

■ A NOTE ON UNITS

The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt , namely, billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

From this table we see that $D'(1990)$ lies somewhere between 257.48 and 348.14 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1980 and 2000.] We estimate that the rate of increase of the national debt of the United States in 1990 was the average of these two numbers, namely

$$D'(1990) \approx 303 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 1990$. □

In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.7 EXERCISES

1. A curve has equation $y = f(x)$.
 - (a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
 - (b) Write an expression for the slope of the tangent line at P .
2. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?
3. (a) Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$
 - (i) using Definition 1
 - (ii) using Equation 2
- (b) Find an equation of the tangent line in part (a).
- (c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistinguishable.
4. (a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$
 - (i) using Definition 1
 - (ii) using Equation 2
- (b) Find an equation of the tangent line in part (a).
- (c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

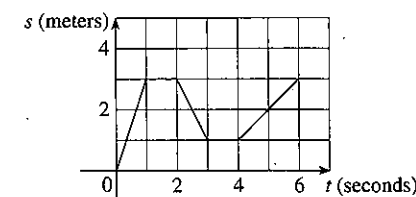
5. $y = \frac{x-1}{x-2}, (3, 2)$
6. $y = 2x^3 - 5x, (-1, 3)$
7. $y = \sqrt{x}, (1, 1)$
8. $y = \frac{2x}{(x+1)^2}, (0, 0)$

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
- (b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.
- (c) Graph the curve and both tangents on a common screen.

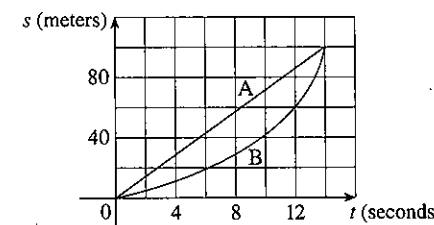
10. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
- (b) Find equations of the tangent lines at the points $(1, 1)$ and $(4, \frac{1}{2})$.
- (c) Graph the curve and both tangents on a common screen.

11. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown. When is the particle moving to the right? Moving to the left? Standing still?

(b) Draw a graph of the velocity function.



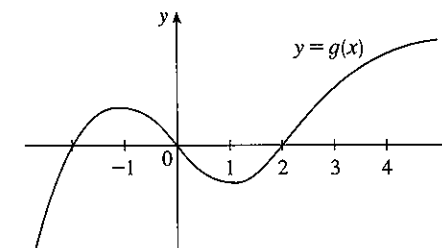
12. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.



- (a) Describe and compare how the runners run the race.
- (b) At what time is the distance between the runners the greatest?
- (c) At what time do they have the same velocity?
13. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
 - (a) Find the velocity of the rock after one second.
 - (b) Find the velocity of the rock when $t = a$.
 - (c) When will the rock hit the surface?
 - (d) With what velocity will the rock hit the surface?
15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a, t = 1, t = 2$, and $t = 3$.
16. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
 - (a) Find the average velocity over each time interval:
 - (i) $[3, 4]$
 - (ii) $[3.5, 4]$
 - (iii) $[4, 5]$
 - (iv) $[4, 4.5]$
 - (b) Find the instantaneous velocity when $t = 4$.
 - (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$$



18. (a) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.
- (b) If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.
19. Sketch the graph of a function f for which $f(0) = 0, f'(0) = 3, f'(1) = 0$, and $f'(2) = -1$.
20. Sketch the graph of a function g for which $g(0) = g'(0) = 0, g'(-1) = -1, g'(1) = 3$, and $g'(2) = 1$.
21. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y = 3x^2 - 5x$ at the point $(2, 2)$.
22. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.
23. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
24. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.
- (b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

25–30 Find $f'(a)$.

25. $f(x) = 3 - 2x + 4x^2$
26. $f(t) = t^4 - 5t$
27. $f(t) = \frac{2t+1}{t+3}$
28. $f(x) = \frac{x^2+1}{x-2}$
29. $f(x) = \frac{1}{\sqrt{x+2}}$
30. $f(x) = \sqrt{3x+1}$

31–36 Each limit represents the derivative of some function f at some number a . State such an f and a in each case.

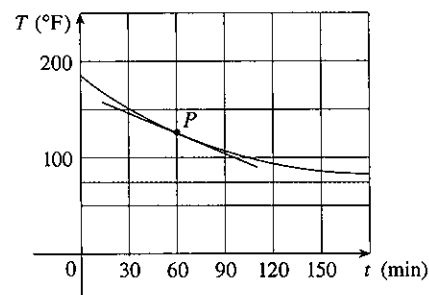
31. $\lim_{h \rightarrow 0} \frac{(1+h)^{10} - 1}{h}$
32. $\lim_{h \rightarrow 0} \frac{\sqrt[3]{16+h} - 2}{h}$
33. $\lim_{x \rightarrow 3} \frac{2^x - 32}{x - 5}$
34. $\lim_{x \rightarrow \pi/4} \frac{\tan x - 1}{x - \pi/4}$
35. $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h}$
36. $\lim_{t \rightarrow 1} \frac{t^4 + t - 2}{t - 1}$

37–38 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 5$.

37. $f(t) = 100 + 50t - 4.9t^2$ 38. $f(t) = t^{-1} - t$

39. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

40. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



41. The table shows the estimated percentage P of the population of Europe that use cell phones. (Midyear estimates are given.)

Year	1998	1999	2000	2001	2002	2003
P	28	39	55	68	77	83

- (a) Find the average rate of cell phone growth
 (i) from 2000 to 2002 (ii) from 2000 to 2001
 (iii) from 1999 to 2000

In each case, include the units.

- (b) Estimate the instantaneous rate of growth in 2000 by taking the average of two average rates of change. What are its units?
 (c) Estimate the instantaneous rate of growth in 2000 by measuring the slope of a tangent.

42. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of June 30 are given.)

Year	1998	1999	2000	2001	2002
N	1886	2135	3501	4709	5886

- (a) Find the average rate of growth
 (i) from 2000 to 2002 (ii) from 2000 to 2001
 (iii) from 1999 to 2000
 In each case, include the units.

- (b) Estimate the instantaneous rate of growth in 2000 by taking the average of two average rates of change. What are its units?
 (c) Estimate the instantaneous rate of growth in 2000 by measuring the slope of a tangent.

43. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

- (a) Find the average rate of change of C with respect to x when the production level is changed
 (i) from $x = 100$ to $x = 105$
 (ii) from $x = 100$ to $x = 101$
 (b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

44. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000 \left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t . What are its units? For times $t = 0, 10, 20, 30, 40, 50,$ and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

45. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

- (a) What is the meaning of the derivative $f'(x)$? What are its units?
 (b) What does the statement $f'(800) = 17$ mean?
 (c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

46. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

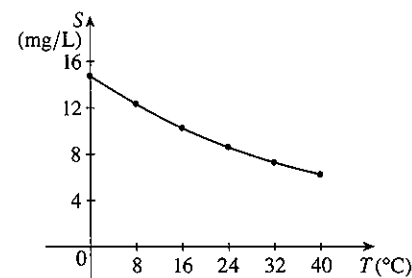
- (a) What is the meaning of the derivative $f'(5)$? What are its units?
 (b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

47. Let $T(t)$ be the temperature (in °F) in Dallas t hours after midnight on June 2, 2001. The table shows values of this function recorded every two hours. What is the meaning of $T'(10)$? Estimate its value.

t	0	2	4	6	8	10	12	14
T	73	73	70	69	72	81	88	91

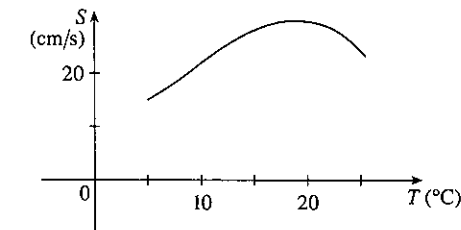
48. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.
 (a) What is the meaning of the derivative $f'(8)$? What are its units?
 (b) Is $f'(8)$ positive or negative? Explain.

49. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T .
 (a) What is the meaning of the derivative $S'(T)$? What are its units?
 (b) Estimate the value of $S'(16)$ and interpret it.



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50. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
 (a) What is the meaning of the derivative $S'(T)$? What are its units?
 (b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



51–52 Determine whether $f'(0)$ exists.

51. $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

52. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

WRITING PROJECT

EARLY METHODS FOR FINDING TANGENTS

The first person to formulate explicitly the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's teacher at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

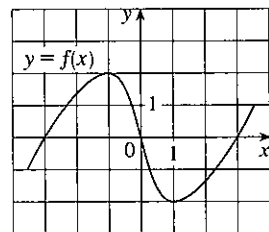
The following references contain explanations of these methods. Read one or more of the references and write a report comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point $(1, 3)$ and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989), pp. 389, 432.
2. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.
3. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
4. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

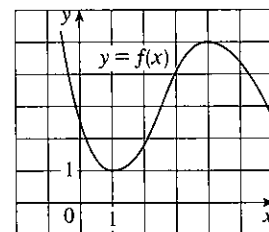
2.8 EXERCISES

1-2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f' .

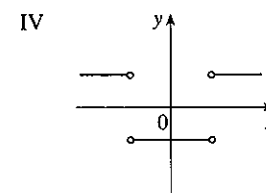
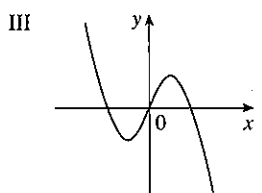
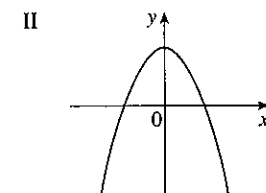
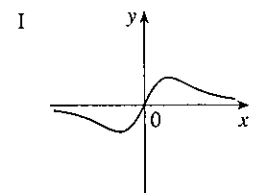
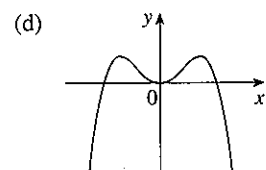
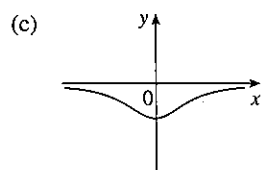
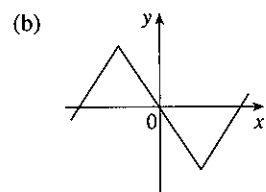
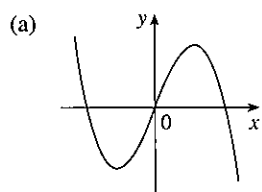
1. (a) $f'(-3)$
 (b) $f'(-2)$
 (c) $f'(-1)$
 (d) $f'(0)$
 (e) $f'(1)$
 (f) $f'(2)$
 (g) $f'(3)$



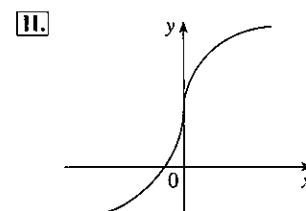
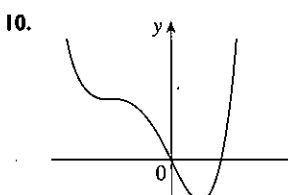
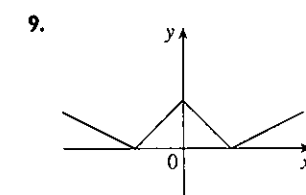
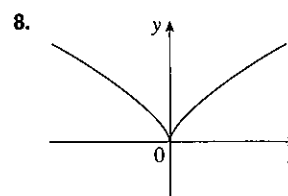
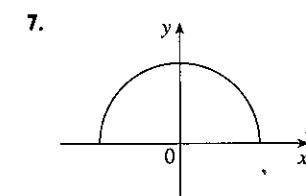
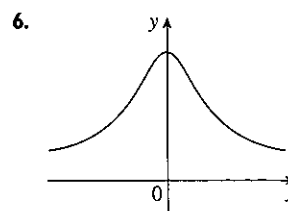
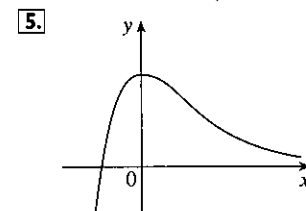
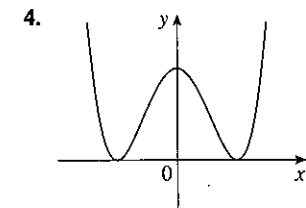
2. (a) $f'(0)$
 (b) $f'(1)$
 (c) $f'(2)$
 (d) $f'(3)$
 (e) $f'(4)$
 (f) $f'(5)$



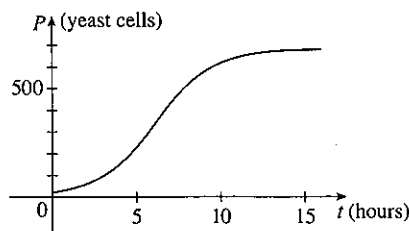
3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



4-11 Trace or copy the graph of the given function f . (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

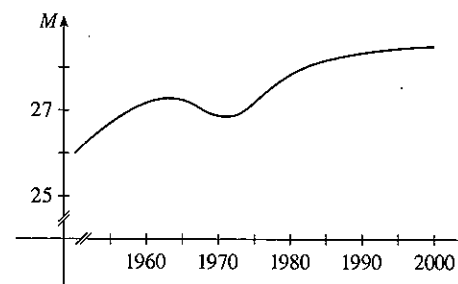


12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to



graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?

13. The graph shows how the average age of first marriage of Japanese men has varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



14-16 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4-11. Can you guess a formula for $f'(x)$ from its graph?

14. $f(x) = \sin x$

15. $f(x) = e^x$

16. $f(x) = \ln x$

17. Let $f(x) = x^2$.

- (a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f .
 (b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
 (c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
 (d) Use the definition of a derivative to prove that your guess in part (c) is correct.

18. Let $f(x) = x^3$.

- (a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f .
 (b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
 (c) Use the values from parts (a) and (b) to graph f' .
 (d) Guess a formula for $f'(x)$.
 (e) Use the definition of a derivative to prove that your guess in part (d) is correct.

19-29 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

19. $f(x) = \frac{1}{2}x - \frac{1}{3}$

20. $f(x) = mx + b$

21. $f(t) = 5t - 9t^2$

22. $f(x) = 1.5x^2 - x + 3.7$

23. $f(x) = x^3 - 3x + 5$

24. $f(x) = x + \sqrt{x}$

25. $g(x) = \sqrt{1 + 2x}$

26. $f(x) = \frac{3 + x}{1 - 3x}$

27. $G(t) = \frac{4t}{t + 1}$

28. $g(t) = \frac{1}{\sqrt{t}}$

29. $f(x) = x^4$

30. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

- (b) Use the graph from part (a) to sketch the graph of f' .
 (c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f' ?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

31. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

32. (a) If $f(t) = t^2 - \sqrt{t}$, find $f'(t)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

33. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the US labor force from 1993 to 2002.

t	U(t)	t	U(t)
1993	6.9	1998	4.5
1994	6.1	1999	4.2
1995	5.6	2000	4.0
1996	5.4	2001	4.7
1997	4.9	2002	5.8

- (a) What is the meaning of $U'(t)$? What are its units?
 (b) Construct a table of values for $U'(t)$.

34. Let $P(t)$ be the percentage of Americans under the age of 18 at time t . The table gives values of this function in census years from 1950 to 2000.

t	P(t)	t	P(t)
1950	31.1	1980	28.0
1960	35.7	1990	25.7
1970	34.0	2000	25.7

- (a) What is the meaning of $P'(t)$? What are its units?
 (b) Construct a table of estimated values for $P'(t)$.
 (c) Graph P and P' .
 (d) How would it be possible to get more accurate values for $P'(t)$?