar{\Omega})$, and $\{d_k\}$ be a sequence of positive constants such that $\mu_1(d_k, h_k) < 0$ for all k (i.e., $\vartheta_k := \vartheta(d_k, h_k) > 0$ exists) and $\lim_{k\to\infty} h_k = h_\infty$ in $C(\bar{\Omega})$. Then*

(a) *if $d_k \to 0$, then $\vartheta_k \to \max\{h_\infty, 0\}$ in $L^\infty(\Omega)$;*

(b) *if $d_k \to \infty$, then $\vartheta_k \to \frac{1}{|\Omega|}\int_\Omega h_\infty$ and $\tilde{\vartheta}_k := \vartheta_k/\|\vartheta_k\|_{L^\infty(\Omega)} \to 1$ in $L^\infty(\Omega)$;*

(c) *if $d_k \to d_\infty \in \boldsymbol{R}^+$, then $\mu_1(d_\infty, h_\infty) \leq 0$. Moreover,*

  (i) *if $\mu_1(d_\infty, h_\infty) = 0$, then $\vartheta_k \to 0$ and $\vartheta_k/\|\vartheta_k\|_{L^\infty(\Omega)} \to \psi_1$ in $L^\infty(\Omega)$, where ψ_1 is the first eigenfunction corresponding to $\mu_1(d_\infty, h_\infty)$, normalized by $\psi_1 > 0$ and $\|\psi_1\|_{L^\infty(\Omega)} = 1$;*

  (ii) *if $\mu_1(d_\infty, h_\infty) < 0$, then $\vartheta_k \to \vartheta(d_\infty, h_\infty)$. In particular, since $\mu_1(d, m) < 0$ for all $d > 0$, $\vartheta_k \to \theta_d$ in $L^\infty(\Omega)$ whenever $d_k \to d$ and $h_k \to m$ in $L^\infty(\Omega)$.*

Remark 2.6. In (b), if we merely assume $\|h_k\|_{L^\infty(\Omega)}$ to be bounded uniformly, then we can still obtain $\vartheta_k/\|\vartheta_k\|_{L^\infty(\Omega)} \to 1$ in $L^\infty(\Omega)$ and, by passing to a subsequence, $\vartheta_k \to c$ for some nonnegative constant c.

Proof. First, we prove (a). If $h_\infty \leq 0$, then for any $\epsilon > 0$, $h_k \leq \epsilon$ in Ω for all k large. Hence as in the proof of Proposition 2.4, we deduce by the strong maximum principle that $\vartheta_k(x) < \|h_k\|_{L^\infty(\Omega)} \leq \epsilon$ in $\bar{\Omega}$ for k sufficiently large; i.e., $\vartheta_k \to 0$ in $L^\infty(\Omega)$.

If $h_\infty > 0$ on a set of positive measure, then so is $h_\infty - \epsilon$ for some $\epsilon > 0$. It is standard (see, e.g., p. 184 in [CC]) that $\vartheta(d_k, h_\infty \pm \epsilon)$ exist for k large. And by Lemma A.1 of [HLM2],

$$\lim_{k\to\infty} \vartheta(d_k, h_\infty \pm \epsilon) = \max\{h_\infty \pm \epsilon, 0\} \quad \text{in } L^\infty(\Omega).$$

By comparison,

$$\vartheta(d_k, h_\infty - \epsilon) \leq \vartheta_k \leq \vartheta(d_k, h_\infty + \epsilon) \quad \text{in } \Omega$$

for all k large. Hence, by letting $k \to \infty$,

$$\max\{h_\infty - \epsilon, 0\} \leq \liminf_{k\to\infty} \vartheta_k \leq \limsup_{k\to\infty} \vartheta_k \leq \max\{h_\infty + \epsilon, 0\} \quad \text{in } \Omega.$$

Letting $\epsilon \to 0$, we have proved (a).

Now we prove (b). By Proposition 2.4, $\|\vartheta_k\|_{L^\infty(\Omega)} < \|h_k\|_{L^\infty(\Omega)}$ is bounded uniformly in k. Therefore by L^p estimates, $\|\vartheta_k\|_{W^{2,p}(\Omega)}$ is bounded uniformly. Dividing

$$d_k \Delta\vartheta_k + \vartheta_k(h_k - \vartheta_k) = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu \vartheta_k = 0 \text{ on } \partial\Omega, \tag{8}$$

by d_k, from elliptic regularity estimates, we deduce that a subsequence, still denoted by ϑ_k, converges to $\bar{\vartheta}$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\gamma}(\Omega)$ for some nonnegative constant $\bar{\vartheta} \geq 0$. Moreover, if we divide (8) by $d_k\|\vartheta_k\|_{L^\infty(\Omega)}$, then $\tilde{\vartheta}_k := \vartheta_k/\|\vartheta_k\|_{L^\infty(\Omega)} \to 1$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\gamma}(\bar{\Omega})$ as $\|\tilde{\vartheta_k}\|_{L^\infty(\Omega)} = 1$.

Next, dividing (8) by $\|\vartheta_k\|_{L^\infty(\Omega)}$, integrating over Ω, and passing to the limit, we have $\int_\Omega \vartheta_k \to \int_\Omega h_\infty$. Hence $\vartheta_k \to \frac{1}{|\Omega|}\int_\Omega h_\infty$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\gamma}(\bar{\Omega})$. Since the limit is independent of subsequences, this proves (b).

Finally, we prove (c). By Lemma 2.3, $\mu_1(d_\infty, h_\infty) \leq 0$. As in (b), $\|\vartheta_k\|_{W^{2,p}(\Omega)}$ is bounded uniformly in k. So we deduce again, up to a subsequence, that $\vartheta_k \to \vartheta_\infty \geq 0$, which must satisfy

$$d_\infty \Delta\vartheta_\infty + \vartheta_\infty(h_\infty - \vartheta_\infty) = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu \vartheta_\infty = 0 \text{ on } \partial\Omega. \tag{9}$$

Assume $\mu_1(d_\infty, h_\infty) = 0$; then multiplying (9) by the first eigenfunction ψ_1' corresponding to $\mu_1(d_\infty, h_\infty)$, and integrating by parts, we have $\int_\Omega \psi_1' \vartheta_\infty^2 = 0$, which implies $\vartheta_k \to 0$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\gamma}(\bar\Omega)$.

Now dividing (8) by $\|\vartheta_k\|_{L^\infty(\Omega)}$, we see that $\tilde\vartheta_k = \vartheta_k/\|\vartheta_k\|_{L^\infty(\Omega)} \to \tilde\vartheta_\infty$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\gamma}(\bar\Omega)$, up to a subsequence. $\tilde\vartheta_\infty \geq 0$ is nontrivial since $\|\tilde\vartheta_\infty\|_{L^\infty(\Omega)} = 1$ for all k, and $\tilde\vartheta_\infty$ satisfies

$$d_\infty \Delta\tilde\vartheta_\infty + h_\infty \tilde\vartheta_\infty = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu \vartheta_\infty = 0 \text{ on } \partial\Omega.$$

So $\tilde\vartheta_\infty$ must coincide with a multiple of the eigenfunction corresponding to the eigenvalue $\mu_1(d_\infty, h_\infty) = 0$.

On the other hand, if $\mu_1(d_\infty, h_\infty) < 0$, by Lemma 2.3, there exists $\epsilon_0 > 0$ such that $\mu_1(d_k, h_\infty - \epsilon_0) < 0$ for all k large. Hence the equation

$$d_k \Delta\underline{\vartheta} + \underline{\vartheta}(h_\infty - \epsilon_0 - \underline{\vartheta}) = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu \underline{\vartheta} = 0 \text{ on } \partial\Omega,$$

has a unique positive solution $\underline{\vartheta}_k$ for all k large, or $\underline{\theta}_\infty$ for $k = \infty$. By comparison,

$$\vartheta_k \geq \underline{\vartheta}_k \geq \frac{1}{2}\underline{\vartheta}_\infty > 0 \quad \text{in } \bar\Omega,$$

where the second inequality follows from Lemma 2.3(b) (by taking $h = h_\infty - \epsilon_0$). Therefore, $\vartheta_k \not\to 0$ and must converge to the unique positive solution $\vartheta(d_\infty, h_\infty)$ of the limiting equation (9). ∎

To study the linear stability of a steady state (U, V) of (1), it suffices to look at (4). We first note that λ is bounded from below independent of d_1, d_2, b, c.

LEMMA 2.7. *There exists $C_1 > 0$ such that for any $d_1, d_2 > 0$ and $b, c \in [0, 1]$, and any steady state (U, V) of* (1)*, the principal eigenvalue λ_1 given by* (4) *satisfies $\lambda_1 \geq -C_1$.*

Proof. Multiplying the first equation of (4) by Φ, and the second equation of (4) by Ψ, integrating by parts, and adding the results, we have

$$\begin{aligned}
&\lambda_1 \int_\Omega (|\Phi|^2 + |\Psi|^2) \\
&\quad \geq \int_\Omega -(m - 2U - cV)\Phi^2 + \int_\Omega (cU + bV)\Phi\Psi - \int_\Omega (m - bU - 2V)\Psi^2 \\
&\quad \geq -C_1 \int_\Omega (|\Phi|^2 + |\Psi|^2)
\end{aligned}$$

by Young's inequality and Proposition 2.4. ∎

We turn to Σ_b and $\bar\Sigma_b$. Recall that

$$\Sigma_b = \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : (\theta_{d_1}, 0) \text{ is linearly stable}\}$$

and

$$\bar\Sigma_b = \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : (\theta_{d_1}, 0) \text{ is linearly neutrally stable}\}.$$

PROPOSITION 2.8.

$$\Sigma_b = \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : \mu_1(d_2, m - b\theta_{d_1}) > 0\}$$

and

$$\bar{\Sigma}_b = \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : \mu_1(d_2, m - b\theta_{d_1}) \geq 0\}.$$

From the definition of linear stability, it suffices to consider (4) with $(U, V) = (\theta_{d_1}, 0)$:

$$\begin{cases} d_1\Delta\Phi + (m - 2\theta_{d_1})\Phi + \lambda\Phi = c\theta_{d_1}\Psi & \text{in } \Omega, \\ d_2\Delta\Psi + (m - b\theta_{d_1})\Psi + \lambda\Psi = 0 & \text{in } \Omega, \\ \partial_\nu\Phi = \partial_\nu\Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

First, by Proposition 2.2,

$$\mu_1(d_1, m - 2\theta_{d_1}) > \mu_1(d_1, m - \theta_{d_1}) = 0 \quad \text{for any } d_1 > 0. \tag{11}$$

Now, Proposition 2.8 follows from (11) and the next result.

LEMMA 2.9. *Every eigenvalue of* (10) *is real, and*

$$\min S = \min\{\mu_1(d_1, m - 2\theta_{d_1}), \mu_1(d_2, m - b\theta_{d_1})\},$$

where S denotes the set of all eigenvalues of (10)*.*

In particular, we see that the principal eigenvalue of (10) exists and has the same sign as the first eigenvalue $\mu_1(d_2, m - b\theta_{d_1})$.

Proof of Lemma 2.9. Let λ be an eigenvalue of (10) with eigenfunction (Φ, Ψ). If $\Psi \not\equiv 0$, then λ, belonging to the spectrum of the self-adjoint operator $d_2\Delta + (m - b\theta_{d_1})$ (with zero Neumann boundary condition), must be real and satisfy $\lambda \geq \mu_1(d_2, m - b\theta_{d_1})$. Alternatively, if $\Psi \equiv 0$, then $\Phi \not\equiv 0$, and λ belongs to the spectrum of $d_1\Delta + (m - 2\theta_{d_1})$ (with zero Neumann boundary condition), which must again be real and satisfy $\lambda \geq \mu_1(d_1, m - 2\theta_{d_1})$. This shows $\lambda \geq \min\{\mu_1(d_1, m - 2\theta_{d_1}), \mu_1(d_2, m - b\theta_{d_1})\}$ for any $\lambda \in S$.

To show that the minimum is assumed, suppose first that $\mu_1(d_1, m - 2\theta_{d_1}) \leq \mu_1(d_2, m - b\theta_{d_1})$. Let ψ_1 be the first eigenfunction corresponding to $\mu_1(d_1, m - 2\theta_{d_1})$; then $\mu_1(d_1, m - 2\theta_{d_1})$ is an eigenvalue of (10) with eigenfunction $(\psi_1, 0)$. Finally assume $\mu_1(d_1, m - 2\theta_{d_1}) > \mu_1(d_2, m - b\theta_{d_1})$, and let ψ_2 be the first eigenfunction corresponding to $\mu_1(d_2, m - b\theta_{d_1})$; then $\mu_1(d_2, m - b\theta_{d_1})$ is an eigenvalue of (10) with eigenfunction

$$(\Phi, \Psi) = ([d_1\Delta + (m - 2\theta_{d_1} + \mu_1(d_2, m - b\theta_{d_1}))]^{-1}[c\theta_{d_1}\psi_2], \psi_2).$$

Here the operator $L = d_1\Delta + (m - 2\theta_{d_1} + \mu_1(d_2, m - b\theta_{d_1}))$, with zero Neumann boundary condition, is invertible, since by definition

$$\mu_1(d_1, m - 2\theta_{d_1} + \mu_1(d_2, m - b\theta_{d_1})) = \mu_1(d_1, m - 2\theta_{d_1}) - \mu_1(d_2, m - b\theta_{d_1}) > 0.$$

Hence every eigenvalue of L is positive. In particular, zero is not an eigenvalue. ∎

By completely analogous arguments, we obtain the following result.

COROLLARY 2.10. *The linear stability of* $(\theta_{d_1}, 0)$*,* $(0, \theta_{d_2})$*, and* $(0, 0)$ *is determined by* $\mu_1(d_2, m - b\theta_{d_1})$*,* $\mu_1(d_1, m - c\theta_{d_2})$*, and* $\min\{\mu_1(d_1, m), \mu_1(d_2, m)\}$*, respectively.*

Remark 2.11. It follows immediately from the variational characterization of the first eigenvalue that $(0,0)$ is always linearly unstable for any $d_1, d_2, b, c > 0$.

Next, we prove Proposition 1.10.

Proof of Proposition 1.10. If $d_1 \geq d_2$, then by Proposition 2.2 and the definition of θ_{d_1},

$$\mu_1(d_2, m - b\theta_{d_1}) \leq \mu_1(d_1, m - b\theta_{d_1}) < \mu_1(d_1, m - \theta_{d_1}) = 0;$$

i.e., $(d_1, d_2) \notin \bar{\Sigma}_b$ for any $b \in (0,1)$, by Proposition 2.8. Therefore,

$$\Sigma_b \subset \bar{\Sigma}_b \subset \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : d_1 < d_2\},$$

by Lemma 2.3.

Next, we claim that $\Sigma_b \subset \Sigma_{b'}$ if $b < b'$. This again follows from Propositions 2.2 and 2.8. Namely, for every $(d_1, d_2) \in \Sigma_b$, we have

$$\mu_1(d_2, m - b'\theta_{d_1}) > \mu_1(d_2, m - b\theta_{d_1}) > 0.$$

Lastly, given any $\epsilon > 0$, let $\mathcal{I}_\epsilon := \{(d_1, d_2) : d_1 \in [\epsilon, 1/\epsilon] \text{ and } d_2 \geq d_1 + \epsilon\}$. It suffices to show that (by Proposition 2.8) for all b sufficiently close to 1, $\mathcal{I}_\epsilon \in \Sigma_b$; i.e.,

$$\mu_1(d_2, m - b\theta_{d_1}) > 0 \text{ for all } (d_1, d_2) \in \mathcal{I}_\epsilon. \tag{12}$$

For each $b \in [0,1]$, define $g_b : [\epsilon, 1/\epsilon] \to \mathbf{R}$ by $g_b(d_1) := \mu_1(d_1 + \epsilon, m - b\theta_{d_1})$. Then

(a) for each $b \in [0,1]$, g_b is continuous in $[\epsilon, 1/\epsilon]$;

(b) as $b \nearrow 1$, $\{g_b\}$ converges pointwise to g_1 (by Lemma 2.3);

(c) $g_b(x) \geq g_{b'}(x)$ in $[\epsilon, 1/\epsilon]$ if $b \geq b'$.

It is standard (e.g., see Theorem 7.13 in [R]) that $g_b \to g_1$ uniformly on $[\epsilon, 1/\epsilon]$ as $b \to 1$. Since $g_1(d_1) = \mu_1(d_1 + \epsilon, m - \theta_{d_1}) > \mu_1(d_1, m - \theta_{d_1}) = 0$ in $[\epsilon, 1/\epsilon]$ (the inequality is strict as $(m - \theta_{d_1})$ is nonconstant), we see that $g_b(x) > 0$ in $[\epsilon, 1/\epsilon]$ for all b sufficiently close to 1. Hence for all b sufficiently close to 1, by Proposition 2.2,

$$\mu_1(d_2, m - b\theta_{d_1}) \geq g_b(d_1) > 0 \quad \text{for all } (d_1, d_2) \in \mathcal{I}_\epsilon. \qquad \square$$

By symmetry, we have the next claim.

COROLLARY 2.12. *Define* $\Sigma'_c = \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : (0, \theta_{d_2})$ *is linearly stable*$\}$ *and* $\bar{\Sigma}'_c = \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : (0, \theta_{d_2})$ *is linearly neutrally stable*$\}$*; then for all* $c \in (0,1)$*,* Σ'_c *is increasing and* $\Sigma'_c \subset \bar{\Sigma}'_c \subset \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : d_1 > d_2\}$*. Moreover, for any* $\epsilon > 0$*,*

$$\{(d_1, d_2) : d_2 \in [\epsilon, 1/\epsilon] \text{ and } d_1 \geq d_2 + \epsilon\} \subset \tilde{\Sigma}_c$$

for all c *sufficiently close to* 1*. In particular,* $\Sigma'_c \nearrow \{(d_1, d_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ : d_1 > d_2\}$*.*

We now apply eigenvalue comparison (Proposition 2.2) to show our assertion in Remark 1.2 that

$$\bar{\Sigma}_b \text{ is the closure of } \Sigma_b \text{ in } \mathbf{R}^2. \tag{13}$$

In fact, we shall establish that the boundary $\partial\Sigma_b$ of Σ_b in $\mathbf{R}^2$ satisfies

$$\partial\Sigma_b = \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : \mu_1(d_2, m - b\theta_{d_1}) = 0\} = \bar{\Sigma}_b \setminus \Sigma_b. \tag{14}$$

First, we state the following observation (Remark 1.11 in section 1 above) from the proof of Theorem 1.7 in [L1].

CLAIM 2.13. *For each $b \in (0,1)$, there exists $\delta > 0$ such that $(d_1, d_2) \notin \Sigma_b$ whenever $0 < d_1 < \delta$.*

To see the claim, by Proposition 2.8, it suffices to show that $\mu_1(d_2, m - b\theta_{d_1}) < 0$ for d_1 sufficiently small, uniformly in $d_2 > 0$. Let $\varphi > 0$ be the first eigenfunction corresponding to $\mu_1(d_2, m - b\theta_{d_1})$, which satisfies

$$d_2\Delta\varphi + (m - b\theta_{d_1})\varphi + \mu_1\varphi = 0 \text{ in } \Omega \qquad \text{and} \qquad \partial_\nu\varphi = 0 \text{ on } \partial\Omega,$$

where $\mu_1 = \mu_1(d_2, m - b\theta_{d_1})$. Dividing by φ and integrating by parts, we have

$$d_2 \int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} + \int_\Omega (m - b\theta_{d_1}) + \mu_1 = 0.$$

Now the first term is positive, and the second term is also positive if d_1 is small (Proposition 2.5(a)). Hence $\mu_1(d_2, m - b\theta_{d_1}) < 0$ for d_1 small, uniformly in d_2.

By the continuous dependence of $\mu_1(d_2, m - b\theta_{d_1})$ on d_1, d_2, and by Claim 2.13, we see that $\partial\Sigma_b \subset \{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : \mu_1(d_2, m - b\theta_{d_1}) = 0\}$. It remains to show that $\{(d_1, d_2) \in \mathbf{R}^+ \times \mathbf{R}^+ : \mu_1(d_2, m - b\theta_{d_1}) = 0\} \subset \partial\Sigma_b$. Given any (d_1, d_2) such that $\mu_1(d_2, m - b\theta_{d_1}) = 0$, by Proposition 2.2, it suffices to show that $m - b\theta_{d_1}$ is nonconstant, since then for all $d_2' > d_2$, $\mu_1(d_2', m - b\theta_{d_1}) > 0$ and hence $(d_1, d_2') \in \Sigma_b$. To this end, we suppose to the contrary that $m - b\theta_{d_1} = C$ for some constant C. Then $w = (1 - b)\theta_{d_1} > 0$ satisfies

$$d_1\Delta w + w(C - w) = 0 \text{ in } \Omega \qquad \text{and} \qquad \partial_\nu w = 0 \text{ in } \partial\Omega.$$

Hence $C > 0$ and $w = C$ (by uniqueness), but then θ_{d_1} is constant. This is a contradiction since m is a nonconstant function, and (14) is proved. This concludes our discussion of Σ_b.

We end this section by discussing three well-known results for two-species competition models. First, we recall the following well-known fact.

LEMMA 2.14. *If $(\tilde{U}, \tilde{V}) = (0, \theta_{d_2})$ (resp., $(0,0)$, $(\theta_{d_1}, 0)$) is a steady state of (1) for some positive d_1, d_2, b, c, and the principal eigenvalue λ_1 of (4) with $(U, V) = (\tilde{U}, \tilde{V})$ is nonzero, then there exist $\delta > 0$ and a neighborhood $\mathcal{O}$ of $(\tilde{U}, \tilde{V})$ in $\{(U, V) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) : U \geq 0$ and $V \geq 0\}$ such that, for $\hat{d}_1, \hat{d}_2, \hat{b}, \hat{c}$ satisfying $\max\{|\hat{d}_1 - d_1|, |\hat{d}_2 - d_2|, |\hat{b} - b|, |\hat{c} - c|\} < \delta$, if $(\hat{U}, \hat{V}) \in \mathcal{O}$ is a steady state of (1), then $(\hat{U}, \hat{V}) = (0, \theta_{\hat{d}_2})$ (resp., $(0,0)$, $(\theta_{\hat{d}_1}, 0)$).*

Proof. Assume that $(\tilde{U}, \tilde{V}) = (0, \theta_{d_2})$ (with some $d_1, d_2, b, c > 0$) is not isolated; then there exists a sequence of coexistence steady states $(U_k, V_k) \to (0, \theta_{d_2})$, with corresponding coefficients satisfying $d_{1,k} \to d_1$, $d_{2,k} \to d_2$, $b_k \to b$, and $c_k \to c$. By passing to a subsequence, $w = \lim_{k\to\infty} U_k / \|U_k\|_{L^\infty(\Omega)}$ exists, satisfying $w \geq 0$ and

$$d_1\Delta w + w(m - c\theta_{d_2}) = 0 \quad \text{in } \Omega$$

with zero Neumann boundary condition. Since $\|w\|_{L^\infty(\Omega)} = 1$, we deduce $\mu_1(d_1, m - c\theta_{d_2}) = 0$. Hence the principal eigenvalue $\lambda_1 = 0$ by Corollary 2.10, a contradiction. The other cases follow similarly. ∎

Second, we state the standard fact that, for monotone dynamical systems, linear stability (resp., linear instability) implies asymptotic stability (resp., instability). We refer to Theorem 7.6.2 of [Sm] for a proof of the result.

PROPOSITION 2.15. *If a steady state $(\tilde{U}, \tilde{V})$ of (1) is linearly stable (resp., linearly unstable), then it is asymptotically stable (resp., unstable).*

Third, we have the following lemma derived from the theory of monotone dynamical systems, which we are going to use repeatedly. (See, e.g., Proposition 9.1 and Theorem 9.2 in [He].)

LEMMA 2.16. *For any $d_1, d_2 > 0$ and any $b, c \geq 0$, assume every coexistence steady state of* (1)*, if it exists, is asymptotically stable; then one of the following alternatives holds:*

(a) *There exists a unique coexistence steady state of* (1) *which is globally asymptotically stable.*

(b) *System* (1) *has no coexistence steady state, and one of $(\theta_{d_1}, 0)$ or $(0, \theta_{d_2})$ is globally asymptotically stable, while the other one is unstable.*

3. Proof of Theorem 1.9. We first prove uniqueness in the special case $d_1 = d_2 = d > 0$.

LEMMA 3.1. *Assume $d_1 = d_2 = d$; then for each $b, c \in (0,1)$,* (1) *has a unique coexistence steady state which is given by $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$.*

Proof. It is straightforward to verify that $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ is a steady state of (1). For uniqueness, assume that $(\hat{U}, \hat{V})$ is any coexistence steady state of (1). We claim that $(1-b)\hat{U} = (1-c)\hat{V}$. Assume to the contrary that $w := (1-b)\hat{U} - (1-c)\hat{V} \not\equiv 0$; then w satisfies

$$d\Delta w + (m - \hat{U} - \hat{V})w = 0 \text{ in } \Omega \qquad \text{and} \qquad \partial_\nu w = 0 \text{ on } \partial\Omega.$$

In other words, $\mu_k(d, m - \hat{U} - \hat{V}) = 0$ for some $k \geq 1$. However,

$$\mu_k(d, m - \hat{U} - \hat{V}) \geq \mu_1(d, m - \hat{U} - \hat{V}) > \mu_1(d, m - \hat{U} - c\hat{V}) = 0,$$

a contradiction, where the last equality follows from the equation satisfied by $\hat{U}$. Therefore, $(1-b)\hat{U} = (1-c)\hat{V}$. Setting $\hat{V} = \frac{1-b}{1-c}\hat{U}$ in the first equation of (1), we have

$$\begin{cases} d\Delta\hat{U} + \hat{U}(m - \hat{U} - c\frac{1-b}{1-c}\hat{U}) = 0 & \text{in } \Omega, \\ \partial_\nu \hat{U} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \hat{U} > 0. \end{cases}$$

Hence by uniqueness, $\hat{U} = \frac{1-c}{1-bc}\theta_d$ and $\hat{V} = \frac{1-b}{1-c}\hat{U} = \frac{1-b}{1-bc}\theta_d$. ◻

LEMMA 3.2. *When $d_1 = d_2 = d > 0$ and $0 < b, c < 1$, $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ is linearly stable, while $(0,0)$, $(\theta_{d_1}, 0)$, and $(0, \theta_{d_2})$ are linearly unstable.*

Proof. For the stability of $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ we consider the eigenvalue problem

$$(15) \qquad \begin{cases} d\Delta\Phi + [m - 2\frac{1-c}{1-bc}\theta_d - c\frac{1-b}{1-bc}\theta_d]\Phi - c\frac{1-c}{1-bc}\theta_d\Psi = -\lambda\Phi & \text{in } \Omega, \\ d\Delta\Psi - b\frac{1-b}{1-bc}\theta_d\Phi + [m - b\frac{1-c}{1-bc}\theta_d - 2\frac{1-b}{1-bc}\theta_d]\Psi = -\lambda\Psi & \text{in } \Omega, \\ \partial_\nu\Phi = \partial_\nu\Psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Simplifying (15), we get

$$(16) \qquad \begin{cases} d\Delta\Phi + [m - (1 + \frac{1-c}{1-bc})\theta_d]\Phi - c\frac{1-c}{1-bc}\theta_d\Psi = -\lambda\Phi & \text{in } \Omega, \\ d\Delta\Psi - b\frac{1-b}{1-bc}\theta_d\Phi + [m - (1 + \frac{1-b}{1-bc})\theta_d]\Psi = -\lambda\Psi & \text{in } \Omega, \\ \partial_\nu\Phi = \partial_\nu\Psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By the Krein–Rutman theorem, the principal eigenvalue (i.e., the simple eigenvalue with least positive real part) $\lambda_1 \in \mathbf{R}$ exists, and the corresponding eigenfunction may be chosen to satisfy $\Phi > 0 > \Psi$. Now let $w := (1-b)\Phi - (1-c)\Psi > 0$. Then w can be regarded as the principal eigenfunction of

$$\begin{cases} d\Delta w + \left[m - (1 + \frac{(1-b)(1-c)}{1-bc})\theta_d\right] w = -\lambda_1 w & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Since $w > 0$ this implies that $\lambda_1 = \mu_1(d, m - (1 + \frac{(1-b)(1-c)}{1-bc})\theta_d) > \mu_1(d, m - \theta_d) = 0$. This shows that $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ is linearly stable, while the linear instability of $(\theta_{d_1}, 0)$, $(0, \theta_{d_2})$, and $(0,0)$ is a consequence of Proposition 1.10, Corollary 2.12, and Remark 2.11. □

The following corollary is immediate from Lemmas 2.16, 3.1, and 3.2.

COROLLARY 3.3. *If $d_1 = d_2 = d > 0$, then for any $b, c \in (0,1)$, $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ is globally asymptotically stable.*

For the sake of completeness, we now present the proof, due to Lou [L3], of Theorem 1.8. This will be needed in establishing our Theorem 1.9 later.

Proof of Theorem 1.8. By Lemma 2.16, it suffices to show that for d_1, d_2 large, $(\theta_{d_1}, 0)$ and $(0, \theta_{d_2})$ are linearly unstable, and every positive steady state is linearly stable.

First we show that $(\theta_{d_1}, 0)$ is unstable. By Corollary 2.10, it suffices to look at $\mu_1(d_2, m - b\theta_{d_1})$. Now by setting the test function $\varphi = \sqrt{1/|\Omega|}$, we have

$$\mu_1(d_2, m - b\theta_{d_1}) = \inf_{H^1(\Omega)} \left\{ \frac{\int_\Omega [d_2|\nabla\varphi|^2 - (m - b\theta_{d_1})\varphi^2]}{\int_\Omega \varphi^2} \right\} \le -\frac{1}{|\Omega|}\int_\Omega (m - b\theta_{d_1}),$$

and the term on the right-hand side tends to $\frac{b-1}{|\Omega|}\int_\Omega m < 0$ since $\theta_d \to \frac{1}{|\Omega|}\int_\Omega m$ as $d \to \infty$ (Proposition 2.5(b)). So $(\theta_{d_1}, 0)$ is unstable if d_1 is large. Similarly, $(0, \theta_{d_2})$ is unstable when d_2 is large.

We proceed to show that every coexistence steady state is linearly stable. Assume to the contrary that for $d_{1,k} \to \infty$ and $d_{2,k} \to \infty$, (1) has a coexistence steady state (U_k, V_k), which is not linearly stable. First we observe that by Remark 2.6, by passing to a subsequence if necessary, we may assume $(U_k, V_k) \to (\bar{U}, \bar{V})$, $U_k/|U_k|_{L^\infty(\Omega)} \to 1$, and $V_k/|V_k|_{L^\infty(\Omega)} \to 1$ in $L^\infty(\Omega)$ for some nonnegative constants $\bar{U}$ and $\bar{V}$.

Moreover, we claim that $(\bar{U}, \bar{V})$ satisfies

$$\begin{cases} \bar{m} - \bar{U} - c\bar{V} = 0, \\ \bar{m} - b\bar{U} - \bar{V} = 0. \end{cases} \tag{18}$$

In particular, $(\bar{U}, \bar{V}) = (\frac{1-c}{1-bc}\bar{m}, \frac{1-b}{1-bc}\bar{m})$ and $\bar{U}, \bar{V} > 0$. To see (18), we first divide the equation of U_k by $|U_k|_{L^\infty(\Omega)}$, integrate over Ω, and pass to the limit. The first equation of (18) follows. Similarly, the second equation of (18) can be obtained by dividing the equation of V_k by $|V_k|_{L^\infty(\Omega)}$ and passing to the limit.

Now, denote the principal eigenvalue of (3) with $(U, V) = (U_k, V_k)$ by λ_k and the corresponding eigenfunction by (Φ_k, Ψ_k) (normalized so that $|\Phi_k|_{L^\infty(\Omega)} + |\Psi_k|_{L^\infty(\Omega)} = 1$). Then λ_k is nonpositive.

By Lemma 2.7, λ_k is bounded from below uniformly; therefore, by passing to a subsequence if necessary, we may assume $\lambda_k \to \bar{\lambda} \le 0$. Standard elliptic estimates guarantee that $\Phi_k \to \bar{\Phi}$ and $\Psi_k \to \bar{\Psi}$ for some constants $\bar{\Phi}$ and $\bar{\Psi}$ satisfying $|\bar{\Phi}| + |\bar{\Psi}| = 1$. However, integrating (4) over Ω and passing to limit, we have

$$\begin{cases} (\bar{m} - 2\bar{U} - c\bar{V})\bar{\Phi} - c\bar{U}\bar{\Psi} + \bar{\lambda}\bar{\Phi} = 0, \\ -b\bar{V}\bar{\Phi} + (\bar{m} - b\bar{U} - 2\bar{V})\bar{\Psi} + \bar{\lambda}\bar{\Psi} = 0, \end{cases}$$

which, in view of (18), becomes

$$\begin{cases} \bar U\bar\Phi + c\bar U\bar\Psi = \bar\lambda\bar\Phi,\\ b\bar V\bar\Phi + \bar V\bar\Psi = \bar\lambda\bar\Psi. \end{cases}$$

Since $\bar U, \bar V > 0$ and $|\bar\Phi| + |\bar\Psi| = 1$, $\bar\lambda$ is an eigenvalue of $\left(\begin{smallmatrix} \bar U & c\bar U \\ b\bar V & \bar V \end{smallmatrix}\right)$ and must be positive. This contradicts the fact that $\bar\lambda = \lim_{k\to\infty}\lambda_k \le 0$, and the theorem is proved. ∎

We are now in a position to prove Theorem 1.9.

Proof of Theorem 1.9. Fix $b, c \in (0,1)$. By Proposition 1.10, Corollary 2.12, and the proof of Theorem 1.8, there exists $\delta > 0$ such that whenever $|d_1 - d_2| < \delta$, both $(\theta_{d_1}, 0)$ and $(0, \theta_{d_2})$ are linearly unstable. By the theory of monotone dynamical systems, it suffices to establish the uniqueness of the coexistence steady state. In view of Theorems 1.7 and 1.8, we need only establish the uniqueness when $d_1, d_2 \to d$ for some $d > 0$.

Assume to the contrary that there exist $d_1 = d_{1,k}$ and $d_2 = d_{2,k}$ such that $d_{1,k}, d_{2,k} \to d$ and (1) has more than one coexistence steady state as $k \to \infty$. By compactness, as $k \to \infty$, every coexistence steady state must converge to some steady state of (1) with $d_1 = d_2 = d$. Since every trivial and semitrivial steady state is linearly unstable (Lemma 3.2), by Lemma 2.14 every steady state converges to $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$.

Next, we apply the implicit function theorem to show that for k large, (U_k, V_k) must lie on the unique branch emanating from $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$. We consider $\mathcal{F}$: $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times \mathbf{R}^+ \times \mathbf{R}^+ \to L^p(\Omega) \times L^p(\Omega)$ $(p > N)$ defined by

$$\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} = \begin{pmatrix} d_1\Delta U + U(m - U - cV) \\ d_2\Delta V + V(m - bU - V) \end{pmatrix}. \tag{19}$$

By Lemma 3.2, the principal eigenvalue of $L = D_{(U,V)}\mathcal{F}|_{(U,V,d_1,d_2)=\left(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d, d, d\right)}$ is positive, which implies that every eigenvalue of L has positive real part and L is invertible. Therefore there exists a neighborhood $\mathcal{O}$ containing $(\frac{1-c}{1-bc}\theta_d, \frac{1-b}{1-bc}\theta_d)$ and a function $(U^*, V^*)(d_1, d_2)$ defined for all d_1, d_2 close to d such that if $(U, V) \in \mathcal{O}$ is a steady state of (1) for some d_1, d_2 close to d, then we must have $(U, V) = (U^*, V^*)(d_1, d_2)$. This contradicts the assumption that (1) has more than one coexistence steady state as $d_{1,k} \to d$ and $d_{2,k} \to d$. ∎

4. Proofs of Theorems 1.12 and 1.5.

First we prove the following result.

LEMMA 4.1. *Let $c \in [0,1]$, $b = 1$, and $d_1 < d_2$; then $(\theta_{d_1}, 0)$ is linearly stable, while $(0, \theta_{d_2})$ and $(0,0)$ are linearly unstable. Moreover, $(\theta_{d_1}, 0)$ is globally asymptotically stable.*

Proof. Since $\mu_1(d_2, m - \theta_{d_1}) > \mu_1(d_1, m - \theta_{d_1}) = 0$, $(\theta_{d_1}, 0)$ is linearly stable by Corollary 2.10. On the other hand, by Corollary 2.10 again, $\mu_1(d_1, m - c\theta_{d_2}) < \mu_1(d_2, m - \theta_{d_2}) = 0$, which gives the instability of $(0, \theta_{d_2})$. The instability of $(0,0)$ follows from Remark 2.11.

The global asymptotic stability of $(\theta_{d_1}, 0)$ follows from Lemma 2.16 if we can rule out coexistence. Assume to the contrary that for some $c \in [0,1]$, $b = 1$, and $d_1 < d_2$ there exists a coexistence steady state $(\tilde U, \tilde V)$. This means in particular that $\mu_1(d_1, m - \tilde U - c\tilde V) = \mu_1(d_2, m - \tilde U - \tilde V) = 0$. But that is impossible, since it also holds that $\mu_1(d_1, m - \tilde U - c\tilde V) < \mu_1(d_2, m - \tilde U - \tilde V)$, by Proposition 2.2. ∎

Next, we prove Theorem 1.12.

Proof of Theorem 1.12. Recall the notation $\mathcal{I}_\epsilon = \{(d_1, d_2) : d_1 \in [\epsilon, 1/\epsilon], d_2 \ge d_1 + \epsilon\}$. By Proposition 1.10, for all $b < 1$ sufficiently close to 1, $(\theta_{d_1}, 0)$ is linearly stable for all $(d_1, d_2) \in \mathcal{I}_\epsilon$. By Lemma 2.16, it suffices to show that, given $\epsilon > 0$, for $b < 1$ sufficiently close to 1 and any $(d_1, d_2) \in \mathcal{I}_\epsilon$, (1) has no coexistence steady state.

Assume to the contrary that for $b_k \nearrow 1$, for some $c_k \to c \in [0,1]$, and for $(d_{1,k}, d_{2,k}) \to (d_1, d_2)$ (d_2 is possibly infinite) such that $0 < d_1 < d_2$, (1) has a coexistence steady state (U_k, V_k).

First we consider the case $d_{2,k} \to d_2 < \infty$. By passing to a subsequence if necessary, $U_k \to U'$ and $V_k \to V'$ in $L^\infty(\Omega)$, where (U_k, V_k) is a steady state of (1). By Lemma 4.1, (U_k, V_k) must converge to one of $(\theta_{d_1}, 0)$, $(0, \theta_{d_2})$, and $(0,0)$. This is impossible by Lemma 2.14, since $(\theta_{d_1}, 0)$ is linearly stable while $(0, \theta_{d_2})$ and $(0,0)$ are linearly unstable.

Secondly, we consider the case $d_{2,k} \to \infty$. By passing to a subsequence, $(U_k, V_k) \to (U', \bar{V})$ in $L^\infty(\Omega)$, where

$$d_1 \Delta U' + U'(m - c\bar{V} - U') = 0 \quad \text{in } \Omega \tag{20}$$

with zero Neumann boundary condition on U', and $\bar{V} = \frac{1}{|\Omega|}\int_\Omega (m - U') \geq 0$. Since $\mu_1(d_{1,k}, m - c_k V_k) \to \mu_1(d_1, m - c\bar{V}) < 0$ (as $\int_\Omega (m - c\bar{V}) \geq 0$ and m is not a constant), we have $U' = \vartheta(d_1, m - c\bar{V}) > 0$ by Proposition 2.5(c)(ii). Now dividing (20) by U' and integrating by parts, we have

$$\int_\Omega (m - c\bar{V} - U') = -d_1 \int_\Omega \frac{|\nabla U'|^2}{U'^2} < 0.$$

This contradicts $\bar{V} = \frac{1}{|\Omega|}\int_\Omega (m - U') \geq 0$. $\square$

Next, we prepare for the proof of Theorem 1.5. We first claim that Theorem 1.5 is a consequence of the following:

(S) there exists some $\bar{c} > 0$ such that for any $0 < d_1 \leq d_2$, $b \in [0,1]$, $c \in [0, \bar{c})$, every coexistence steady state, if it exists, is linearly stable.

To see the claim, we first assume that (S) is true. Since $(0, \theta_{d_2})$ is unstable whenever $c \in [0,1)$ and $d_1 \leq d_2$, we have the following three possibilities, depending on the linear stability of $(\theta_{d_1}, 0)$:

(i) $(d_1, d_2) \in \Sigma_b$; i.e., $(\theta_{d_1}, 0)$ is linearly stable while $(0, \theta_{d_2})$ is linearly unstable.
(ii) $(d_1, d_2) \notin \bar{\Sigma}_b$; i.e., both $(\theta_{d_1}, 0)$ and $(0, \theta_{d_2})$ are linearly unstable.
(iii) $(d_1, d_2) \in \partial\Sigma_b$; i.e., $(0, \theta_{d_2})$ is linearly unstable and the principal eigenvalue of $(\theta_{d_1}, 0)$ is zero.

Fix any $c \in [0, \bar{c})$ and $b \in [0,1]$. By Lemma 2.16, $(\theta_{d_1}, 0)$ is globally asymptotically stable in case (i). In case (ii), there exists a unique coexistence steady state that is globally asymptotically stable. We claim that one can rule out the possibility of coexistence in case (iii); hence $(\theta_{d_1}, 0)$ is in fact globally asymptotically stable (Lemma 2.16). Assume to the contrary that there exists one coexistence steady state $(\tilde{U}, \tilde{V})$. By (S), $(\tilde{U}, \tilde{V})$ is linearly stable. By the implicit function theorem, for all $(\tilde{d}_1, \tilde{d}_2)$ that is close to (d_1, d_2), (1) has a coexistence steady state. In particular this is the case for all $(\tilde{d}_1, \tilde{d}_2) \in \Sigma_b$ that is close to $(d_1, d_2) \in \partial\Sigma_b$. (The existence of such $(\tilde{d}_1, \tilde{d}_2)$ follows from (14).) But this contradicts case (i). Hence, it suffices to show (S).

Assume to the contrary that for some sequences of $b_k \in [0,1]$, $b_k \to b_\infty \in [0,1]$, $c_k \searrow 0$, and $d_{1,k} \leq d_{2,k}$, (1) has a coexistence steady state (U_k, V_k) which is not linearly stable. In other words, $\lambda = \lambda_k \leq 0$ in (4). By the Krein–Rutman theorem, we may choose the eigenfunctions of (4) to satisfy $\Phi_k > 0 > \Psi_k$ in $\bar{\Omega}$ and be normalized by $\int_\Omega (\Phi_k^2 + \Psi_k^2) = 1$. For simplicity, we suppress the subscript k in the rest of this proof whenever it does not cause any confusion. We consider the following cases separately:

(A) $d_1 \to 0$, $d_2 \to 0$.
(B) $d_1 \to 0$, $d_2 \to d_{2,\infty} \in (0, \infty]$.
(C) $d_1 \to \infty$, $d_2 \to \infty$.
(D) For some $d \in \mathbf{R}^+$, $d_1 \to d$, $d_2 \to d$.
(E) $d_1 \to d_{1,\infty} \in \mathbf{R}^+$ and $d_2 \to \infty$.
(F) $b_\infty < 1$ and $d_1 \to d_{1,\infty}$, $d_2 \to d_{2,\infty}$ for some $(d_{1,\infty}, d_{2,\infty}) \in \partial\Sigma_{b_\infty}$.
(G) $b_\infty < 1$ and $d_1 \to d_{1,\infty}$, $d_2 \to d_{2,\infty}$ for some $(d_{1,\infty}, d_{2,\infty}) \notin \partial\Sigma_{b_\infty}$.

Note that if $b_\infty = 1$, then by Theorem 1.12, cases (A)–(D) exhaust all the possibilities. But if $b_\infty < 1$, then we have to consider in addition cases (E), (F), and (G).

First, we further divide the most delicate case (A) into (A_1): $d_1, d_2, b, c \to 0$, and (A_2): $d_1, d_2, c \to 0$ and $b \to b_\infty \in (0, 1]$.

We first prepare with two lemmas.

LEMMA 4.2. *Assume (A_2) and that the principal eigenvalue λ of (4) is nonpositive. Then $\Phi < -\Psi$ in Ω for all k sufficiently large, where we choose the principal eigenfunction (Φ, Ψ) of (4) so that $\Phi > 0 > \Psi$.*

Proof. Since $U \to m$ in $L^\infty(\Omega)$ by Proposition 2.5(a), we see that eventually $(m - 2U - cV + \lambda) < 0$ in $\bar{\Omega}$. By the strong maximum principle (as in the proof of Proposition 2.4), it is sufficient to show

$$\begin{cases} d_1\Delta(\Phi + \Psi) + (m - 2U - cV + \lambda)(\Phi + \Psi) > 0 & \text{in } \Omega, \\ \partial_\nu(\Phi + \Psi) \le 0 & \text{on } \partial\Omega, \end{cases}$$

which, in view of the first equation of (4), is equivalent to

$$\begin{cases} d_1\Delta(-\Psi) + (m - 2U - cV + \lambda)(-\Psi) < cU\Psi & \text{in } \Omega, \\ \partial_\nu(-\Psi) \ge 0 & \text{on } \partial\Omega. \end{cases}$$

The boundary condition is met immediately. Now

$$\begin{aligned} & d_1\Delta(-\Psi) + (m - 2U - cV + \lambda)(-\Psi) - cU\Psi \\ =& \frac{d_1}{d_2}\left[-bV\Phi + (m - bU - 2V)\Psi + \lambda\Psi\right] - (m - 2U - cV + \lambda)\Psi - cU\Psi \\ =& -\frac{d_1}{d_2}bV\Phi - \lambda\left(1 - \frac{d_1}{d_2}\right)\Psi + \left[\frac{d_1}{d_2}(m - bU - 2V) - (m - 2U - cV) - cU\right]\Psi \\ <& \, 0, \end{aligned}$$

since $U \to m$, $V \to (1-b_\infty)m$, $0 < d_1 \le d_2$, and $c \searrow 0$, the terms in the square bracket converge to $[1 - \frac{d_1}{d_2}(1 - b_\infty)]m > 0$ in $\bar{\Omega}$, while the first two terms are nonpositive. ∎

LEMMA 4.3. *Assuming that one of (B)–(G) holds, then passing to a subsequence if necessary, there exist $U_\infty, \tilde{V}_\infty > 0$ in $\bar{\Omega}$ such that*

$$U_k \to U_\infty \quad \textit{and} \quad \tilde{V}_k := V_k/\|V_k\|_{L^\infty(\Omega)} \to \tilde{V}_\infty \quad \textit{in } L^\infty(\Omega). \tag{21}$$

Proof. Case (B): By Proposition 2.5(a), $U \to m$ in $L^\infty(\Omega)$. On the one hand, V satisfies the single equation

$$d_2\Delta V + (m - bU - V)V = 0 \text{ in } \Omega \qquad \text{and} \qquad \partial_\nu V = 0 \text{ on } \partial\Omega.$$

Hence, by Proposition 2.5,

$$V \to \begin{cases} (1 - b_\infty)\theta_{\frac{d_{2,\infty}}{1-b_\infty}} & \text{if } d_{2,\infty} < \infty, \\ \frac{1-b_\infty}{|\Omega|}\int_\Omega m & \text{if } d_{2,\infty} = \infty, \end{cases}$$

in $L^\infty(\Omega)$. On the other hand, $\tilde V = V/\|V\|_{L^\infty(\Omega)}$ satisfies

$$\begin{cases} d_2\Delta\tilde V + (m - bU - V)\tilde V = 0 & \text{in } \Omega,\\ \partial_\nu \tilde V = 0 \quad \text{on } \partial\Omega \quad \text{and} & \|\tilde V\|_{L^\infty(\Omega)} = 1,\end{cases}$$

and hence by Proposition 2.5

$$\tilde V \to \begin{cases} 1 & \text{if } b_\infty = 1 \text{ or } d_{2,\infty} = \infty,\\ \theta_{\frac{d_{2,\infty}}{1-b_\infty}}/|\theta_{\frac{d_{2,\infty}}{1-b_\infty}}|_{L^\infty(\Omega)} & \text{if } b_\infty < 1 \text{ and } d_{2,\infty} < \infty,\end{cases}$$

in $L^\infty(\Omega)$. This shows that (21) holds in case (B).

Case (C): We claim that $U \to \bar m := \frac{1}{|\Omega|}\int_\Omega m$ and $\tilde V \to 1$ in $L^\infty(\Omega)$. That $U \to \bar m$ follows from Proposition 2.5(b). Similarly, one can see that $\tilde V := V/\|V\|_{L^\infty(\Omega)} \to 1$ in $L^\infty(\Omega)$ by Remark 2.6.

Case (D): First we see that $U \to \theta_d$ in $L^\infty(\Omega)$ by Proposition 2.5(c)(ii). Next, we claim that

$$V \to (1 - b_\infty)\theta_d \quad \text{and} \quad \tilde V := \frac{V}{\|V\|_{L^\infty(\Omega)}} \to \frac{\theta_d}{\|\theta_d\|_{L^\infty(\Omega)}} \quad \text{in } W^{2,p}(\Omega). \tag{22}$$

On the one hand, if $b_\infty < 1$, then $m - bU \to m - b_\infty\theta_d$ with $\mu_1(d, m - b_\infty\theta_d) < \mu_1(d, m - \theta_d) = 0$. Hence by Proposition 2.5(c)(ii), (22) is proved. On the other hand, if $b_\infty = 1$, then $m - bU \to m - \theta_d$ with $\mu_1(d, m - \theta_d) = 0$, and (22) follows from Proposition 2.5(c)(i), since $\theta_d/\|\theta_d\|_{L^\infty(\Omega)}$ is the (normalized) first eigenfunction of $\mu_1(d, m - \theta_d)$.

Case (E): We claim that $U \to \theta_{d_{1,\infty}}$ and $\tilde V \to 1$ in $L^\infty(\Omega)$. That $U \to \theta_{d_{1,\infty}}$ follows from Proposition 2.5(c)(ii), while $\tilde V \to 1$ follows from Remark 2.6.

Case (F): We claim that $U \to \theta_{d_{1,\infty}}$, $V \to 0$, and $\tilde V \to \varphi_1$ in $L^\infty(\Omega)$, where $\varphi_1 > 0$ is the normalized principal eigenfunction corresponding to $\mu_1(d_{2,\infty}, m - b\theta_{d_{1,\infty}}) = 0$. Now $U \to \theta_{d_{1,\infty}}$ by Proposition 2.5(c)(ii). Since $(d_{1,\infty}, d_{2,\infty}) \in \partial\Sigma_{b_\infty}$, $\mu_1(d_{2,\infty}, m - b_\infty\theta_{d_{1,\infty}}) = 0$ (by (14)), by Proposition 2.5(c)(i), $V \to 0$ and $\tilde V \to \varphi_1$ in $L^\infty(\Omega)$, where φ_1 is the normalized first eigenfunction of $\mu_1(d_{2,\infty}, m - b_\infty\theta_{d_{1,\infty}})$.

Case (G): $\mu_1(d_{2,\infty}, m - b_\infty\theta_{d_{1,\infty}}) \neq 0$. Then by Proposition 2.5(c), $\mu_1(d_{2,\infty}, m - b_\infty\theta_{d_{1,\infty}}) < 0$, $U \to \theta_{d_{1,\infty}}$, and $V \to V'$ in $L^\infty(\Omega)$, where $V' = \vartheta(d_{2,\infty}, m - b_\infty\theta_{d_1,\infty})$, following the notation of section 2. Hence $\tilde V \to V'/\|V'\|_{L^\infty(\Omega)}$, which is positive in $\bar\Omega$. ∎

Finally, we prove (S), which implies Theorem 1.5.

Proof of (S). For (A_1), we are going to show that every coexistence steady state is linearly stable for d_1, d_2, b, c sufficiently small, which is (S). Although this result $(b, c \to 0)$ was not covered in [HLM2], one can observe that it follows from the same arguments. For the sake of completeness, we include a simple proof here for the special case we need.

Multiplying the first equation in (3) by U and the second equation by V, and integrating by parts, we have

$$\begin{cases} \int_\Omega(-U^2\Phi - cU^2\Psi + \lambda U\Phi) = 0,\\ \int_\Omega(-bV^2\Phi - V^2\Psi + \lambda V\Psi) = 0.\end{cases}$$

Subtracting the second equation from the first gives

$$\int_\Omega(-U^2 + bV^2)\Phi + \int_\Omega(V^2 - cU^2)\Psi + \lambda\int_\Omega(U\Phi - V\Psi) = 0.$$

Now $b, c \to 0$, and $U \to m$ and $V \to m$ in $L^\infty(\Omega)$ by Proposition 2.5(i). So the first two integrals are negative. Since $\int_\Omega (U\Phi - V\Psi) > 0$ (because $\Phi > 0 > \Psi$), it follows that $\lambda > 0$. This contradiction proves (S) in case (A_1).

Before we treat the remaining cases, multiply the second equation of (4) by Ψ and integrate by parts. Then we have

$$\int_\Omega \left[-d_2|\nabla\Psi|^2 + (m - bU - V)\Psi^2\right] - \int_\Omega (bV\Phi\Psi + V\Psi^2) + \lambda \int_\Omega \Psi^2 = 0. \tag{23}$$

Since $\mu_1(d_2, m - bU - V) = 0$ by the equation for V, the first integral involving the square bracket is nonpositive by variational characterization as $\mu_1(d_2, m-bU-V) = 0$. The last term $\lambda \int_\Omega \Psi^2 \le 0$ since $\lambda \le 0$ by assumption. We see that a contradiction to (23) is in order if one can show

$$\int_\Omega \left(b\frac{V}{\|V\|_{L^\infty(\Omega)}}\Phi\Psi + \frac{V}{\|V\|_{L^\infty(\Omega)}}\Psi^2 \right) > 0. \tag{24}$$

Now we take up case (A_2). We are going to show (24), which gives the contradiction. It is clear that (24) is a consequence of $\Phi > 0 > \Psi$ and Lemma 4.2. We thus arrive at a contradiction in case (A_2).

To show (S) for the remaining cases (B) to (G), we multiply the first equation of (4) by Φ and integrate by parts:

$$\int_\Omega \left[-d_1|\nabla\Phi|^2 + (m - U - cV + \lambda)\Phi^2\right] = \int_\Omega U\Phi^2 + \int_\Omega (cU\Phi\Psi).$$

Now by the equation for U, we see that $\mu_1(d_1, m - U - cV) = 0$. As it also holds that $\lambda \le 0$, the left-hand side of the preceding equation is nonpositive. Hence,

$$\int_\Omega (U\Phi^2) \le c\int_\Omega (U\Phi|\Psi|) \le \frac{1}{2}\int_\Omega (U\Phi^2) + \frac{c^2}{2}\int_\Omega (U\Psi^2),$$

by Young's inequality. Cancelling, we have

$$\int_\Omega (U\Phi^2) \le c^2 \int_\Omega (U\Psi^2). \tag{25}$$

By Lemma 4.3, there exist positive constants C_1, C_2 independent of k such that

$$C_1 \le U \le C_2 \quad \text{and} \quad C_1 \le \frac{V}{\|V\|_{L^\infty(\Omega)}} \le C_2 \quad \text{in } \bar{\Omega}. \tag{26}$$

And by Young's inequality, we have for k sufficiently large (hence c small)

$$\int_\Omega \left(b\frac{V}{\|V\|_{L^\infty(\Omega)}}\Phi|\Psi| \right) \le \frac{1}{2}\int_\Omega \frac{V}{\|V\|_{L^\infty(\Omega)}}\Psi^2 + \frac{b^2}{2}\int_\Omega \frac{V}{\|V\|_{L^\infty(\Omega)}}\Phi^2 < \int_\Omega \frac{V}{\|V\|_{L^\infty(\Omega)}}\Psi^2,$$

where the strict inequality follows from (25) and (26). Therefore (24) is proved, and we have a contradiction. This finishes the proof of (S). ∎

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