# MSLC – Math 151 SP 09
# Exam 1 Review Solutions

**PROBLEMS**

1. What is the domain of tan($x$)?

*All real numbers except where cos(x)=0.*

*That is, all real numbers except* $x = \frac{\pi}{2} + k\pi$ *where k is an integer.*

2. Use the laws of logarithms to rewrite the expression $\ln\left(x^5\sqrt{\frac{(y+7)^3}{z^9}}\right)$ in a form with no exponents.

$$\ln\left(x^5\sqrt{\frac{(y+7)^3}{z^9}}\right)$$

$$\ln\left(x^5\right) + \ln\left(\sqrt{\frac{(y+7)^3}{z^9}}\right)$$

$$5\ln(x) + \tfrac{1}{2}\ln\left(\frac{(y+7)^3}{z^9}\right)$$

$$5\ln(x) + \tfrac{1}{2}\left[\ln\left((y+7)^3\right) - \ln\left(z^9\right)\right]$$

$$5\ln(x) + \tfrac{1}{2}\left[3\ln(y+7) - 9\ln(z)\right]$$

3. Solve $e^{5x+1} = 17$ for x

$$\ln(e^{5x+1}) = \ln(17)$$

$$5x + 1 = \ln(17)$$

$$5x = \ln(17) - 1$$

$$x = \frac{\ln(17) - 1}{5}$$

4. $f(x) = \frac{8x-5}{12x+1}$. Find $f^{-1}(x)$.

$$y = \frac{8x-5}{12x+1}$$

$$y(12x+1) = (8x-5)$$

$$12xy + y = 8x - 5$$

$$12xy - 8x = -5 - y$$

$$x(12y-8) = -5 - y$$

$$x = \frac{-5-y}{12y-8} = \frac{5+y}{8-12y}$$

5. Find the domains of the following functions:

a. $f(x)=\sqrt{\dfrac{x-1}{x+1}}$

*We can't divide by 0 so* $x \neq -1$. *We can't take the square root of a negative number so*

$\dfrac{x-1}{x+1} \geq 0$

$(x-1 \geq 0 \quad \text{and} \quad x+1 \geq 0) \quad OR \quad (x-1 \leq 0 \quad \text{and} \quad x+1 \leq 0)$.

$(x \geq 1 \quad \text{and} \quad x \geq -1) \quad OR \quad (x \leq 1 \quad \text{and} \quad x \leq -1)$

$x \geq 1 \quad OR \quad x \leq -1$

*So we get* $(-\infty,-1) \cup [1,\infty)$.

b. $f(x)=\ln\left(x^2-4\right)$

*We can't take the log of a negative number or zero so*

$x^2-4>0$

$x^2>4$

$2<x \quad or \quad x<-2$

$(-\infty,-2) \cup (2,\infty)$

c. $f(x)=\sqrt{1-\sqrt{9-x^2}}$

*We can't take the square root of a negative number so:*

$1-\sqrt{9-x^2} \geq 0 \quad and \quad 9-x^2 \geq 0$

$1 \geq \sqrt{9-x^2} \quad and \quad 9 \geq x^2$

$1 \geq 9-x^2 \quad and \quad -3 \leq x \leq 3$

$-8 \geq -x^2 \quad and \quad -3 \leq x \leq 3$

$8 \leq x^2 \quad and \quad -3 \leq x \leq 3$

$\sqrt{8} \leq x \quad or \quad x \leq -\sqrt{8} \quad and \quad -3 \leq x \leq 3$

$-3 \leq x \leq -\sqrt{8} \quad or \quad \sqrt{8} \leq x \leq 3$

$[-3,-\sqrt{8}] \cup [\sqrt{8},3]$

6. Given $f(x)=\sqrt{x}$ and $g(x)=x^2-5x+6$ find the functions:

a) $h(x)=g\left[f(x)\right]=\left(\sqrt{x}\right)^2-5\sqrt{x}+6=x-5\sqrt{x}+6$

b) $k(x)=f\left[g(x)\right]=\sqrt{x^2-5x+6}$

c) $l(x)=f\left[f(x)\right]=\sqrt{\sqrt{x}}=\left(x^{1/2}\right)^{1/2}=x^{1/4}=\sqrt[4]{x}$

7. Given $f(x)=\dfrac{6}{5-\sqrt{x-1}}$, find

a. $f(12)=\dfrac{6}{5-\sqrt{12-1}}=\dfrac{6}{5-\sqrt{11}}$

b. domain of *f*

*We can't take the square root of a negative so*

$x-1\geq 0$

$x\geq 1$

*We can't divide by 0 so*

$5-\sqrt{x-1}\neq 0$

$5\neq \sqrt{x-1}$

$25\neq x-1$

$x\neq 26$

*Which gives us* $[1,26)\cup(26,\infty)$

c. all asymptotes of *f*

$x=26$

d. $f^{-1}(x)$

$$y=\frac{6}{5-\sqrt{x-1}}$$

$$y(5-\sqrt{x-1})=6$$

$$5-\sqrt{x-1}=\frac{6}{y}$$

$$-\sqrt{x-1}=\frac{6}{y}-5$$

$$\sqrt{x-1}=-\frac{6}{y}+5$$

$$x-1=\left(-\frac{6}{y}+5\right)^2$$

$$x=\left(-\frac{6}{y}+5\right)^2+1$$

8. Find the following limits:

a. $\lim\limits_{x\to 7^+}\dfrac{x^2-4x-21}{x^2-2x-35}=\lim\limits_{x\to 7^+}\dfrac{(x-7)(x+3)}{(x-7)(x+5)}=\lim\limits_{x\to 7^+}\dfrac{\cancel{(x-7)}(x+3)}{\cancel{(x-7)}(x+5)}=\dfrac{10}{12}=\dfrac{5}{6}$

b.

$$\lim_{x\to 0^+} \frac{1}{1-e^{\sqrt{x}}} = -\infty$$

$\sqrt{x}$ is going to be number very close to 0 and positive as $x \to 0^+$.

$e^{\sqrt{x}}$ is going to be a number very close to $e^0=1$, but bigger then 1 as $x \to 0^+$.

Therefore, $1-e^{\sqrt{x}}$ is going to approach 0 as $x \to 0^+$ but be less than 0.

Thus, $\dfrac{1}{\text{(a number slightly less then 0)}} = -\infty$

c. $$\lim_{x\to 6} \frac{|6-x|}{6-x} = \lim_{x\to 6} \begin{cases} \dfrac{6-x}{6-x} & 6-x \ge 0 \\ \dfrac{-(6-x)}{6-x} & 6-x < 0 \end{cases} = \lim_{x\to 6} \begin{cases} 1 & 6 \ge x \\ -1 & 6 < x \end{cases} = \begin{cases} \lim\limits_{x\to 6^-}(1) = 1 \\ \lim\limits_{x\to 6^+}(-1) = -1 \end{cases} = DNE$$

d. $$\lim_{x\to 0} \frac{\cos^2(x)-1}{\cos(x)-1} = \lim_{x\to 0} \frac{(\cos(x)+1)(\cos(x)-1)}{\cos(x)-1} = \lim_{x\to 0} \frac{(\cos(x)+1)\,\cancel{(\cos(x)-1)}}{\cancel{(\cos(x)-1)}} = \cos(0)+1 = 2$$

e.

$$\lim_{x\to 1^-} e^{\frac{1}{x-1}} = 0$$

$x-1$ gets close to 0, but is less then 0 as $x \to 1^-$. $\frac{1}{x-1}$ goes to $-\infty$ as $x \to 1^-$. Thus, $e^{\frac{1}{x-1}}$ goes to $e^{-\infty} = \dfrac{1}{e^{\infty}} = 0$ as $x \to 1^-$.

f. $$\lim_{t\to -2} \frac{7-t}{t^2-t-6} \approx \frac{9}{0} \Rightarrow \lim_{t\to -2} \frac{7-t}{t^2-t-6} = \begin{cases} \lim\limits_{t\to -2^+} \dfrac{7-t}{t^2-t-6} \\ \lim\limits_{t\to -2^-} \dfrac{7-t}{t^2-t-6} \end{cases} = \begin{cases} \lim\limits_{t\to -2^+} \dfrac{9}{0^-} = -\infty \\ \lim\limits_{t\to -2^-} \dfrac{9}{0^+} = \infty \end{cases} = DNE$$

g. $$\lim_{x\to -\infty} \frac{2x^3-3x^2+4}{x^3+3x^2-1} = \lim_{x\to -\infty} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \frac{(2x^3-3x^2+4)}{(x^3+3x^2-1)} = \lim_{x\to -\infty} \frac{\frac{2x^3}{x^3} - \frac{3x^2}{x^3} + \frac{4}{x^3}}{\frac{x^3}{x^3} + \frac{3x^2}{x^3} - \frac{1}{x^3}} = \lim_{x\to -\infty} \frac{2 - \frac{3}{x} + \frac{4}{x^3}}{1 + \frac{3}{x} - \frac{1}{x^3}} = \frac{2-0+0}{1+0-0} = 2$$

h. $\lim\limits_{\theta\to\infty} \cos\theta = DNE$ *because cosine oscillates between -1 and 1 and doesn't approach just one number*

i. $\lim_{x\to-\infty}\frac{x^3}{\sqrt{4x^6-7}}=\lim_{x\to-\infty}\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\frac{x^3}{\sqrt{4x^6-7}}=\lim_{x\to-\infty}\frac{1}{-\sqrt{\frac{4x^6-7}{x^6}}}=\lim_{x\to-\infty}\frac{1}{-\sqrt{4-\frac{7}{x^6}}}=\frac{1}{-\sqrt{4+0}}=-\frac{1}{2}$

j. $\lim_{x\to0}x^2\sin\left(\frac{1}{x}\right)=?$

$-1\le\sin(\frac{1}{x})\le1$

$-x^2\le x^2\sin\left(\frac{1}{x}\right)\le x^2$

$\lim_{x\to0}(-x^2)\le\lim_{x\to0}(x^2\sin\left(\frac{1}{x}\right))\le\lim_{x\to0}(x^2)$

$0\le\lim_{x\to0}(x^2\sin\left(\frac{1}{x}\right))\le0$

$\lim_{x\to0}(x^2\sin\left(\frac{1}{x}\right))=0$

*by the Squeeze Theorem*

k. $\lim_{x\to3}\frac{2x-\sqrt{3x^2+2x+3}}{x-3}=\lim_{x\to3}\frac{\left(2x-\sqrt{3x^2+2x+3}\right)\left(2x+\sqrt{3x^2+2x+3}\right)}{(x-3)\left(2x+\sqrt{3x^2+2x+3}\right)}$

$=\lim_{x\to3}\frac{\left(4x^2-\left(3x^2+2x+3\right)\right)}{(x-3)\left(2x+\sqrt{3x^2+2x+3}\right)}=\lim_{x\to3}\frac{\left(4x^2-3x^2-2x-3\right)}{(x-3)\left(2x+\sqrt{3x^2+2x+3}\right)}$

$=\lim_{x\to3}\frac{\left(x^2-2x-3\right)}{(x-3)\left(2x+\sqrt{3x^2+2x+3}\right)}=\lim_{x\to3}\frac{(x-3)(x+1)}{(x-3)\left(2x+\sqrt{3x^2+2x+3}\right)}$

$=\lim_{x\to3}\frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}\left(2x+\sqrt{3x^2+2x+3}\right)}=\frac{4}{12}=\frac{1}{3}$

l.

$\lim_{x\to\infty}2x-\sqrt{4x^2-x}=\lim_{x\to\infty}\frac{\left(2x-\sqrt{4x^2-x}\right)\left(2x+\sqrt{4x^2-x}\right)}{\left(2x+\sqrt{4x^2-x}\right)}=\lim_{x\to\infty}\frac{\left(4x^2-4x^2+x\right)}{\left(2x+\sqrt{4x^2-x}\right)}$

$=\lim_{x\to\infty}\frac{x}{\left(2x+\sqrt{4x^2-x}\right)}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x}}\frac{x}{\left(2x+\sqrt{4x^2-x}\right)}=\lim_{x\to\infty}\frac{1}{\left(2+\sqrt{\frac{4x^2-x}{x^2}}\right)}=\lim_{x\to\infty}\frac{1}{\left(2+\sqrt{4-\frac{1}{x}}\right)}=\frac{1}{2+\sqrt{4}}=\frac{1}{4}$

9. Let $f(x)=\dfrac{x^3-2x^2-5x+6}{x^2+x-2}$. Find all discontinuities. Find the limits at the places where $f(x)$ is discontinuous.

*Discontinuities are where the bottom equals 0.*

$$x^2+x-2=0$$
$$(x-1)(x+2)=0$$
$$x=1 \quad or \quad x=-2$$

*Limits at the discontinuities:*

$$\lim_{x\to 1}\frac{x^3-2x^2-5x+6}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)(x^2-x+6)}{(x-1)(x+2)}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2-x+6)}{\cancel{(x-1)}(x+2)}=\frac{6}{3}=2$$

$$\lim_{x\to -2}\frac{x^3-2x^2-5x+6}{x^2+x-2}=\lim_{x\to -2}\frac{(x-1)(x^2-x+6)}{(x-1)(x+2)}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2-x+6)}{\cancel{(x-1)}(x+2)}=\frac{12}{0}\Rightarrow$$

$$\lim_{x\to -2}\frac{\cancel{(x-1)}(x^2-x+6)}{\cancel{(x-1)}(x+2)}=\begin{cases}\lim\limits_{x\to -2^+}\dfrac{(x^2-x+6)}{(x+2)}\\[2ex] \lim\limits_{x\to -2^-}\dfrac{(x^2-x+6)}{(x+2)}\end{cases}=\begin{cases}\dfrac{12}{0^+}=\infty\\[2ex] \dfrac{12}{0^-}=-\infty\end{cases}=DNE$$

10. Let $f(x)=\begin{cases}\dfrac{12}{x-3} & x<-1\\ x+2 & -1\le x\le 3\\ a-x & 3<x\end{cases}$ Find the following:

a. $f(-1)=(-1)+2=1$

b. $f(3)=3+2=5$

c. $\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}(x+2)=3$

d. $\lim\limits_{x\to -1}f(x)=\begin{cases}\lim\limits_{x\to -1^+}(x+2)=1\\[1ex] \lim\limits_{x\to -1^-}\left(\dfrac{12}{x-3}\right)=-3\end{cases}=DNE$

e. What value must $a$ have in order for $f(x)$ to be continuous at $x=3$?

$$f(x)=\begin{cases}\dfrac{12}{x-3} & x<-1\\ x+2 & -1\le x\le 3\\ a-x & 3<x\end{cases}$$

$$\lim_{x\to 3^-}f(x)=\lim_{x\to 3^+}f(x)$$
$$\lim_{x\to 3^-}(x+2)=\lim_{x\to 3^+}(a-x)$$
$$5=a-3$$
$$a=8$$

11. Let $f(x)=\frac{1}{x+1}$. Find $\lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2}$.

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{\frac{1}{x+1}-\frac{1}{3}}{x-2}=\lim_{x\to 2}\frac{\frac{3-(x+1)}{3(x+1)}}{x-2}=\lim_{x\to 2}\frac{2-x}{3(x+1)(x-2)}$$

$$=\lim_{x\to 2}\frac{-\cancel{(x-2)}}{3(x+1)\cancel{(x-2)}}=\frac{-1}{9}$$

12. Let $f(x)=x^2-3x+2$. Using $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$ find $f'(1)$. Give the equation of the tangent line at $x=1$.

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\left[(1+h)^2-3(1+h)+2\right]-0}{h}=\lim_{h\to 0}\frac{(1+2h+h^2)-3-3h+2}{h}=\lim_{h\to 0}\frac{h^2-h}{h}=\lim_{h\to 0}h-1=-1$$

*So the slope of the tangent line is -1. The tangent line goes through the point (1, f(1))=(1,0).*
*Thus, we get:* $y-0=-1(x-(1))$

$y=-x+1$

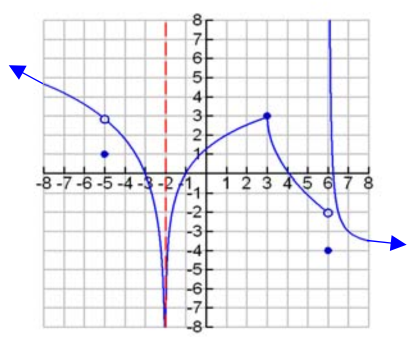
13. Let $f(x)=\sqrt{x-4}$. Using $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$ find $f'(6)$.

$$f'(x)=\lim_{h\to 0}\frac{f(6+h)-f(6)}{h}=\lim_{h\to 0}\frac{\sqrt{(6+h)-4}-\sqrt{6-4}}{h}=\lim_{h\to 0}\frac{\sqrt{h+2}-\sqrt{2}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{h+2}-\sqrt{2}\right)\left(\sqrt{h+2}+\sqrt{2}\right)}{h\left(\sqrt{h+2}+\sqrt{2}\right)}=\lim_{h\to 0}\frac{((h+2)-(2))}{h\left(\sqrt{h+2}+\sqrt{2}\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{h+2}+\sqrt{2}\right)}=\frac{1}{2\sqrt{2}}$$

$f(x)$




14. Let $f(x)$ be the graph to the right. Find:

a) $\lim\limits_{x\to\infty}f(x)=-4$ b) $\lim\limits_{x\to-\infty}f(x)=\infty$ c) $\lim\limits_{x\to-5}f(x)=3$

d) $\lim\limits_{x\to-2^-}f(x)=-\infty$ e) $\lim\limits_{x\to-2^+}f(x)=-\infty$ f) $\lim\limits_{x\to 3}f(x)=3$

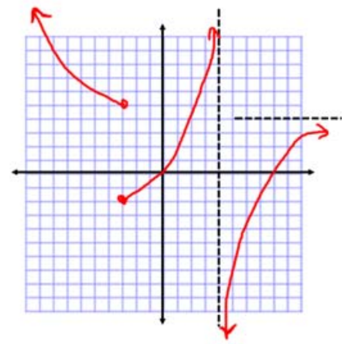
g) $\lim\limits_{x\to 6^-}f(x)=-2$ h) $\lim\limits_{x\to 6^+}f(x)=\infty$ i) $f(-5)=1$

j) $f(-2)=undefined$ k) $f(3)=3$ l) $f(6)=-4$

15. Sketch a graph with the following properties:

$\lim_{x\to\infty} f(x) = 4,$ $\lim_{x\to-\infty} f(x) = +\infty,$ $\lim_{x\to 4^-} f(x) = \infty$

$\lim_{x\to 4^+} f(x) = -\infty,$ $\lim_{x\to -3^-} f(x) = 5,$ $\lim_{x\to -3^+} f(x) = -2$



16. Use the Intermediate Value Theorem to prove that there exists a positive number $c$ such that $c^2 = 2$. (Hint: Let $f(x) = x^2$. Choose your endpoints to be $x = 1$ and $x = 2$.) Clearly state each condition of the Intermediate Value Theorem and why this set-up satisfies the condition.

*The intermediate value theorem says: If $f(x)$ is continuous on the closed interval between the two points a and b on the x-axis, if N is a number that is between $f(a)$ and $f(b)$ on the y-axis, and if $f(a) \neq f(b)$, then there is number c between a and b on the x- axis such that $f(c) = N$*

*Let $f(x) = x^2$. Then f(x) is continuous for all real numbers, particularly from* [1,2].

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

So $f(1) \neq f(2)$

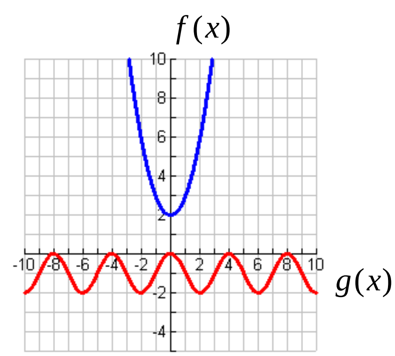
*and* $f(1) < 2 < f(2)$

*So, by the Intermediate Value Theorem, there is some number c such that $f(c)=c^2=2$.*

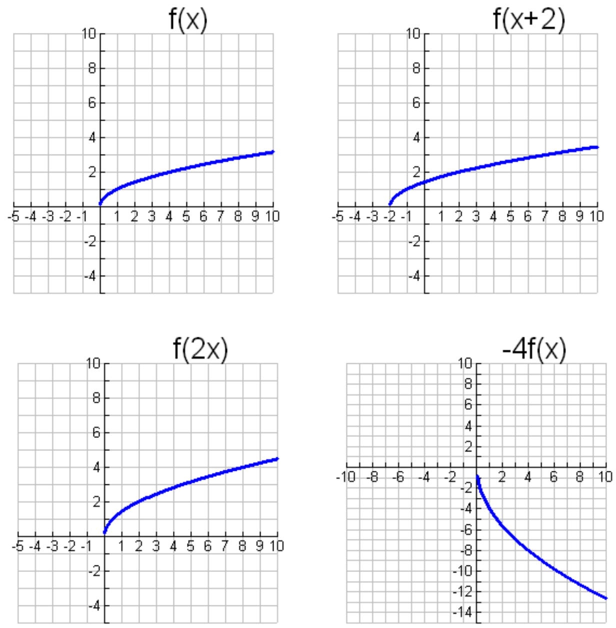
17. Let $f(x)$ and $g(x)$ be the functions given in the graph to the right. Find the following values:

a) $g(f(2)) = g(6) = -2$  b) $f(g(2)) = f(-2) = 6$

c) $g(g(6)) = g(-2) = -2$



18. Sketch the graph of $f(x)=\sqrt{x}$. Now sketch $f(x+2)$, $f(2x)$, and $-4f(x)$.




19. Use the derivative rules to find the derivatives of the following functions:

(a)

$$f(x)=x^{24}+7x^3$$

$$f'(x)=24x^{24-1}+7(3)x^{3-1}=24x^{23}+21x^2$$

(b)

$$g(x)=56e^x-8x^4$$

$$g'(x)=56e^x-8(4)x^{4-1}=56e^x-32x^3$$

(c)

$$h(x)=\frac{5}{x^3}+\sqrt[4]{x}=5x^{-3}+x^{1/4}$$

$$h'(x)=5(-3)x^{-3-1}+(1/4)x^{1/4-1}=-15x^{-4}+\tfrac{1}{4}x^{-3/4}$$

## Quick Summary of the Material

We strongly suggest you make your own study guide using your book and class notes. Here are a few things you may want to include:

1. A review of all the basic functions and their properties found in section 1.2 of your book.

2. The properties of exponential and logarithmic functions.

3. Odd/Even Functions:
   - An odd function is one that is symmetric about the origin. ex. $y = x$ , $y = x^3$ , $y = \sin x$
   - An even function is one that is symmetric about the y-axis. ex. $y = c$ , $y = x^2$ , $y = \cos x$

4. Domain & Range: Domain is the "input" into a function. Range is the "output" of a function.

5. Limits:
   - Intuitive definition – the limit L is the number the function approaches as x approaches the value a.
   - Limit Laws – Look in Section 2.3 for all of the limit laws.

6. The three essential parts of Continuity:
   A function $f(x)$ is continuous at $x = a$ ($a$ need not be in the domain of $f(x)$) if
   1. $f(a)$ exists as a finite value
   2. $\lim_{x \to a} f(x)$ exists (which means $\begin{cases} \lim_{x \to a^-} f(x) \to L \\ \lim_{x \to a^+} f(x) \to M \end{cases}$ are both finite values and $L = M$ )
   3. $f(a) = \lim_{x \to a} f(x)$

   If ALL three conditions hold, then $f(x)$ is continuous at $x = a$ .

7. Average Rate of Change/Instantaneous Rate of Change:
   - The "average rate of change" is the rate of change of a function (or slope of the secant line) between two points $x = a$ and $x = b$ on a continuous function. This is calculated by $\frac{f(b) - f(a)}{b - a}$ .
   - The instantaneous rate of change is the rate of change (or slope of the tangent line) that occurs at a single point, say $x = a$ , of a function. It is calculated by $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided this limit exists. This is also called the derivative of the function at $x = a$ , and we denote it $f'(a)$ .

8. Intermediate Value Theorem:
If $f(x)$ is continuous on the closed interval between the two points *a* and *b* on the *x*-axis, if *N* is a number that is between $f(a)$ and $f(b)$ on the *y*-axis, and if $f(a) \neq f(b)$ , then there is number *c* between *a* and *b* on the *x*- axis such that $f(c) = N$ .

9. This chart of special trig values:

| Θ | 0 (0) | π/6 (30) | π/4 (45) | π/3 (60) | π/2 (90) |
|---|---|---|---|---|---|
| Sin Θ | 0 | 1/2 | √2/2 | √3/2 | 1 |
| Cos Θ | 1 | √3/2 | √2/2 | 1/2 | 0 |
| Tan Θ | 0 | √3/3 | 1 | √3 | UND |

10. The differentiation rules in section 3.1.

11. Anything the book has in a red box.