MSLC – Math 151
Exam 2 Review Solutions

Disclaimer: This review should NOT be used as your only guide for what to study.

1. Compute the following derivatives using the **differentiation rules**:

a.)

$$\frac{d}{d\theta}\left(\frac{\cot(4\theta)}{\theta^3+2}\right)=\frac{\frac{d}{d\theta}\left(\cot(4\theta)\right)\left(\theta^3+2\right)-\left(\cot(4\theta)\right)\frac{d}{d\theta}\left(\theta^3+2\right)}{\left(\theta^3+2\right)^2}$$

$$=\frac{\left(-\csc^2(4\theta)\cdot 4\right)\left(\theta^3+2\right)-\left(\cot(4\theta)\right)\left(3\theta^2\right)}{\left(\theta^3+2\right)^2}$$

b.) $\frac{d}{dx}\left(\cos^3(5x)\right)=3\cos^2(5x)\cdot\frac{d}{dx}\left(\cos(5x)\right)=3\cos^2(5x)\cdot\left(-\sin(5x)\cdot 5\right)$

c.)

$$\frac{d}{d\theta}\left(\theta^{\sin(\theta)}\right)$$

$$y=\theta^{\sin(\theta)}$$

$$\ln(y)=\ln(\theta^{\sin(\theta)})$$

$$\ln(y)=\sin(\theta)\cdot\ln(\theta)$$

$$\frac{1}{y}\cdot\frac{dy}{d\theta}=\frac{d}{d\theta}\left(\sin(\theta)\right)\cdot\left(\ln(\theta)\right)+\left(\sin(\theta)\right)\frac{d}{d\theta}\left(\ln(\theta)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{d\theta}=\left(\cos(\theta)\right)\cdot\left(\ln(\theta)\right)+\left(\sin(\theta)\right)\left(\frac{1}{\theta}\right)$$

$$\frac{dy}{d\theta}=\left[\left(\cos(\theta)\right)\cdot\left(\ln(\theta)\right)+\left(\sin(\theta)\right)\left(\frac{1}{\theta}\right)\right]\left(\theta^{\sin(\theta)}\right)$$

d.) $\frac{d}{dx}\left(\sin\left(e^{x^2}\right)\right)=\cos\left(e^{x^2}\right)\cdot\frac{d}{dx}\left(e^{x^2}\right)=\cos\left(e^{x^2}\right)\cdot\left(e^{x^2}\cdot(2x)\right)$

e.)

$$\frac{d}{dy}\left(\left(y^{1/4}-\frac{1}{\sqrt{y}}\right)\tan(y)\right)=\frac{d}{dy}\left(y^{1/4}-\frac{1}{\sqrt{y}}\right)\left(\tan(y)\right)+\left(y^{1/4}-\frac{1}{\sqrt{y}}\right)\frac{d}{dy}\left(\tan(y)\right)$$

$$=\left(\tfrac{1}{4}y^{-3/4}+\tfrac{1}{2}y^{-3/2}\right)\left(\tan(y)\right)+\left(y^{1/4}-\frac{1}{\sqrt{y}}\right)\left(\sec^2(y)\right)$$

f.) $\frac{d}{dt}\left(t^3-3^t+\ln(3)\right)=3t^2-3^t\ln(3)+0$

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g.)

$$\frac{d}{dx}\left[\left(5x^2+3x\right)^{-10}+2\pi-\sec\left(6x\right)-\sqrt{x+3}+\cos\left(\tfrac{\pi}{4}\right)\right]$$

$$=-10\left(5x^2+3x\right)^{-11}\cdot\left(10x+3\right)+0-\sec(6x)\tan(6x)\cdot(6)-\tfrac{1}{2}(x+3)^{-1/2}+0$$

h.)

$$\cos\left(xy^2\right)=y^3+x$$

$$-\sin\left(xy^2\right)\left(y^2+x(2y)\frac{dy}{dx}\right)=3y^2\frac{dy}{dx}+1$$

$$-y^2\sin\left(xy^2\right)-x(2y)\frac{dy}{dx}\sin\left(xy^2\right)=3y^2\frac{dy}{dx}+1$$

$$-x(2y)\frac{dy}{dx}\sin\left(xy^2\right)-3y^2\frac{dy}{dx}=1+y^2\sin\left(xy^2\right)$$

$$\left(-x(2y)\sin\left(xy^2\right)-3y^2\right)\frac{dy}{dx}=1+y^2\sin\left(xy^2\right)$$

$$\frac{dy}{dx}=\frac{1+y^2\sin\left(xy^2\right)}{\left(-x(2y)\sin\left(xy^2\right)-3y^2\right)}$$

2. Find the following:
   a. The equation of the tangent line to the curve $y=(3x-1)^5$ at the point where $x=1$.

      The slope of the tangent line is the derivative evaluated at x=1.

      $y=(3x-1)^5$

      $y'=5(3x-1)^4(3)$

      $y'(1)=5(2)^4(3)=240$

      So the equation for the tangent line is:

      $y-32=240(x-1)$

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b. The **equations of the lines** tangent to $y = \frac{x}{x-2}$ and perpendicular to the line through the points (6, 6) and (12, 18).
(Hint: If a line has a slope of m, a line perpendicular to the first line has a slope of $\frac{-1}{m}$*).*

The line through (6,6) and (12, 18) has a slope of $\frac{18-6}{12-6} = \frac{12}{6} = 2$ *so we want the equations of all tangent lines with a slope of* $\frac{-1}{2}$ *.*

$$y = \frac{x}{x-2}$$
$$y' = \frac{(x-2)-x}{(x-2)^2} = \frac{-1}{2}$$
$$\frac{-2}{(x-2)^2} = \frac{-1}{2}$$
$$-4 = -(x-2)^2$$
$$4 = (x-2)^2$$
$$\pm 2 = x-2$$
$$x = 0 \quad or \quad x = 4$$

So we want the tangent lines at (0,0) and (4, 2), which are:

$$y - 0 = \tfrac{-1}{2}(x-0) \quad and \quad y - 2 = \tfrac{-1}{2}(x-4)$$

c. Find the equation of the tangent line to the curve $x + xy + y^2 = 7$ at the point (1, 2).

$$x + xy + y^2 = 7$$
$$1 + (y + x\frac{dy}{dx}) + 2y\frac{dy}{dx} = 0$$
$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -1 - y$$
$$\frac{dy}{dx} = \frac{-1-y}{x+2y}$$
$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-1-2}{1+2(2)} = -\frac{3}{5}$$

So the equation for the tangent line is: $y - 2 = -\frac{3}{5}(x-1)$

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3. Find the points on the curve $2x^2 + x + 3y^2 = 4$ where the tangent lines are horizontal and vertical.

$$2x^2 + x + 3y^2 = 4$$
$$4x + 1 + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-4x-1}{6y}$$

So the tangent line is horizontal when

$$-4x - 1 = 0$$
$$x = \tfrac{-1}{4}$$
$$2\left(\tfrac{-1}{4}\right)^2 + \left(\tfrac{-1}{4}\right) + 3y^2 = 4$$
$$3y^2 = \tfrac{33}{8}$$
$$y^2 = \tfrac{11}{8}$$
$$y = \pm\sqrt{\tfrac{11}{8}}$$
$$\left(\tfrac{-1}{4}, \sqrt{\tfrac{11}{8}}\right) \quad \left(\tfrac{-1}{4}, -\sqrt{\tfrac{11}{8}}\right)$$

And the tangent line is vertical when

$$6y = 0$$
$$y = 0$$
$$2x^2 + x + 0 = 4$$
$$2x^2 + x - 4 = 0$$
$$x = \frac{-1 \pm \sqrt{33}}{4}$$
$$\left(\frac{-1+\sqrt{33}}{4}, 0\right) \quad \left(\frac{-1-\sqrt{33}}{4}, 0\right)$$

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4. The position of a particle moving along the *y*-axis after *t* seconds is given by $y=\sqrt{3t+1}$.
   a. What is the velocity of the particle after 1 second?

$$y=\sqrt{3t+1}$$

$$y'=\tfrac{1}{2}(3t+1)^{-1/2}(3)=\frac{3}{2\sqrt{3t+1}}$$

$$y'(1)=\frac{3}{2\sqrt{3(1)+1}}=\frac{3}{4}$$

   b. After 1 second, what is the acceleration of the particle?

$$y'=\tfrac{1}{2}(3t+1)^{-1/2}(3)$$

$$y''=\tfrac{1}{2}\left(\tfrac{-1}{2}\right)(3t+1)^{-3/2}(3)(3)=\frac{-9}{4\left(\sqrt{3t+1}\right)^3}$$

$$y''(1)=\frac{-9}{4\left(\sqrt{3(1)+1}\right)^3}=\frac{-9}{4(8)}=\frac{-9}{32}$$

   c. When is the particle stationary?

$$y'=\frac{3}{2\sqrt{3t+1}}=0 \ \text{ if } \ 3=0 \ \textit{which never happens, so the particle is never stationary}$$

5. Let $H(x)=f(f(x))$, and $G(x)=(H(x))^2$. Also let $f(5)=14$, $f(14)=3$, $f'(14)=10$, and $f'(5)=7$.

   a. Find

$$H'(x)=\frac{d}{dx}(f(f(x))=f'(f(x))\cdot f'(x)$$

$$H'(5)=f'(f(5))\cdot f'(5)=f'(14)\cdot 7=10\cdot 7=70$$

   b. Find

$$G'(x)=\frac{d}{dx}(H(x))^2=2(H(x))\cdot(H'(x))$$

$$G'(5)=2(H(5))\cdot(H'(5))=2f(f(5))\cdot 70=2f(14)\cdot 70=2\cdot 3\cdot 70=420$$

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6. Find the following limits. Your answer should be a number, $\pm\infty$, or DNE.

a.

$$\lim_{x\to0}\left(\frac{\sin 3x}{\tan 7x}\right)=\lim_{x\to0}\left(\frac{\sin 3x}{\tan 7x}\cdot\frac{3x}{3x}\right)=\lim_{x\to0}\left(\frac{\sin 3x}{3x}\cdot\frac{3x}{\tan 7x}\right)=\left(\lim_{x\to0}\frac{\sin 3x}{3x}\right)\cdot\left(\lim_{x\to0}\frac{3x}{\tan 7x}\right)$$

$$=1\cdot\left(\lim_{x\to0}\frac{3x}{\tan 7x}\right)=\lim_{x\to0}\frac{3x}{\left(\frac{\sin 7x}{\cos 7x}\right)}=\lim_{x\to0}\left(\frac{3x\cos(7x)}{\sin(7x)}\right)=\left(\lim_{x\to0}\frac{3x}{\sin(7x)}\right)\cdot\left(\lim_{x\to0}\cos(7x)\right)=$$

$$\left(\lim_{x\to0}\frac{3x}{\sin(7x)}\right)\cdot1=\lim_{x\to0}\left(\frac{3x}{\sin(7x)}\cdot\frac{7x}{7x}\right)=\left(\lim_{x\to0}\frac{7x}{\sin(7x)}\right)\cdot\left(\lim_{x\to0}\frac{3x}{7x}\right)=1\cdot\left(\lim_{x\to0}\frac{3\cancel{x}}{7\cancel{x}}\right)=\frac{3}{7}$$

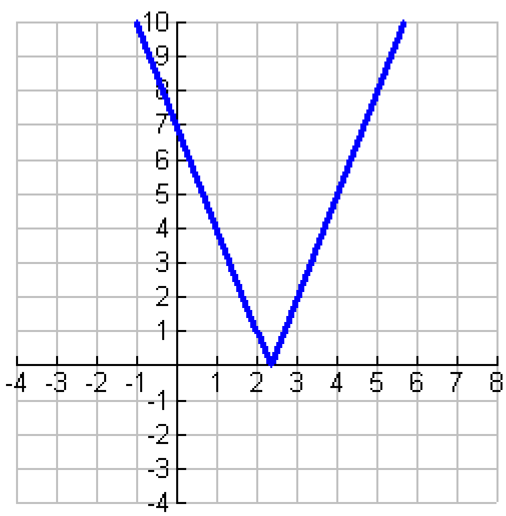
b.

$$\lim_{\theta\to0}\frac{\sin\theta(\cos\theta-1)}{\theta^2}=\lim_{\theta\to0}\left(\frac{\sin\theta}{\theta}\cdot\frac{(\cos\theta-1)}{\theta}\right)=\left(\lim_{\theta\to0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to0}\frac{(\cos\theta-1)}{\theta}\right)$$

$$=1\cdot0=0$$

7. Let $f(x)$ be defined by the graph on the right.

Let $g(x)=5x^3-2x$. Let $h(x)=\dfrac{f(x)}{g(x)}$



a) Find $\dfrac{d}{dx}\big(f(g(x))\big)$ when $x=1$.

$$\frac{d}{dx}\big(f(g(x))\big)=f'(g(x))\cdot g'(x)$$

$$g(x)=5x^3-2x \quad so \quad g(1)=5(1)^3-2(1)=3$$

$$g'(x)=15x^2-2 \quad so \quad g'(1)=15(1)^2-2=13$$

Thus, we have

$$f'(g(1))\cdot g'(1)=f'(3)\cdot(13)=3\cdot(13)=39$$

b) Find $\dfrac{d}{dx}\big(h(x)\big)$ when $x=2$.

$$\frac{d}{dx}\big(h(x)\big)=\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{[g(x)]^2}$$

$$g(x)=5x^3-2x \quad so \quad g(2)=5(2)^3-2(2)=36$$

$$g'(x)=15x^2-2 \quad so \quad g'(2)=15(2)^2-2=58$$

$$\frac{f'(2)\cdot g(2)-f(2)\cdot g'(2)}{[g(2)]^2}=\frac{(-3)\cdot(36)-(1)\cdot(58)}{[36]^2}\approx-0.1280864198$$

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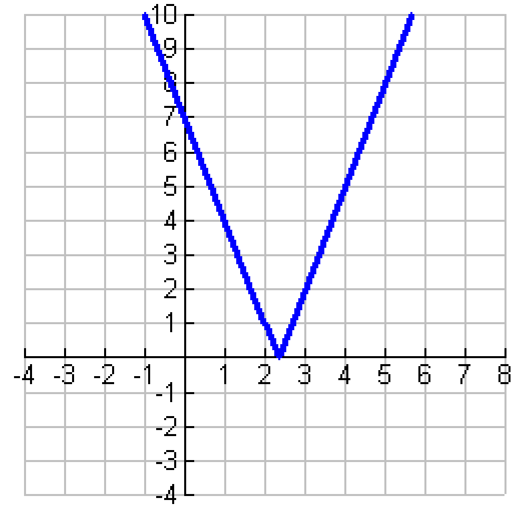
c) Find $\frac{d^2}{dx^2}\left(h(x)\right)$ when $x = 2$.



$$\frac{d}{dx}\left(h(x)\right) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$$

$$\frac{d^2}{dx^2}\left(h(x)\right) = \frac{d}{dx}\left(\frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}\right)$$

$$= \frac{\left[\left(f''(x)\cdot g(x) + f'(x)g'(x)\right) - \left(f'(x)\cdot g'(x) + f(x)\cdot g''(x)\right)\right]\left[g(x)\right]^2 - \left[f'(x)\cdot g(x) - f(x)\cdot g'(x)\right]\left[2\left[g(x)\right]\cdot g'(x)\right]}{\left[g(x)\right]^4}$$

$g(x) = 5x^3 - 2x \quad so \quad g(2) = 5(2)^3 - 2(2) = 36$

$g'(x) = 15x^2 - 2 \quad so \quad g'(2) = 15(2)^2 - 2 = 58$

$g''(x) = 30x \quad so \quad g''(2) = 30(2) = 60$

$f(2) = 1, \; f'(2) = -3, \; f''(2) = 0$

$$\left.\frac{d^2}{dx^2}\right|_{x=2}\left(h(x)\right) = \frac{\left[\left(f''(2)\cdot g(2) + f'(2)g'(2)\right) - \left(f'(2)\cdot g'(2) + f(2)\cdot g''(2)\right)\right]\left[g(2)\right]^2 - \left[f'(2)\cdot g(2) - f(2)\cdot g'(2)\right]\left[2\left[g(2)\right]\cdot g'(2)\right]}{\left[g(2)\right]^4}$$

$$= \frac{\left[\left((0)\cdot(36) + (-3)(58)\right) - \left((-3)\cdot(58) + (1)\cdot(60)\right)\right]\left[36\right]^2 - \left[(-3)\cdot(36) - (1)\cdot(58)\right]\left[2\left[36\right]\cdot(58)\right]}{\left[36\right]^4}$$

$$= \frac{\left[-174 - \left(-174 + 60\right)\right]\left[1296\right] - \left[-108 - 58\right]\left[4176\right]}{1679616}$$

$$= \frac{\left[60\right]\left[1296\right] + \left[166\right]\left[4176\right]}{1679616} = \frac{77760 + 693216}{1679616} = \frac{770976}{1679616} \approx .459019$$

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8. Compute the following derivatives using **logarithmic differentiation:**

$$\frac{d}{dx}\left(\frac{\left(x^2+e^x\right)^{10}\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}}{\sqrt[3]{e^{\pi x}}}\right)$$

$$y=\frac{\left(x^2+e^x\right)^{10}\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}}{\sqrt[3]{e^{\pi x}}}$$

$$\ln(y)=\ln\left(\frac{\left(x^2+e^x\right)^{10}\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}}{\sqrt[3]{e^{\pi x}}}\right)$$

$$\ln(y)=\ln\left(\left(x^2+e^x\right)^{10}\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}\right)-\ln\left(\sqrt[3]{e^{\pi x}}\right)$$

$$\ln(y)=\ln\left(\left(x^2+e^x\right)^{10}\right)+\ln\left(\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}\right)-\ln\left(e^{\pi x/3}\right)$$

$$\ln(y)=10\ln\left(x^2+e^x\right)-\pi\ln\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]-\tfrac{\pi x}{3}\ln(e)$$

$$\ln(y)=10\ln\left(x^2+e^x\right)-\pi\ln\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]-\tfrac{\pi x}{3}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=10\frac{1}{\left(x^2+e^x\right)}\cdot\left(2x+e^x\right)-\pi\frac{1}{\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]}\cdot\left[-\sin\left(\pi x^3\right)\cdot\left(3\pi x^2\right)-\frac{1}{\sqrt{x}}\cdot\left(\tfrac{1}{2}x^{-1/2}\right)\right]-\frac{\pi}{3}$$

$$\frac{dy}{dx}=\left(10\frac{1}{\left(x^2+e^x\right)}\cdot\left(2x+e^x\right)-\pi\frac{1}{\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]}\cdot\left[-\sin\left(\pi x^3\right)\cdot\left(3\pi x^2\right)-\frac{1}{\sqrt{x}}\cdot\left(\tfrac{1}{2}x^{-1/2}\right)\right]-\frac{\pi}{3}\right)\cdot\left(\frac{\left(x^2+e^x\right)^{10}\left[\cos\left(\pi x^3\right)-\ln\sqrt{x}\right]^{-\pi}}{\sqrt[3]{e^{\pi x}}}\right)$$

9. A particle is moving along a curve $y=\sqrt{3x+1}$. As the particle passes through the point $(1,2)$ its x-coordinate is increasing at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

The distance formula gives us

$$D(x,y)=\sqrt{(x-0)^2+(y-0)^2}$$

$$y=\sqrt{3x+1}$$

$$D(x,y)=\sqrt{x^2+(3x+1)}$$

$$\frac{dD}{dt}=\tfrac{1}{2}\left(x^2+(3x+1)\right)^{-1/2}\left(2x\frac{dx}{dt}+3\frac{dx}{dt}\right)$$

$$\frac{dD}{dt}=\tfrac{1}{2}\left((1)^2+(3(1)+1)\right)^{-1/2}\left(2(1)(3)+3(3)\right)$$

$$\frac{dD}{dt}=\frac{15}{2\sqrt{5}}$$

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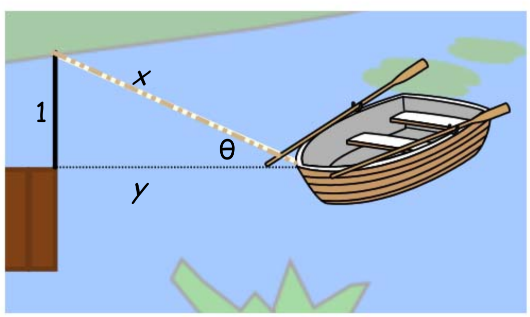
10. The mass of the part of a metal rod that lies between its left end and a point *x* meters to the right is $3x^2$ kg. Linear density, ρ, is defined as the rate of change of mass with respect to length. Find the linear density when *x* is 2 meters.

    The mass of the rod, m, is a function of the length of the rod, x. Thus, we have $m(x) = 3x^2$. We want to find the rate of change of mass with respect to length when x is 2, which means that we want to find $\frac{dm}{dx} = m'(x) = \rho(x)$ when $x = 2$. $m'(x) = \rho(x) = 6x$ so $\rho(2) = 12$ so the linear density is 12 kg/m.

11. A curve passes through the point (0,5) and has the property that the slope of the curve at every point *P* is twice the *y*-coordinate of *P*. What is the equation of the curve?

    Since the slope (i.e. derivative) at every point is twice the *y*-coordinate of that point, we have the equation $\frac{dy}{dx} = 2y$. There is a theorem in the book which states that the only solution to this type of equation is $y(t) = Ce^{2t}$. But we know that $y(0) = 5$ so the equation of the curve must satisfy $5 = Ce^{(2)0}$ so $C = 5$. Thus, the equation of the curve is $y(t) = 5e^{2t}$.

12. A boat is being pulled to a dock by a rope attached to the bow of the boat and is passing through a pulley on the end of the dock that is 1 meter higher than the bow of the boat. The rope is being pulled at a constant rate of 1 meter per second.



a) How fast is the boat approaching the dock when the boat is 8 meters away from the dock?

$$1^2 + y^2 = x^2, \quad \frac{dx}{dt} = -1$$

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$1^2 + 8^2 = x^2 \quad \text{so} \quad x = \sqrt{65}$$

$$2(8)\frac{dy}{dt} = 2(\sqrt{65})(-1) \quad \text{so} \quad \frac{dy}{dt} = \frac{-\sqrt{65}}{8}$$

So the boat is approaching at $\frac{\sqrt{65}}{8}$ *meters per second.*

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b) How fast is the angle between the rope and the horizontal line from the bow of the boat to the dock changing at this moment?

$$\sin(\theta) = \frac{1}{x}$$

$$\cos(\theta)\frac{d\theta}{dt} = \frac{-1}{x^2}\frac{dx}{dt}$$

$$\sin(\theta) = \frac{1}{\sqrt{65}}$$

$$\left(\frac{8}{\sqrt{65}}\right)\frac{d\theta}{dt} = \frac{-1}{\left(\sqrt{65}\right)^2}(-1)$$

$$\frac{d\theta}{dt} = \frac{1}{8\sqrt{65}}$$

13. Suppose sand is being poured onto a conical pile at the rate of 2 cubic feet per second.

a. Assume the height is twice the radius of the base. What rate does the radius need to be changing in order to keep this ratio constant? (This should depend on the length of the radius!)

$$V = \tfrac{1}{3}\pi r^2 h$$

$$h = 2r$$

$$V = \tfrac{1}{3}\pi r^2(2r) = \tfrac{2}{3}\pi r^3$$

$$\frac{dV}{dt} = 2\pi r^2\frac{dr}{dt}$$

$$2 = 2\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{\pi r^2}$$

b. What is the rate of change of the radius when the pile is 10 feet high?

$$\frac{dr}{dt} = \frac{1}{\pi(5)^2}$$

Quick Summary of the Material

We strongly suggest you make your own study guide using your book and class notes. Here are a few things you may want to include:

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1. Anything from Exam 1 that you are still unsure about.

2. **Table of Key Derivatives:**

Exponent and Log functions	$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}a^x = a^x \ln a$	$\frac{d}{dx}\ln x = \frac{1}{x}$
Trigonometric functions	$\frac{d}{dx}\sin x = \cos x$ $\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\cos x = -\sin x$ $\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\tan x = \sec^2 x$ $\frac{d}{dx}\cot x = -\csc^2 x$
Inverse Trig functions	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

3. **Special Limits**

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin(x)}\right) = 1 \qquad \lim_{x\to 0}\left(\frac{\cos(x)-1}{x}\right) = 0$$

4. **Implicit Differentiation**

Used when it is impossible or impractical to solve a function in two variables to be in terms of one variable. If we want to find $\frac{dy}{dx}$, we think of y as implicitly defined as a function of x. We differentiate both sides of the equation in terms of x. When we differentiate x, we get 1, but when we differentiate y, we get $\frac{dy}{dx}$ or y' (either is fine). Then we solve for $\frac{dy}{dx}$. To evaluate the derivative, we would need a point (x, y) on the curve.

Example: $x^2 + y^2 = 1$.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1) \quad \Rightarrow \quad 2x + \underbrace{2y\frac{dy}{dx}}_{\text{Chain rule here}} = 0 \quad \Rightarrow \quad 2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

5. Logarithmic Differentiation

Used when the function is complicated or for functions with an x in base and in the exponent.

<u>Option 1</u>: Take the log of both sides, simplify with log properties, differentiate (implicit chain rule on y will always happen on the left side), then solve for y'.

Ex. $y = x^x \Rightarrow \ln y = \ln x^x = x \ln x \Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \Rightarrow \frac{1}{y}y' = 1 * \ln x + x * \frac{1}{x} = \ln x + 1$

$\Rightarrow \quad y' = y(\ln x + 1) = x^x(\ln x + 1)$

<u>Option 2</u>: Take e^ln(your equation), simplify with log properties, differentiate (not implicit). Option 2 is better if you only want to take the log of part of your function, such as with $y = x^x + \sin x$

Ex. $y = x^x \Rightarrow y = e^{\ln x^x} = e^{x\ln x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{x\ln x}\right) \Rightarrow y' = e^{x\ln x}\left(1 * \ln x + x * \frac{1}{x}\right) = e^{x\ln x}(\ln x + 1)$

6. **Derivative Rules**

- $\frac{d}{dx}c = 0$ — derivative of ANY constant (anything without an x)
- $\frac{d}{dx}(c \cdot f) = c \cdot f'$ — derivative of a constant times a function

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- $\frac{d}{dx}(x^n) = nx^{n-1}$ the Power Rule
- $(f \pm g)' = f' \pm g'$ sum or difference of functions
- $\left(f \cdot g\right)' = f' \cdot g + f \cdot g'$ the Product Rule
- $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ the Quotient Rule
- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ the Chain Rule

7. Word Problems:

a. *Applications of Derivatives (Section 3.7)*

These are the most straight-forward word problems. These are problems where you are asked to find the rate of change of one quantity with respect to another quantity. (i.e. cost in terms of items produced). While some of these problems ask the rate of change with respect to time, they differ from related rates because the only rate involved is the one you are asked to find. To solve these problems, identify the equation which is changing, differentiate (implicit should not be needed), and, if necessary, plug in to get the rate at a specific value. You might also be asked to find an average rate of change, which would just be like finding a slope of a secant line.

Example: *Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of pressure and volume remains constant: PV =c. Find the rate of change of volume with respect to pressure.*

b. *Exponential Growth and Decay (Section 3.8)*

Whenever the rate of change is proportional to the size, then this is an exponential growth and decay problem. You will have an equation $\frac{dy}{dt} = ky$ where k is the relative growth rate. The theorem section 3.8 says that the only solution to this differential equation is $y(t) = y(0)e^{kt}$. You just need to figure out the relative growth rate and the value of $y(0)$ and you will have the equation you need.

Example: *A bacteria culture grows with constant relative growth rate. After 2 hour there are 600 bateria and after 8 hours the count is 75,000. When will the population reach 200,000?*

c. *Related Rates (Section 3.9)*

These are problems where you are given one (or more) rate(s) and asked to find another rate. These rates are almost always given in terms of time. You figure out an equation from the geometry of the problem which includes all the variables you want the rates of change of and no other variables. You make sure that in this step you only put in constants for values that never change. Then you differentiate implicitly in terms of time. Finally you solve for the rate you want by plugging in for all the other values at some *particular* moment in time, including the other rates.

Example: *A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of $60\,{}^{cm}\!/_{s}$. Find the rate the area within the circle is increasing after 3 seconds.*