PRINCIPAL EIGENVALUE AND EIGENFUNCTION OF ELLIPTIC OPERATOR WITH LARGE ADVECTION AND ITS APPLICATION TO A COMPETITION MODEL

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Abstract. The asymptotic behavior, as the coefficient of the advection term approaches infinity, of the principal eigenvalue of an elliptic operator is determined. The limiting profiles of the corresponding eigenfunctions are also given. As an application a Lotka-Volterra reaction-diffusion-advection model for two competing species in a heterogeneous environment is investigated. The two species are assumed to be identical except their dispersal strategies: one disperses by random diffusion only, and the other by both random diffusion and advection along environmental gradient. When the advection is strong relative to random dispersal, both species can coexist. In some situations, it is further shown that the density of the species with large advection in the direction of resources is concentrated at the spatial location with maximum resources; detailed asymptotic profiles of all coexistence states are also given.

Keywords: Principal eigenvalue/eigenfunction, large advection, competition, coexistence.

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1. Introduction

In this paper, we first study the asymptotic behaviors, as $s \to +\infty$, of the principal eigenvalue $\lambda(s)$ and its eigenfunction to

$$\begin{cases} -\Delta\varphi - 2s\nabla m \cdot \nabla\varphi + c\,\varphi = \lambda(s)\varphi & \text{in } \Omega, \\ \partial_n\varphi = 0 \quad \text{on } \partial\Omega, \quad \varphi > 0 \quad \text{on } \bar{\Omega}, & \int_\Omega e^{2sm}\varphi^2 dx = 1, \end{cases} \tag{1.1}$$

where m and c are given smooth functions on $\bar{\Omega}$, Ω is a domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\partial_n\varphi(x) = \mathbf{n}(x) \cdot \nabla\varphi(x)$, and $\mathbf{n}(x)$ is the unit exterior normal to $\partial\Omega$ at x.

There have been a large literature on the asymptotic behaviors of the principal eigenvalue and its eigenfunctions, but almost exclusively on Dirichlet boundary conditions; see, for example, [2, 9, 10, 11, 18] and references therein. The current study of the principal eigenvalue and its eigenfunctions under the Neumann boundary condition is motivated by the recent work [4, 5] on a Lotka-Volterra two species competition model with diffusion and advection along resource gradient.

In [2] elliptic eigenvalue problems with large drift and Neumann boundary conditions are also investigated, with its focus on the situation when the drift velocity field $\mathbf{v}$ is divergence free and $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Among other things, connections between the limit of the principal eigenvalue and the first integrals of the velocity field $\mathbf{v}$ are established in [2]. For (1.1) the velocity field $\mathbf{v}$ is a gradient field, i.e., $\mathbf{v} = \nabla m$ for some function m. Clearly, if the gradient field is divergence free

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and satisfies $\nabla m \cdot \mathbf{n} = 0$ on $\partial\Omega$, the only possibility is $m \equiv$ *constant*. Hence, the results of [2] do not apply to our problem since we always assume that m is a non-constant function in order to reflect the spatial heterogeneity of the environment.

By calculus of variation, the principal eigenvalue $\lambda(s)$ is characterized by

$$\begin{aligned} \lambda(s) &= \min_{\int_\Omega e^{2sm}\varphi^2 dx=1} \int_\Omega e^{2sm}(|\nabla\varphi|^2 + c\varphi^2)dx \\ &= \min_{\int_\Omega w^2 dx=1} \int_\Omega \Big\{|\nabla w - sw\nabla m|^2 + cw^2\Big\}dx, \end{aligned} \tag{1.2}$$

where the second equation is obtained by the substitution $\varphi = e^{-sm}w$. Note that $w = e^{sm}\varphi$ satisfies

$$\begin{cases} -\Delta w + (s^2|\nabla m|^2 + s\Delta m + c - \lambda)w = 0 \quad \text{in } \Omega, \\ \partial_n w - s\, w\, \partial_n m = 0 \quad \text{on } \partial\Omega, \quad w > 0 \text{ on } \bar{\Omega}, \quad \int_\Omega w^2 dx = 1. \end{cases} \tag{1.3}$$

We denote by $w(s,\cdot)$ the unique solution w to (1.3). Since $\{w^2(s,\cdot)\}_{s>0}$ is weakly compact, we denote by $\{w^2(s_j,\cdot)\}_{j=1}^\infty$ ($\lim_{j\to\infty} s_j = \infty$) a generic sequence that converges weakly to a certain probability measure μ in the following sense:

$$\lim_{j\to\infty} \int_\Omega w^2(s_j,x)\zeta(x)dx = \int_{\bar{\Omega}} \zeta(x)\mu(dx) \qquad \forall\, \zeta \in C(\bar{\Omega}). \tag{1.4}$$

By investigating the support of the measure μ, we can find the asymptotic behavior of $\lambda(s)$.

Theorem 1. *Assume that all critical points of m are non-degenerate. Let $\mathcal{M}$ be the set of points of local maximum of m. Then the support of μ is contained in $\mathcal{M}$ and*

$$\lim_{s\to\infty} \lambda(s) = \min_{\mathcal{M}} c.$$

It remains unknown whether the support of μ is precisely given by $\mathcal{M}$. Concerning the profile of the eigenfunction near the support of μ, we have

Theorem 2. *Assume that $x_0 \in \bar{\Omega}$, $|\nabla m(x_0)| = 0, D^2 m(x_0) < 0$ and $\mu(\{x_0\}) > 0$.*
(i) *If $x_0 \in \Omega$, then*

$$\lim_{j\to\infty} s_j^{-N/4} w(s_j, x_0 + y/\sqrt{s_j}) \;=\; \sqrt{\frac{\mu(\{x_0\})}{\det(-D^2 m(x_0))\pi^N}}\; e^{\frac{1}{2}yD^2 m(x_0)y^T}$$

uniformly on any compact subset of $\mathbb{R}^N$.
(ii) *If $x_0 \in \partial\Omega$, then*

$$\lim_{j\to\infty} s_j^{-N/4} w(s_j, x_0 + y/\sqrt{s_j}) = 2\sqrt{\frac{\mu(\{x_0\})}{\det(-D^2 m(x_0))\pi^N}}\; e^{\frac{1}{2}yD^2 m(x_0)y^T}$$

uniformly on any compact subset of $\{y \in \mathbb{R}^N \ ;\ y \cdot \mathbf{n}(x_0) < 0\}$.

Next we turn to applications of previous results and consider steady states of the parabolic system

$$
\left\{
\begin{array}{ll}
u_t = \nabla \cdot [\mu \nabla u - \alpha u \nabla m] + [m(x) - u - v]u & \text{in } \Omega \times (0, \infty), \\
v_t = \nabla \cdot [\nu \nabla v] + [m(x) - u - v]v & \text{in } \Omega \times (0, \infty), \\
\mu \partial_n u - \alpha u \partial_n m = \partial_n v = 0 & \text{on } \partial\Omega \times (0, \infty).
\end{array}
\right.
\tag{1.5}
$$

This system was introduced in [4, 5] to model the competition of two species whose density at location $x \in \bar{\Omega}$ and $t \geqslant 0$ are denoted by $u = u(x, t)$ and $v = v(x, t)$ respectively. On the one hand, the two species have identical growth rate, $m(x) - u - v$, where $m(x)$ represents the intrinsic growth rate at the spatial location $x \in \bar{\Omega}$. On the other hand, the two species have different dispersal strategies: the species with density v disperses only by random diffusion, with a positive constant rate ν; the other species disperses not only by random diffusion, with a positive constant rate μ, but also by a directed movement towards more favorable habitats, modeled by the term $\alpha u \nabla m$, where α is a positive parameter. The third equation in (1.5) is a no-flux boundary condition which means that no individuals cross the boundary. The goal is to understand how different dispersal strategies affect the outcome of competition in a heterogeneous environment. In particular, we are interested in steady state of (1.5), i.e. a solution $(U, V) = (U(x), V(x))$ to

$$
\left\{
\begin{array}{ll}
\mu \Delta U - \alpha \nabla \cdot (U \nabla m) + [m(x) - U - V]U = 0 & \text{in } \Omega, \\
\nu \Delta V + [m(x) - U - V]V = 0 & \text{in } \Omega, \\
\mu \partial_n U - \alpha U \partial_n m = \partial_n V = 0 & \text{on } \partial\Omega.
\end{array}
\right.
\tag{1.6}
$$

Of course, we are interested in non-negative solutions. Then by maximum principle, either $U \equiv 0$ or $U > 0$ in $\bar{\Omega}$ and either $V \equiv 0$ or $V > 0$ in $\bar{\Omega}$. Of importance are coexistence states, i.e. solutions to (1.6) satisfying $U > 0, V > 0$ on $\bar{\Omega}$.

When $\alpha = 0$, it is well-known [8] that there are coexistence states if and only if $\mu = \nu$. In particular, if $\mu < \nu$, then starting from any non-trivial non-negative initial data, the solution (u, v) to (1.5) approaches, as $t \to \infty$, a steady state (U, V) of (1.5) with $U > 0$ and $V \equiv 0$ in $\bar{\Omega}$; that is, the slower diffusing species always wins the competition. When $\alpha > 0$, it seems that the species u becomes "smarter" and hence will continue to win the competition. However, for sufficiently large α, there are coexistence states.

Theorem 3. *Suppose that $\int_\Omega m(x)dx > 0$ and the set of critical points of $m(\cdot)$ has Lebesgue measure zero. Then for every $\mu > 0$ and $\nu > 0$, there exists a positive constant α_1 such that if $\alpha \geq \alpha_1$, the system* (1.6) *has at least one stable positive solution.*

This theorem was established in [5] under the extra condition that m has at least one isolated global maximum.

From biological point of view Theorem 3 is surprising at the first look. If $\mu < \nu$ and α is small, the species u always wins the competition. As α increases, the species u has the tendency to move toward more favorable regions, so it has more competitive advantage than the species v and should still be the sole winner of the competition. However, the above theorem tells us that "smarter" species may not necessarily win the competition. A possible explanation for such coexistence is that as α becomes large, the "smarter" competitor moves toward and concentrates at places of locally most favorable environments, leaving enough room for the other species to survive in places with less resources. Our primary goal here is to study profiles of all positive solutions to (1.6) and to justify the above intuitive argument, at least in some cases of $m(x)$.

It is shown in [5] that for any positive solution (U, V) of (1.6), $\|U\|_{L^2(\Omega)} \to 0$ as $\alpha \to \infty$, i.e. the total population size of species u, becomes sufficiently small if α is large. It is natural to inquire whether the density function $U \to 0$ in $L^\infty(\Omega)$ as $\alpha \to \infty$. As shown in our next result, the answer is negative in general. Theorem 1 suggests that U is concentrated at the local maximum of $m(x)$, and this will also be established in the next theorem for the case when $m(x)$ has a unique local maximum (and thus it must be the global maximum) in $\bar\Omega$.

When $\int_\Omega m(x)\,dx > 0$, we denote by θ the unique solution to

$$\nu\Delta\theta + \theta(m-\theta) = 0 \;\text{ in }\; \Omega, \qquad \theta > 0 \;\text{ on }\; \bar\Omega, \qquad \partial_n\theta = 0 \;\text{ on }\; \partial\Omega.$$

Theorem 4. *Suppose that* $\int_\Omega m(x)\,dx > 0$ *and all critical points of* m *are non-degenerate. Let* $\mathcal{M}$ *be the set of points of local maximum of* m. *Then for any positive solution of* (1.6),

$$\varliminf_{\alpha\to\infty} \max_{\bar\Omega} U \ge \max_{\mathcal{M}}[m-\theta] > 0.$$

Assume further that $m(x)$ *satisfies*

$$\text{(1.7)} \qquad \partial_n m \le 0 \text{ on } \partial\Omega,\ \exists x_0 \in \Omega\ \ni \{x \in \overline{\Omega} : \nabla m(x) = 0\} = \{x_0\},\ \det(D^2 m(x_0)) \ne 0.$$

Then for any positive solution $(U, V) = (U(\alpha,\cdot), V(\alpha,\cdot))$ *of* (1.6),

$$\lim_{\alpha\to\infty} \Big\|V(\alpha,\cdot) - \theta(\cdot)\Big\|_{C^{1+\beta}(\bar\Omega)} = 0 \quad \forall\beta \in (0,1),$$
$$\lim_{\alpha\to\infty} \Big\|U(\alpha,x)e^{\alpha[\max_{\bar\Omega} m - m(x)]/\mu} - 2^{N/2}[m(x_0)-\theta(x_0)]\Big\|_{L^\infty(\Omega)} = 0.$$

Theorem 4 shows that if m has a unique local maximum in $\bar\Omega$ which is also non-degenerate, then the "smarter" species is concentrated near the local maximum. We conjecture that if m has multiple local maxima, for any positive solution (U, V) of (1.6), U concentrates at all local maxima of $m(x)$ in $\bar\Omega$.

The rest of the paper is organized as follows. In §2 we provide definitions of boundary and interior critical points and establish an upper bound of the principal eigenvalue $\lambda(s)$. In §3 and §4 we show that the support of μ is contained in $\mathcal{M}$. This yields the lower bound of $\lambda(s)$, and thus completes the proof of Theorem 1. §5 and §6 are devoted to the proof of Theorem 2, i.e., determining the asymptotic profile of the principal eigenfunctions. Finally, in §7 we apply results and ideas from previous sections to establish Theorems 3 and 4.

2. Preliminaries

In the sequel, we use $B(x,\delta)$ to denote the ball $\{y\in\mathbb{R}^N \mid |y-x|<\delta\}$. Also, $\mathbb{S}^{N-1}$ is the unit sphere $\{\mathbf{e}\in\mathbb{R}^N \mid |\mathbf{e}|=1\}$. We shall regard $x\in\mathbb{R}^N$ and $\nabla m(x)$ as row vectors, whereas x^T is a column vector being the transpose of x.

2.1. Classification of Critical Points.

We introduce the following:

Definition 1. *Assume that* $m\in C^2(\bar\Omega)$.

1. *An* **interior critical point** *of* m *is a point* $x\in\Omega$ *satisfying* $\nabla m(x)=\mathbf{0}$. *An interior critical point,* x*, is called* **non-degenerate** *if* $\det(D^2m(x))\neq 0$.
2. *A* **boundary critical point** *of* m *is a point* $x\in\partial\Omega$ *that satisfies* $|\nabla m(x)|=|\partial_n m(x)|$. *A boundary critical point,* x*, is called* **non-degenerate** *if*
$$\det\Big(D^2_{\partial\Omega}m(x)\Big)\neq 0,\qquad \textit{either } |\nabla m(x)|\neq 0 \textit{ or } \det\Big(D^2m(x)\Big)\neq 0.$$
3. *A* **point of local maximum** *of* m *is an* $x\in\bar\Omega$ *that satisfies* $m(x)\geq m(y)$ *for every* y *in a small neighborhood of* x*, and there exists some sequence* $\{r_j\}$ *of positive numbers such that*
$$m(x)>\max_{\bar\Omega\cap\partial B(x,r_j)} m\quad \forall\, j\in\mathbb{N},\qquad \lim_{j\to\infty} r_j=0.$$
We denote the set of points of local maximum by $\mathcal{M}$. *The reason for such a definition of local maximum is that we want to avoid the scenario when the set of local maximum contains some flat piece.*

Remark 2.1. We explain our definition.

1. If $x\in\Omega$ is a non-degenerate interior critical point of m, then for small z,
$$\nabla m(x+z)=z\,A,\quad A:=\int_0^1 D^2m(x+tz)dt.$$
Since $\det D^2m(x)\neq 0$, we see that $|\nabla m(x+z)|>0$ for every small $z\neq\mathbf{0}$. A generic non-degenerate interior critical point, x, can be classified into three types:
    - (a) **Interior Local minimum**: $|\nabla m(x)|=0$ and $D^2m(x)$ is positive definite;
    - (b) **Interior Local maximum**: $|\nabla m(x)|=0$ and $D^2m(x)$ is negative definite;
    - (c) **Interior Saddle**: $|\nabla m(x)|=0$ and $D^2m(x)$ admits both positive and negative eigenvalues.
2. To consider the boundary critical points, we introduce the signed distance function
$$d(x)=\begin{cases}\min\{|x-y| \mid y\in\partial\Omega\} & \text{if } x\notin\Omega,\\ -\min\{|x-y| \mid y\in\partial\Omega\} & \text{if } x\in\Omega.\end{cases}$$
Suppose $x\in\partial\Omega$. Then $\mathbf{n}(x):=\nabla d(x)$ is the unit exterior normal to $\partial\Omega$ at x and $\partial_n m(x)=\mathbf{n}(x)\cdot\nabla m(x)$. The operator $\nabla_{\partial\Omega}:=\nabla-\mathbf{n}(\mathbf{n}\cdot\nabla)$ is the gradient restricted to $\partial\Omega$ so the condition $|\nabla m|=|\partial_n m|$ is equivalent to $|\nabla_{\partial\Omega}m|=0$. Hence,

    a boundary critical point is a critical point of the function restricted to $\partial\Omega$.

Here $D^2_{\partial\Omega}m$ is the Hessian of the function m restricted to $\partial\Omega$. To avoid the use of differential geometry, we define it here as follows. Let $z \in \partial\Omega$. Make a rigid rotation such that $\mathbf{n}(z) = -\mathbf{e}_N := (0, \cdots, 0, -1)$. Then locally $\partial\Omega$ can be written as a graph $x_N = \psi(x')$ where $x' = (x_1, \cdots, x_{N-1})$ and $\psi_{x_i}(z') = 0$ for $i = 1, \cdots, N-1$. Then $m|_{\partial\Omega}$ corresponds to the function $m(x', \psi(x'))$. Hence,

$$\begin{aligned} \nabla_{\partial\Omega} m(z) &= \nabla m - \mathbf{n}(\mathbf{n}\cdot\nabla m) = (m_{x_1}(z), \cdots, m_{x_{N-1}}(z), 0), \\ D^2_{\partial\Omega} m(z) &= \Big(m_{x_i x_j}(z) + m_{x_N}(z)\psi_{x_i x_j}(z')\Big)_{(N-1)\times(N-1)}. \end{aligned}$$

(3) A generic non-degenerate boundary critical point, x, can be classified as follows:
   (a) **Boundary Local Minimum:** either $\{|\nabla m(x)| = -\partial_n m(x) > 0 \ \text{ and } \ D^2_{\partial\Omega}m(x) > 0\}$ or $\{|\nabla m(x)| = 0 \ \text{ and } \ D^2 m(x) > 0\}$;
   (b) **Boundary Local Maximum:** either $\{|\nabla m(x)| = \partial_n m(x) > 0 \ \text{ and } \ D^2_{\partial\Omega}m(x) < 0\}$ or $\{|\nabla m(x)| = 0 \ \text{ and } \ D^2 m(x) < 0\}$;
   (c) **Boundary Saddle:**
   $\{|\nabla m(x)| = \partial_n m(x) > 0 \ \text{ and } \ \exists\, \mathbf{t} \in \mathbb{S}^N \ni \ \mathbf{t} \perp \mathbf{n}(x),\ (\mathbf{t}\cdot\nabla n)^2 m(x) > 0\}$,
   or $\{|\nabla m(x)| = -\partial_n m(x) > 0 \ \text{ and } \ \exists\, \mathbf{t} \in \mathbb{S}^N \ni \ \mathbf{t} \perp \mathbf{n}(x),\ (\mathbf{t}\cdot\nabla n)^2 m(x) < 0\}$,
   or $\{|\nabla m(x)| = 0 \ \text{ and } \ \exists\, \mathbf{e}_1, \mathbf{e}_2 \in \mathbb{S}^N \ni \ (\mathbf{e}_1\cdot\nabla)^2 m(x) > 0 > (\mathbf{e}_2\cdot\nabla)^2 m(x)\}$.

The following lemma is a direct consequence of the above discussion.

Lemma 2.1. *Assume that every critical point of m is non-degenerate. Then $\mathcal{M}$ consists of points of interior and boundary local maxima.*

2.2. **An Upper Bound of the Principal Eigenvalue.** Since we do not a priori know the existence of the limit $\lim_{s\to\infty}\lambda(s)$, we define

$$\lambda^* = \overline{\lim_{s\to\infty}}\, \lambda(s), \qquad \lambda_* = \underline{\lim}_{s\to\infty}\, \lambda(s).$$

The upper bound of λ^* is very easy to establish:

Lemma 2.2. *Let $\mathcal{M}$ be the collection of points of local maximum. Then*

$$\lambda^* := \overline{\lim_{s\to\infty}}\, \lambda(s) \leqslant \min_{\mathcal{M}} c. \tag{2.1}$$

Proof. Fix $z \in \mathcal{M}$. Let $\{r_i\}$ be the sequence given in Definition 1, part (3). Then there exists a sequence $\{\alpha_i\}$ such that

$$0 < \alpha_i < r_i, \qquad \min_{\bar{B}(z,\alpha_i)\cap\bar{\Omega}} m =: m_i > \max_{\bar{\Omega}\cap\partial\bar{B}(z,r_i)} m.$$

Hence, we can find another sequence $\{\beta_i\}_{i\in\mathbb{N}}$ such that

$$0 < \alpha_i < \beta_i < r_i, \qquad \min_{\bar{B}(z,\alpha_i)\cap\bar{\Omega}} m =: m_i > M_i := \max_{\bar{\Omega}\cap\bar{B}(z,r_i)\setminus B(z,\beta_i)} m, \qquad \lim_{i\to\infty} r_i = 0.$$

Let

$$u_i(x) = \begin{cases} 1 & \text{if } \ x \in B(z,\beta_i), \\ (r_i - |x|)/(r_i - \beta_i) & \text{if } \ x \in B(z,r_i)\setminus B(z,\beta_i), \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus B(z,r_i). \end{cases}$$

Then

$$\begin{aligned}\lambda(s) &\leqslant \frac{\int_\Omega e^{2sm} c u_i^2}{\int_\Omega e^{2sm} u_i^2} + \frac{\int_\Omega e^{2sm} |\nabla u_i|^2}{\int_\Omega e^{2sm} u_i^2} \\ &\leqslant \max_{\bar{B}(z,r_i)} c + \frac{e^{2sM_i} r_i^N}{|r_i - \beta_i|^2 \alpha_i^N e^{2sm_i}}.\end{aligned}$$

First sending $s \to \infty$ then sending $i \to \infty$ we the obtain $\overline{\lim}_{s\to\infty} \lambda(s) \leqslant c(z)$. This completes the proof. □

To study lower bound, we need special techniques, to be presented in the next two sections.

3. Lower Bound

To show that $\lambda_* = \lambda^*$, we first select a sequence $\{s_i\}$ such that

$$\lim_{i\to\infty} s_i = \infty, \qquad \lim_{i\to\infty} \lambda(s_i) = \lambda_*, \qquad \lim_{i\to\infty} w^2(s_i, \cdot) = \mu \ \text{ (in measure)}.$$

From (1.2), we obtain

$$\lambda_* := \underline{\lim}_{s\to\infty} \lambda(s) \geqslant \lim_{j\to\infty} \int_\Omega c(x) w^2(s_j, x) dx = \int_\Omega c(x) \mu(dx). \tag{3.1}$$

If we can show that μ is supported on $\mathcal{M}$, then Theorem 1 follows from (3.1) and (2.1).

For convenience, introduce $c_* = \min_{\bar{\Omega}} c$, $c^* = \max_{\bar{\Omega}} c$. The characterization (1.2) immediately gives the bounds

$$c_* \leqslant \lambda(s) \leqslant c^* \qquad \forall\, s \in \mathbb{R}.$$

The key to study the support of μ is the following inequality:

$$c^* - c_* \geqslant \int_\Omega |\nabla w(s,x) - s w(s,x) \nabla m(x)|^2 dx \quad \forall\, s \in \mathbb{R}. \tag{3.2}$$

3.1. Non-Critical Interior Points.

Denote the set of non-critical interior points by

$$\Omega_1 = \{z \in \Omega \mid |\nabla m(z)| > 0\}.$$

Lemma 3.1. $\mu(\Omega_1) = 0$.

Proof. Fix an arbitrary $z \in \Omega_1$. There exist positive δ and R such that

$$|\nabla m| > \delta \ \text{ in } \ B(z, 2R) \subset \Omega.$$

Let ρ be a smooth cut-off function such that

$$\rho = 1 \text{ in } \ B(0,1), \qquad \rho = 0 \text{ in } \ \mathbb{R}^N \setminus B(0,2), \qquad 0 \leqslant \rho \leqslant 1, \ \ |\nabla \rho| \leqslant 2 \text{ in } \ B(0,2).$$

Set $\zeta(x) = \rho([x-z]/R)$. Then

$$\begin{aligned} c^* - c_* &\geqslant \int_\Omega \zeta^2(x)|\nabla w(s,x) - sw(s,x)\nabla m(x)|^2 dx \\ &= \int_\Omega \zeta^2 \Big\{ |\nabla w|^2 + s^2 w^2 |\nabla m|^2 \Big\} - s\int_\Omega \zeta^2 \nabla w^2 \cdot \nabla m \\ &= \int_\Omega \zeta^2 \Big\{ |\nabla w|^2 + [s^2|\nabla m|^2 + s\Delta m] w^2 \Big\} + 2s\int_\Omega w^2 \zeta \nabla\zeta \cdot \nabla m \\ &\geqslant \int_\Omega \Big\{ \tfrac{1}{2}s^2|\nabla m|^2 + s\Delta m \Big\} w^2\zeta^2 - 2\int_\Omega |\nabla \zeta|^2 w^2 \\ &\geqslant \int_\Omega \Big\{ \tfrac{1}{2}s^2|\nabla m|^2 + s\Delta m \Big\} w^2\zeta^2 - \frac{8}{R^2}, \end{aligned}$$

where we have used integration by parts in the second equation for the second integral, the Cauchy inequality $2s|\zeta\nabla\zeta\cdot\nabla m| \leqslant \frac{1}{2}s^2\zeta^2|\nabla m|^2 + 2|\nabla\zeta|^2$ in the second inequality, and finally the estimate $|\nabla\zeta| \leqslant 2/R$ and the identity $\int_\Omega w^2 = 1$ in the last inequality.

Thus, when $s \geqslant 4\|\Delta m\|_\infty/\delta^2$, we have $\frac{1}{2}s^2|\nabla m|^2 + s\Delta m \geqslant s^2\delta^2/4$ in the support of ζ so that

$$\int_\Omega \zeta^2(x) w^2(s,x) dx \leqslant \frac{4}{s^2\delta^2}\int_\Omega \Big\{ \tfrac{1}{2}s^2|\nabla m|^2 + s\Delta m \Big\} w^2\zeta^2 dx \leqslant \frac{4}{\delta^2 s^2}\Big(c^* - c_* + \frac{8}{R^2}\Big).$$

Sending $s \to \infty$ we then obtain $\mu(B(z,R)) \leqslant \lim_{s\to\infty}\int_\Omega w^2(s,x)\zeta^2(x)dx = 0$. Thus, $\mu(\Omega_1) = 0$. □

3.2. **Non-Degenerate Interior Critical Points which are not Local Maxima.** We denote by $\mathbb{S}^{N-1}$ the unit sphere and define the set of non-degenerate interior critical points which are not local maxima by

$$\Omega_2 := \{ z \in \Omega \mid \nabla m(z) = \mathbf{0}, \quad \exists\, \mathbf{e} \in \mathbb{S}^{N-1} \ni (\mathbf{e}\cdot\nabla)^2 m(x) > 0 \}.$$

Note that if $x_0 \notin \Omega_2$ is a non-degenerate interior critical point, then as $\det(D^2 m(x_0)) \neq 0$, all eigenvalues of $D^2 m(x_0)$ are negative so x_0 is an interior local maximum.

Lemma 3.2. $\mu(\Omega_2) = 0$.

Proof. Fix $z \in \Omega_2$. Let $\mathbf{e} \in \mathbb{S}^{N-1}$ satisfy $(\mathbf{e}\cdot\nabla)^2 m(z) > 0$. By rotation assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \cdots, 0)$. Then there exist positive R and δ such that

$$(\mathbf{e}\cdot\nabla)^2 m = m_{x_1x_1} > \delta \text{ in } B(x, 2R) \subset \Omega.$$

Let $\zeta(x) = \rho([x-z]/R)$. Then

$$\begin{aligned} c^* - c_* &\geqslant \int_\Omega \zeta^2 |\nabla w - sw\nabla m|^2 dx \geqslant \int_\Omega \zeta^2 |w_{x_1} - swm_{x_1}|^2 dx \\ &= \int_\Omega \zeta^2 \Big\{ w_{x_1}^2 + s^2 m_{x_1}^2 w^2 + s m_{x_1x_1} w^2 \Big\} + 2s\int_\Omega w^2 \zeta\zeta_{x_1} m_{x_1} \\ &\geqslant s\int_\Omega m_{x_1x_1} w^2 \zeta^2 - \int_\Omega \zeta_{x_1}^2 w^2 \\ &\geqslant s\delta \int_{B(z,R)} w^2\zeta^2 - \frac{4}{R^2}. \end{aligned}$$

This implies that

$$\int_{B(z,R)} w^2 \leqslant \frac{1}{s\delta}\Big\{c^* - c_* + \frac{4}{R^2}\Big\}.$$

Sending $s \to \infty$ we obtain as before $\mu(B(z,R)) = 0$. Thus, $\mu(\Omega_2) = 0$. $\square$

3.3. **Non-Critical Boundary Points.** We define the set of non-critical boundary points by

$$\Omega_3 = \{z \in \partial\Omega \mid \quad |\nabla m(z)| > |\partial_n m(z)|\}.$$

Lemma 3.3. *$\mu(\Omega_3) = 0$.*

Proof. Let $z \in \Omega_3$. By translation and rotation, we assume that $z = \mathbf{0}$, $\mathbf{n}(z) = -\mathbf{e}_N := (0, \cdots, 0, -1)$ and $\nabla m(z) = -[\partial_n m(z)]\mathbf{e}_N + 2\delta \mathbf{e}_1$ where $\delta = \frac{1}{2}\sqrt{|\nabla m(z)|^2 - |\partial_n m(z)|^2} > 0$ and $\mathbf{e}_1 = (1, 0, \cdots, 0)$.

Locally, we can represent $\partial\Omega$ as a graph $x_N = \psi(x')$ where $x' = (x_1, \cdots, x_{N-1})$ and $\psi(0') = 0$, $\nabla_{x'}\psi(0') = 0'$. The boundary $\partial\Omega$ can be locally flattened by the transformation

$$y = Y(x) := (x', x_N - \psi(x')) \qquad \Leftrightarrow \qquad x = X(y) := (y', y_N + \psi(y')).$$

Note that $D_x Y(\mathbf{0})$ and $D_y X(\mathbf{0})$ are the identity matrix. In a small neighborhood of the origin,

$$\|D_y X(y))\| \leqslant \sqrt{2}, \qquad |\nabla_y f(X(y))|^2 = |\nabla_x f(x)\, D_y X(y)|^2 \leqslant 2|\nabla_x f(x)|^2 \quad \forall\, f \in C^1.$$

By taking R small, we can assume that

$$m_{y_1}(X(y)) > \delta \quad \forall\, y \in B^+(\mathbf{0}, 2R) := \{y \in B(\mathbf{0}, 2R) \mid y_N > 0\} = Y(\Omega) \cap B(\mathbf{0}, 2R).$$

Let $\zeta(y) = \rho(y/R)$. Then

$$\begin{aligned}
c^* - c_* &\geqslant \int_\Omega \zeta^2(Y(x))|\nabla w - sw\nabla m|^2 dx = \int_\Omega \zeta^2 w^2 |\nabla_x(\ln w - sm)|^2 dx \\
&\geqslant \tfrac{1}{2}\int_{B^+(\mathbf{0},2R)} \zeta^2 w^2 |\nabla_y(\ln w - sm)|^2 \, |\det(D_y X(y))| dy \\
&\geqslant \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} |\nabla_y w - sw\nabla_y m|^2 \zeta^2 dy \\
&\geqslant \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} |w_{y_1} - swm_{y_1}|^2 \zeta^2 dy \\
&= \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} \{w_{y_1}^2 + s^2 m_{y_1}^2 w^2 - s\zeta^2 m_{y_1}[w^2]_{y_1}\}\zeta^2 dy \\
&= \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} \Big\{[w_{y_1}^2 + (s^2 m_{y_1}^2 + s m_{y_1 y_1})w^2]\zeta^2 + 2s\zeta\zeta_{y_1} m_{y_1} w^2\Big\}\, dy \\
&\geqslant \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} \Big\{[\tfrac{1}{2}s^2 m_{y_1}^2 + s m_{y_1 y_1}]w^2\zeta^2 - 2|\nabla\zeta|^2 w^2\Big\} dy \\
&\geqslant \tfrac{1}{4}\int_{B^+(\mathbf{0},2R)} \Big\{\tfrac{1}{2}s^2 m_{y_1}^2 + s m_{y_1 y_1}\Big\} w^2 \zeta^2 dy - \frac{4}{R^2}.
\end{aligned}$$

When s is large enough, we have $\frac{1}{2}s^2|m_{y_1}|^2 - s|m_{y_1y_1}| > \frac{1}{4}s^2\delta^2$ in $B^+(0,2R)$. It then follows that

$$\int_{B(z,R)\cap\Omega} w^2(s;x)dx \leqslant 2\int_{B^+(\mathbf{0},2R)} w^2\zeta^2 dy \leqslant \frac{32}{s^2\delta^2}\Big\{c^* - c_* + \frac{4}{R^2}\Big\}.$$

Sending $s \to \infty$ we then obtain $\mu(B(z,R)) = 0$. Thus, $\mu(\Omega_3) = 0$.

3.4. Non-Degenerate Boundary Critical Points which are not Local Maxima.

We divide all boundary critical points which are not local maxima into three categories:

$$\begin{aligned}
\Omega_4 &= \{z \in \partial\Omega \mid \ |\nabla m(z)| = -\partial_n m(z) > 0\},\\
\Omega_5 &= \{z \in \partial\Omega \mid |\nabla m(z)| = \partial_n m(z) > 0,\ \exists\, \mathbf{t} \in \mathbb{S}^{N-1}\ \ni\ \mathbf{t} \perp \mathbf{n}(z), (\mathbf{t}\cdot\nabla)^2 m(z) > 0\},\\
\Omega_6 &= \{z \in \partial\Omega \mid \ |\nabla m(z)| = 0, \quad \exists\, \mathbf{e} \in \mathbb{S}^N \ni (\mathbf{e}\cdot\nabla)^2 m(z) > 0\}.
\end{aligned}$$

Note that if z is a non-degenerate boundary critical point and $z \notin \Omega_4\cup\Omega_5\cup\Omega_6$, then z is a point of boundary local maximum.

Lemma 3.4. $\mu(\Omega_4\cup\Omega_5) = 0$.

Proof. Let $z \in \Omega_4$. There exist small positive δ and R such that

$$|\nabla m| > \delta \text{ in } B(z,2R)\cap\Omega, \qquad \partial_n m < 0 \text{ on } B(z,2R)\cap\partial\Omega.$$

Same as before, setting $\zeta(x) = \rho((x-z)/R)$ we have

$$\begin{aligned}
c^* - c_* &\geqslant \int_\Omega \zeta^2|\nabla w - sw\nabla m|^2 dx\\
&= -\int_{\partial\Omega}\zeta^2 w^2\partial_n m + \int_\Omega\Big(\zeta^2\{|\nabla w|^2 + (s^2|\nabla m|^2 + s\Delta m)w^2\} + 2s\zeta w^2\nabla\zeta\nabla m\Big).
\end{aligned}$$

Dropping the boundary integral and using the same calculation as before we obtain $\mu(B(x,R)) = 0$. Hence $\mu(\Omega_4) = 0$.

The proof of $\mu(\Omega_5) = 0$ follows the same proof as that for Ω_3, just in the final step using $m_{y_1y_1} > 0$. ☐

The proof of $\mu(\Omega_6) = 0$ is quite complicated, and hence is left to the next section.

4. BOUNDARY CRITICAL POINTS

In this section we consider a generic point x_0 that satisfies

$$x_0 \in \partial\Omega, \qquad |\nabla m(x_0)| = 0, \qquad \det(D^2(m(x_0)) \neq 0, \qquad \det(D^2_{\partial\Omega}m(x_0)) \neq 0.$$

We denote by $a_1, \cdots, a_{N-1}$ the eigenvalues of $D^2_{\partial\Omega}m(x_0)$ and by $a_1, \cdots, a_{N-1}, a_N$ that of $D^2 m(x_0)$. Also,

$$a = \min\{|a_i|^2 \mid i = 1, \cdots, N\}, \qquad A = \sum_{i=1}^N a_i = \Delta m(x_0).$$

4.1. **Normalization.** Locally we flatten the boundary $\partial\Omega$ and normalize m into a diagonal bilinear form via the following steps.

(i) By a rigid rotation, we can assume that $\mathbf{n}(x_0) = -\mathbf{e}_N$ and

$$D_{x_ix_j}m(x_0) = a_i\delta_{ij} \quad \forall\, i,j = 1,\cdots,N-1, \qquad \det(D^2_{\partial\Omega}m(x_0)) = \prod_{i=1}^{N-1} a_i \neq 0.$$

(ii) Let R be a small positive constant and $Y : x \in \mathbb{R}^N \to Y(x) \in \mathbb{R}^N$ be a diffeomorphism that flattens the boundary $\partial\Omega \cap B(x_0, 4R)$:

$$Y(x_0) = \mathbf{0}, \qquad D_xY(x_0) = \mathbf{I}, \qquad Y(\partial\Omega \cap B(x_0,4R)) \subset \{(y_1,\cdots,y_N) \mid y_N = 0\}.$$

(iii) Let $x = X(y)$ be the inverse of $y = Y(x)$. Since $\nabla m(x_0) = \mathbf{0}$ and $D_xY(x_0) = \mathbf{I}$, we have

$$\begin{aligned} m(X(y)) - m(x_0) &= \tfrac{1}{2}\sum_{i=1}^{N-1}\Big\{a_iy_i^2 + 2a_{iN}y_iy_N\Big\} + \tfrac{1}{2}a_{NN}y_N^2 + O(|y|^3) \\ &= \tfrac{1}{2}\sum_{i=1}^{N-1} a_i\Big(y_i + \frac{a_{iN}}{a_i}y_N\Big)^2 + \tfrac{1}{2}y_N^2\Big(a_{NN} - \sum_{i=1}^{N-1}\frac{a_{1N}^2}{a_i}\Big) + O(|y|^3). \end{aligned}$$

Hence, after a further diffeomorphism, $Z : y \in \mathbb{R}^N \to z = Z(y) \in \mathbb{R}^N$ defined by

$$z_N = y_N[1 + O(y)], \qquad z_i = y_i + a_{iN}y_N/a_i + O(|y|^2) \quad \forall\, i = 1,\cdots,N-1,$$

we have

$$m \circ X \circ Z^{-1}(z) = m(x_0) + \tfrac{1}{2}\sum_{i=1}^{N} a_iz_i^2, \qquad \prod_{i=1}^{N} a_i = \det(D^2m(x_0)) \neq 0,$$
$$Z \circ Y(\partial\Omega \cap B(\mathbf{0},4R)) \subset \{z \mid z_N = 0\}.$$

(iv) Now for any cut-off function ζ that is supported in $B(\mathbf{0},4R)$, we have

$$\begin{aligned} c^* - c_* &\geqslant \int_\Omega \zeta^2|\nabla_x w - sw\nabla_x m|^2dx = \int_\Omega \zeta^2w^2|\nabla_x(\ln w - sm)|^2dx \\ &= \int_{B^+(\mathbf{0},4R)} \zeta^2w^2|\nabla_z(\ln w - sm)D_xZ|\det(D_zX)dz \\ &\geqslant \frac{1}{C(\Omega,m)}\int_{B^+(\mathbf{0},4R)} \zeta^2w^2|\nabla_z(\ln w - sm)|^2dz \\ &= \frac{1}{C(\Omega,m)}\int_{B^+(\mathbf{0},4R)} \zeta^2|\nabla_z w - sw\nabla_z m|^2dz. \end{aligned}$$

Note that

$$m_{z_N} = 0 \quad \text{on} \quad \{z \in B(\mathbf{0},4R) \mid z_N = 0\}. \tag{4.1}$$

4.2. **Proof of $\mu(\Omega_6) = 0$.** Assume that $x_0 \in \Omega_6$. Then there exists $i \in \{1, \cdots, N\}$ such that $a_i > 0$. Let $\zeta(z) = \rho(z/R)$. Then

$$\begin{aligned}
(c^* - c_*)C(\Omega, m) &\geqslant \int_{B^+(\mathbf{0},2R)} \zeta^2(z)|w_{z_i} - sw\nabla_{z_i} m|^2 dz \\
&= \int_{B^+(\mathbf{0},2R)} \zeta^2[w_{z_i}^2 + s^2 m_{z_i}^2 w^2 - s(w^2)_{z_i} m_{z_i}] dz \\
&= \int_{B^+(\mathbf{0},2R)} \Big\{\zeta^2[w_{z_i}^2 + s^2 m_{z_i}^2 w^2 + sw^2 m_{z_i z_i}] + 2s\zeta w^2 \zeta_{z_i} m_{z_i}\Big\} dz \\
&\geqslant s\int_{B^+(\mathbf{0},2R)} \zeta^2 w^2 m_{z_i z_i} - \int_{B^+(\mathbf{0},2R)} |\nabla\zeta|^2 w^2 \\
&\geqslant a_i s \int_{B^+(\mathbf{0},R)} w^2\zeta^2 dz - \frac{4}{R^2}\int_{B^+(x_0,2R)} w^2 dz.
\end{aligned}$$

Here in the second equation we have used the integration by parts and the fact that $m_{z_N} = 0$ on $\{z_N = 0\}$ in case $i = N$. It then follows that

$$\int_{B^+(\mathbf{0},R)} w^2(s, X(Z^{-1}(z)))dz \leqslant \frac{C}{s}.$$

Sending $s \to \infty$ we then conclude that $\mu(B(x_0, R')) = 0$ where R' is a small positive number such that $B(x_0, R') \subset \Omega X(Z^{-1}(B^+(\mathbf{0}, R)))$. We have proven the following:

Lemma 4.1. *$\mu(\Omega_6) = 0$.*

4.3. **Proof of Theorem 1.** By the estimate we obtained so far, we know that μ is supported on $\mathcal{M}$, the set of points of local maxima. Hence,

$$\lambda_* \geqslant \int_{\bar{\Omega}} c(x)\mu(dx) = \int_{\mathcal{M}} c(x)\mu(dx) \geqslant \min_{\mathcal{M}} c \geqslant \lambda^* \geqslant \lambda_*.$$

The assertion of Theorem 1 thus follows.

4.4. **Boundary Local Maximum.** For later application, we derive a useful estimate here. Assume that x_0 is a boundary local maximum with $|\nabla m(x_0)| = 0$, $\det(D^2 m(x_0)) \neq 0$. Let $K = \sqrt{A/a}$. Assume that $s \gg 1$. Let $\zeta : \mathbb{R}^N \to [0, 1]$ be a smooth function such that

$$\zeta = 1 \text{ in } B(\mathbf{0}, R) \setminus B(\mathbf{0}, 2K/\sqrt{s}), \qquad \zeta = 0 \text{ in } B(\mathbf{0}, K\sqrt{s}) \cup B(\mathbf{0}, 2R)^c, \quad |\nabla\zeta| \leqslant 2\sqrt{s}/K.$$

Then

$$\begin{aligned}
(c^* - c_*)C &\geqslant \int_{B^+(\mathbf{0},2R)} \zeta^2(z)|\nabla_z w - sw\nabla_z m|^2 dz \\
&= \int_{B^+(\mathbf{0},2R)} \zeta^2[|\nabla_z w|^2 + s^2|\nabla_z m|w^2 - s\nabla_z w^2 \cdot \nabla_z m]dz \\
&= \int_{B^+(\mathbf{0},2R)} \Big\{\zeta^2[|\nabla_z w|^2 + s^2|\nabla_z m|^2 w^2 + sw^2\Delta_z m] + 2s\zeta w^2 \nabla_z\zeta \cdot \nabla_z m\Big\} dz \\
&\geqslant \int_{B^+(\mathbf{0},2R)} \zeta^2 w^2\{\tfrac{1}{2}s^2|\nabla_z m|^2 + s\Delta m\}dz - 2\int_{B^+(\mathbf{0},2R)} |\nabla_z\zeta|^2 w^2 dz \\
&\geqslant \int_{B^+(\mathbf{0},R)\setminus B^+(\mathbf{0},2K/\sqrt{s})} s[\tfrac{1}{2}sa|z|^2 - A]w^2 dz - \frac{8s}{K^2}\int_{B^+(0,2R)} w^2 dz.
\end{aligned}$$

Here in the second equation, we have used integration by parts and the fact that $\zeta^2 m_{z_N} = 0$ on $\partial B^+(\mathbf{0}, 2R)$. For $|z| > 2K/\sqrt{s}$, we have $\frac{1}{2}sa|z|^2 - A > \frac{1}{4}sa|z|^2$. Hence,

$$\frac{as^2}{4}\int_{B^+(\mathbf{0},R)\setminus B^+(\mathbf{0},2K/\sqrt{s})} |z|^2 w^2\, dz \leqslant \frac{8s}{K^2}\int_{B^+(\mathbf{0},2R)} w^2\, dz + (c^* - c_*)C.$$

Translating into the x variable, we have the following:

Lemma 4.2. *Assume that* $x_0 \in \bar{\Omega}$ *satisfies* $|\nabla m(x_0)| = 0$ *and* $\det(D^2 m(x_0)) \neq 0$. *Then there exist positive constants* R *and* $C(\Omega, m)$ *such that*

$$\int_{B(x_0,R)\cap\Omega} s|x - x_0|^2 w^2(s, x)dx \leqslant C(\Omega, m) \quad \forall\, s \geqslant 1. \tag{4.2}$$

Here the case $x_0 \in \partial\Omega$ is just what we have considered. The case $x_0 \in \Omega$ follows from a similar but much simpler calculation.

5. Limiting Profile of the Eigenfunction: Interior Local Maximum

Here in this section, we consider the profile of the eigenfunction near interior local maximum point and establish Theorem 2(i). By scaling, we shall assume without loss of generality that

$$|\nabla m|^2_{L^\infty(\Omega)} + |\Delta m|_{L^\infty(\Omega)} + (\max_{\bar{\Omega}} c - \min_{\bar{\Omega}} c) \leqslant 1.$$

We define

$$q(s; x) = s^2|\nabla m(x)|^2 + s\Delta m(x) + c(x) - \lambda(s).$$

5.1. L^∞ estimate.

Lemma 5.1. *There exists a constant* $M > 0$ *such that*

$$\|w(s, \cdot)\|_{L^\infty(\Omega)} \leqslant M s^{N/2} \quad \forall\, s \geqslant 1. \tag{5.1}$$

Proof. Set $W(y) = w(s, y/s)$ and $\Omega_s = \{sx \mid x \in \Omega\}$. Then

$$\frac{|\Delta_y W|}{W} = \frac{|q|}{s^2} \leqslant 1 \quad \text{in } \Omega_s, \qquad\qquad \frac{|\partial_n W|}{W} \leqslant |\nabla m| \leqslant 1 \quad \text{on } \partial\Omega_s.$$

It then follows from local elliptic estimate [12] that there exists a positive constant M such that

$$\forall\, y \in \Omega_s, \qquad W(y) \leqslant M\|W\|_{L^2(B(y,1)\cap\Omega_s)} \leqslant M s^{N/2}\|w\|_{L^2(\Omega)} = M s^{N/2}.$$

The assertion of the Lemma thus follows. ☐

5.2. **An Exponential Decay Estimate.**

Lemma 5.2. *Let k, r be positive constants and W be a C^2 function satisfying*

$$\Delta W(x) = Q(x)W(x) > 0, \quad Q(x) \geqslant (N+1)k^2 \quad \forall\, x \in B(\mathbf{0}, r).$$

Then

$$W(\mathbf{0}) \leqslant e^{1-kr} \max\{W(x) \mid |x| = r\}.$$

Proof. Consider the function $\bar{W}(x) = W(x)e^{-\sqrt{1+|kx|^2}}$. It is easy to verify that $\bar{W}$ satisfies

$$\Delta\bar{W} + \frac{2k^2\, x\cdot\nabla\bar{W}}{\sqrt{|kx|^2+1}} = \bar{W}\Big\{Q(x) - k^2\frac{|kx|^2}{|kx|^2+1} - \frac{k^2}{\sqrt{|kx|^2+1}}\frac{(N-1)|kx|^2+N}{|kx|^2+1}\Big\} > 0.$$

Thus, $\bar{W}$ cannot attain its maximum at any interior point. Therefore,

$$W(\mathbf{0}) = e\bar{W}(\mathbf{0}) \leqslant e \max_{\partial B(\mathbf{0},r)} \bar{W} = e^{1-\sqrt{1+|kr|^2}} \max_{\partial B(\mathbf{0},r)} W \leqslant e^{1-kr} \max_{\partial B(\mathbf{0},r)} W.$$

The assertion of the lemma thus follows. ☐

Lemma 5.3. *Assume that for some positive constants a and R,*

$$|\nabla m(x)|^2 \geqslant a|x-x_0|^2 \quad \forall\, x \in B(x_0, 4R) \subset \Omega. \tag{5.2}$$

Then for every $s \geqslant 1$,

$$w(s,x) \leqslant e^{1+\sqrt{(N+2)/a}-\sqrt{s}|x-x_0|} \max_{B(x_0,2|x-x_0|)} w \quad \forall\, x \in B(x_0, 2R). \tag{5.3}$$

Proof. The assertion is trivially true if $\sqrt{s}|x-x_0| \leqslant \sqrt{(N+2)/a}$, i.e., $x \in B(x_0, \sqrt{(N+2)/as})$. Hence, we consider $2R > \sqrt{(N+2)/as}$ and $x \in B(x_0, 2R) \setminus B(x_0, \sqrt{(N+2)a/s})$. For every $z \in B(x_0, 4R) \setminus B(x_0, \sqrt{(N+2)/as})$ we have

$$q(s;z) \geqslant s^2 a|z-x_0|^2 - s \geqslant (N+2)s - s = (N+1)s.$$

Thus, applying Lemma 5.2 to $W(z) = w(s, x+z)$ in $B(\mathbf{0}, |x-x_0| - \sqrt{(N+2)/as})$ with $k = \sqrt{s}$, we obtain

$$\begin{aligned} w(s,x) &\leqslant e^{1-\sqrt{s}(|x-x_0|-\sqrt{(N+2)/as})} \max_{B(x,|x-x_0|-\sqrt{(N+2)/as})} w \\ &\leqslant e^{1+\sqrt{(N+2)/a}} e^{-\sqrt{s}|x-x_0|} \max_{B(x_0,2|x-x_0|)} w. \end{aligned}$$

The assertion of the lemma thus follows. ☐

5.3. **Interior Critical Points.** Let $x_0 \in \Omega$ be an interior non-degenerate critical point of m:

$$|\nabla m(x_0)| = 0, \quad \det(D^2 m(x_0)) \neq 0.$$

Then there exist positive constants R, a such that (5.2) holds. Thus, w satisfies (5.3).

Now for each $s \geqslant 1$, consider the functions

$$\begin{aligned} W(x_0, s; y) &:= s^{-N/4} w(s, x_0 + y/\sqrt{s}), \\ Q(x_0, s; y) &:= s|\nabla m(x_0 + y/\sqrt{s})|^2 + \Delta m(x_0 + y/\sqrt{s}) + \frac{c(x_0 + y/\sqrt{s}) - \lambda(s)}{s}. \end{aligned}$$

Then we have the following:

$$\begin{gathered} \Delta_y W = Q(s, y) W \qquad \forall\, y \in B(\mathbf{0}, \sqrt{s}R), \\ a|y|^2 - 1 < |Q(s, y)| \leqslant M_2 |y|^2 + 1 \quad \forall y \in B(\mathbf{0}, \sqrt{s}R), \\ \int_{B(\mathbf{0}, \sqrt{s}R)} W^2(x_0, s; y) dy = \int_{B(x_0, R)} w^2(s, x) dx, \end{gathered}$$

where $M_2 = \|D^2 m\|^2_{L^\infty(\Omega)}$.

Let $y_0 \in \bar{B}(0, 2\sqrt{s}R)$ such that

$$W(x_0, s; y_0) = \bar{M}(x_0, s, R) := \max_{\bar{B}(\mathbf{0}, 2\sqrt{s}R)} W.$$

Then in view of (5.3), we have

$$W(x_0, s; y) \leqslant \bar{M}(x_0, R, s) e^{1 + \sqrt{(N+2)/a} - |y|} \quad \forall\, y \in B(\mathbf{0}, \sqrt{s}R).$$

We now show that $\bar{M}(x_0, s, R)$ is bounded, uniformly in s. Consider two cases.

(i) Suppose $y_0 \in \partial B(\mathbf{0}, 2\sqrt{s}R)$, then by (5.3) and (5.1), we have

$$\begin{aligned} \bar{M} &= s^{-N/4} w(s, x_0 + y_0/\sqrt{s}) \leqslant s^{-N/4} e^{1 + \sqrt{(N+2)/a} - |y_0|} M s^{N/2} = M e^{1 + \sqrt{(N+2)/a} - 2\sqrt{s}R} s^{N/4} \\ &\leqslant M e^{1 + \sqrt{(N+2)/a}} R^{-N/2} \sup_{t > 0} e^{-2t} t^{N/2} = M e^{1 + \sqrt{(N+2)/a} - N/2} (\tfrac{N}{4})^{N/2} R^{-N/2}. \end{aligned}$$

(ii) Suppose $y_0 \in B(0, 2\sqrt{s}R)$. Then $\Delta_y W(x_0, s; y_0) \leqslant 0$. Consequently, $Q(x_0, s; y_0) \leqslant 0$, so that $y_0 \in B(0, \sqrt{1/a})$. By Harnack inequality [12], there exists a constant $C(a, M_2)$ such that

$$\max_{B(\mathbf{0}, \sqrt{1/a})} W \leqslant C(a, M_2) \min_{B(\mathbf{0}, \sqrt{1/a})} W.$$

It then follows that

$$1 \geqslant \int_{B(\mathbf{0}, \sqrt{1/a})} W^2 dy \geqslant \frac{|B(\mathbf{0}, \sqrt{1/a})|}{C^2(a, M_2)} \bar{M}^2, \qquad \text{i.e.} \qquad \bar{M}(x_0, s, R) \leqslant \frac{C(a, M_2)}{\sqrt{|B(\mathbf{0}, \sqrt{1/a})|}}.$$

Thus, $\bar{M}$ is bounded, uniformly in $s \geqslant 1$.

Assume that $\mu(B(x_0, R)) > 0$. Then there exist a sequence $\{s_j\}$ and a function W^* such that

$$\lim_{j \to \infty} W(x_0, s_j; y) = W^*(y) \quad \text{locally uniformly in } \mathbb{R}^N,$$

$$\int_{\mathbb{R}^N} W^{*2}(y) dy = \lim_{j \to \infty} \int_{B(x_0, R)} w^2(s_j, x)\, dx = \mu(B(x_0, R)) = \mu(\{x_0\}) > 0.$$

In addition, W^* solves the following equation:

$$\Delta W^*(y) = \Big\{|y\, D^2 m(x_0)|^2 + \Delta m(x_0)\Big\} W^*(y) \quad \text{in } \mathbb{R}^N,$$
$$0 < W^*(y) < C(m,\Omega) e^{-|y|} \quad \forall\, y \in \mathbb{R}^n.$$

Since $\det(D^2 m(x_0)) \neq 0$, all eigenvalues of $D^2 m(x_0)$ are non-zero. One can show that this equation has a solution if and only if $D^2 m(x_0)$ is negative definite. In such a case, the solution is unique and is given by

$$W^*(y) = \sqrt{\frac{\mu(\{x_0\})}{\det(-D^2 m(x_0))\pi^n}}\; e^{\frac{1}{2} y D^2 m(x_0) y^T}.$$

This proves Theorem 2 (i).

6. LIMITING PROFILE OF EIGENFUNCTION: BOUNDARY LOCAL MAXIMUM

This section is devoted to the proof of Theorem 2 for the case $x_0 \in \partial\Omega$. Hence, let $x_0 \in \partial\Omega$ be a boundary local maximum that satisfies

$$|\nabla m(x_0)| = 0, \qquad D^2 m(x_0) < 0. \tag{6.1}$$

Denote

$$\begin{aligned} \mu(s,R) &= \int_{B(x_0,R)\cap\Omega} w^2(s,x)dx, \\ W(s,y) &= \frac{1}{s^{N/4}\sqrt{\mu(s,R)}} w(s, x_0 + y/\sqrt{s}), \\ \Omega_s &= \{y \mid x_0 + y/\sqrt{s} \in \Omega\}, \end{aligned}$$

where $R > 0$ is chosen such that $\mu(B(x_0,R)\cap\Omega) = \mu(\{x_0\})$. Note that $\lim_{s\to 0}\mu(s,R) = \mu(\{x_0\}) > 0$. Then we have

$$\begin{aligned} \int_{\Omega_s \cap B(\mathbf{0}, R\sqrt{s})} W^2(s,y)dy &= \frac{1}{\mu(s,R)} \int_{\Omega\cap B(x_0,R)} w^2(s,x)ds = 1, \\ \int_{\Omega_s \cap B(\mathbf{0}, R\sqrt{s})} |y|^2 W^2(s,y)dy &= \frac{1}{\mu(s,R)} \int_{\Omega\cap B(x_0,R)} s|x-x_0|^2 w^2(s,x)dx \leqslant C(\Omega, m). \end{aligned}$$

Also, denote

$$Q(x_0,s;y) \;:=\; s|\nabla m(x_0 + y/\sqrt{s})|^2 + \Delta m(x_0 + y/\sqrt{s}) + \frac{c(x_0 + y/\sqrt{s}) - \lambda(s)}{s}.$$

Then we have the following:

$$\Delta_y W(s,y) = Q(x_0,s;y) W(s,y) \qquad \forall\, y \in B(\mathbf{0}, \sqrt{s}R) \cap \Omega_s,$$
$$a|y|^2 - 1 < |Q(x_0,s;y)| \leqslant M_2|y|^2 + 1 \quad \forall y \in B(\mathbf{0}, \sqrt{s}R) \cap \Omega_s,$$

where $M_2 = \|D^2 m\|^2_{L^\infty(\Omega)}$. In addition, for $y \in \partial\Omega_s \cap B(x_0, \sqrt{s}R)$, denoting by $\mathbf{N}(y)$ the unit exterior normal of Ω_s at y, we have

$$\begin{aligned}
\mathbf{N}(y) &= \mathbf{n}(x_0 + y/\sqrt{s}) = \mathbf{n}(x_0) + O(y/\sqrt{s}),\\
\frac{\mathbf{N}(y)\cdot\nabla W(s,y)}{W(s,y)} &= \frac{1}{\sqrt{s}}\frac{\mathbf{n}(x)\cdot\nabla w(s,x))}{w(s,x)}\Big|_{x=x_0+y/\sqrt{s}}\\
&= \sqrt{s}\mathbf{n}(x)\cdot\nabla m(x)\Big|_{x=x_0+y/\sqrt{s}}\\
&= \sqrt{s}\Big(\mathbf{n}(x_0) + O(y/\sqrt{s})\Big)\Big(D^2 m(x_0) y^T/\sqrt{s} + O(y^2/s)\Big)\\
&= \mathbf{n}(x_0) D^2 m(x_0) y^T + O(y^2/\sqrt{s}) + O(y^3/s).
\end{aligned}$$

For simplicity, we assume that $\mathbf{n}(x_0) = -\mathbf{e}_N$. It then follows from local elliptic estimate that along a subsequence $\{s_j\}$ and for some smooth function W^*,

$$\begin{aligned}
\lim_{j\to\infty} s_j &= \infty,\\
\lim_{j\to\infty} W(s_j, y) &= W^*(y) \qquad \text{locally uniformly in } \mathbb{R}^{N+} := \{y\in\mathbb{R}^N \mid y_N > 0\}.
\end{aligned}$$

In addition,

$$\begin{gathered}
\Delta W^*(y) = \Big(|yD^2 m(x_0)|^2 + \Delta m(x_0)\Big) W^*(y) \quad \text{in } \mathbb{R}^{N+},\\
W^* > 0, \qquad \textstyle\int_{\mathbb{R}^{N+}} y^2 (W^*)^2(y)dy \leqslant C(\Omega, m), \qquad \textstyle\int_{\mathbb{R}^{N+}} (W^*)^2(y)dy = 1,\\
W^*_{y_N}(y', 0) = e_N D^2 m(x_0)(y', 0)^T \quad \forall\, y' \in \mathbb{R}^{N-1}.
\end{gathered}$$

Since the solution is positive, one can use Harnack inequality to show that $W^* = O(1/|y|^2)$ as $|y| \to \infty$. One then can use comparison to show that the solution is given by

$$W^*(y) = \frac{2e^{\frac{1}{2}y^T D^2 m(x_0) y}}{\sqrt{\det(-D^2 m(x_0))\pi^N}}.$$

This establishes Theorem 2 (ii) and completes the proof of Theorem 2.

7. Proof of Theorems 3 and 4

This section is devoted to the proof of Theorems 3 and 4. Theorem 3 follows from Lemma 7.4, and Theorem 4 are consequences of Lemmas 7.6, 7.9, and 7.11. Without loss of generality and also for simplicity, we assume that $\nu > 0$ is fixed and $\mu = 1$ for the rest of this section. We denote

$$m^* = \max_{x\in\bar\Omega} m(x).$$

7.1. **Existence of Coexistence State.** We first establish an existence result for positive solutions of (1.6), which generalizes results in [5]. We also provide some uniform lower bound of $\max_{\bar\Omega} U$ for any positive solution (U, V) of (1.6).

Lemma 7.1. *Assume that $m(\cdot)$ is not a constant function and $\int_\Omega m(x)dx > 0$. Then there exists a unique positive solution θ to*

$$\nu\Delta\theta + \theta(m-\theta) = 0 \qquad \text{in } \Omega, \qquad \partial_n\theta = 0 \qquad \text{on } \partial\Omega. \tag{7.1}$$

In addition, $0 < \theta(x) < m^*$ for every $x \in \bar\Omega$.

Also, for every $\alpha \geqslant 0$, there exists a unique solution $\bar U = \bar U(\alpha, \cdot)$ to

$$\nabla\cdot(\nabla\bar U - \alpha\bar U\nabla m) + (m - \bar U)\bar U = 0 \;\; \text{in} \;\; \Omega, \quad \partial_n\bar U - \alpha\bar U\partial_n m = 0 \;\text{on}\; \partial\Omega. \tag{7.2}$$

Proof. The proof can be found, for example, in [1, 3, 6]. Here for reader's convenience, we provide an alternative proof.

Non-negative solutions to (7.1) are given by critical points of the energy functional

$$\mathbf{E}[v] = \int_\Omega \Big\{\frac{\nu}{2}|\nabla v|^2 + \frac{1}{3}|v|^3 - \frac{m}{2}\,v^2\Big\}dx, \qquad v \in H^1(\Omega).$$

This energy is bounded from below, so in $H^1(\Omega)$ there exists at least a minimizer, say θ. Since $\mathbf{E}[\theta] = \mathbf{E}[|\theta|]$, $|\theta|$ is also a minimizer so we can assume that $\theta \geqslant 0$. Since $\int_\Omega m(x)dx > 0$, for sufficiently small positive ε, $\mathbf{E}[\varepsilon] = O(\varepsilon^3) - \frac{1}{2}\varepsilon^2\int_\Omega m(x)dx < 0$. Hence, the minimizer is non-trivial. It then follows from strong maximum principle [17] that $\theta > 0$. Thus, (7.1) admits a positive solution.

Suppose $\hat\theta$ is another positive solution and $\theta \not\equiv \hat\theta$. Then by exchange the role of θ and $\hat\theta$ if necessary, there exists a constant $k > 1$ and $y \in \bar\Omega$ such that

$$k\hat\theta \geqslant \theta \;\;\text{on}\;\; \Omega, \quad k\hat\theta(y) = \theta(y).$$

This implies that $\Delta(k\hat\theta - \theta)(y) = k(1-k)\hat\theta(y)^2 < 0$. But this contradicts maximum principle (recall $\partial_n\theta = \partial_n\hat\theta = 0$ on $\partial\Omega$). Hence, (7.1) admits a unique positive solution.

The assertion $\theta < m^*$ also follows from strong maximum principle.

To show the existence of a solution to (7.2), we consider the energy functional

$$\mathbf{E}[\alpha; u] = \int_\Omega \Big\{\frac{e^{\alpha m}}{2}|\nabla(e^{-\alpha m}u)|^2 + \frac{e^{-\alpha m}}{3}|u|^3 - \frac{me^{-\alpha m}}{2}u^2\Big\}.$$

Since the function $\alpha \in \mathbb{R} \to \int_\Omega me^{\alpha m}dx$ is strictly increasing, the assumption $\int_\Omega m(x)dx > 0$ implies that $\int_\Omega me^{\alpha m}dx > 0$ for every $\alpha \geqslant 0$. Hence, for small positive ε, the energy is negative for the test function $\varepsilon e^{\alpha m}$. Therefore, using a similar argument as above, one can show that (7.2) admits a unique positive solution $\bar U$. □

Lemma 7.2. *There exists a positive constant $\alpha_0 = \alpha_0(\nu, m)$ such that*

$$\int_\Omega [m-\theta]e^{\alpha m}dx > 0 \qquad \forall\,\alpha \geqslant \alpha_0. \tag{7.3}$$

Consequently, for every $\alpha \geqslant \alpha_0$, there exists a unique positive solution $\underline{U} = \underline{U}(\alpha, \cdot)$ to

$$\nabla\cdot[\nabla\underline{U} - \alpha\underline{U}\nabla m] + (m - \theta - \underline{U})\underline{U} = 0 \quad \text{in } \Omega, \quad \partial_n\underline{U} - \alpha\underline{U}\partial_n m = 0 \quad \text{on } \partial\Omega. \tag{7.4}$$

Proof. Since $\|\theta\|_\infty := \max_{\bar{\Omega}} \theta < m^*$, we see that

$$\lim_{\alpha\to\infty} \int_\Omega [m-\theta] e^{\alpha[m-\|\theta\|_\infty]} dx = \infty.$$

Hence, there exists $\alpha_0 > 0$ such that (7.3) holds for every $\alpha \geqslant \alpha_0$. By considering

$$\mathbf{E}[\alpha,\nu,u] = \int_\Omega \Big\{ \frac{e^{\alpha m}}{2} |\nabla(e^{-\alpha m}u)|^2 + \frac{e^{-\alpha m}}{3}|u|^3 - \frac{[m-\theta]e^{-\alpha m}}{2} u^2 \Big\},$$

the existence of a unique $\underline{U}$ follows the same proof as that for θ. ☐

Lemma 7.3. *Suppose that $\int_\Omega m(x)\,dx > 0$ and the set of critical points of m has Lebesgue measure zero. Then there exists a positive constant $\alpha_1 = \alpha_1(\nu, m)$ such that*

$$\int_\Omega [m(x) - \bar{U}(\alpha,x)]dx > 0 \qquad \forall\, \alpha \geqslant \alpha_1. \tag{7.5}$$

Consequently, for every $\alpha \geqslant \alpha_1$, there exists a unique positive solution $\underline{V}$ to

$$\nu\Delta \underline{V} + [m - \bar{U} - \underline{V}]\underline{V} = 0 \;\; in \;\; \Omega, \quad \partial_n \underline{V} = 0 \;\; on \;\; \partial\Omega.$$

Proof. It is shown in [5] that if the set of critical points of m has measure zero, then $\int_\Omega \bar{U} \to 0$ as $\alpha \to +\infty$. Since $\int_\Omega m(x)\,dx > 0$, there exists $\alpha_1 > 0$ such that (7.5) holds for $\alpha \geq \alpha_1$. The existence of a unique $\underline{V}$ follows from (7.5) and a similar argument as that for θ. ☐

Lemma 7.4. *Assume that the set of critical points of m has Lebesgue measure zero and $\int_\Omega m(x)dx > 0$. Then for every $\alpha \geqslant \alpha_1$ the elliptic system* (1.6) *admits at least one stable positive solution. In addition, every positive solution (U,V) to* (1.6) *satisfies*

$$\underline{V}(x) < V(x) < \theta(x), \quad \underline{U}(x) < U(x) < \bar{U}(x) \quad \forall\, x \in \bar{\Omega}.$$

Clearly, Theorem 3 is a direct consequence of the above Lemma.

Proof. Consider the parabolic system (1.5) with initial data

$$u(x,0) = \underline{U}(x), \qquad v(x,0) = \theta(x) \qquad \forall\, x \in \bar{\Omega}.$$

One can use comparison and maximum principle to show that

$$v_t(x,t) < 0 < u_t(x,t), \quad \underline{U}(x) \leqslant u(x,t) < \bar{U}(x), \quad \underline{V}(x) < v(x,t) \leqslant \theta(x) \quad \forall\, x \in \bar{\Omega}, t \geqslant 0.$$

Thus, the limit $\lim_{t\to\infty}(u(x,t), v(x,t))$ exists and is a positive solution to (1.6). In fact, one can observe that $(\bar{U},\theta)$ and $(\underline{U}, \underline{V})$ are a pair of supersolution and subsolution to (1.6), and the existence of a stable positive solution follow from supersolution and subsolution method and monotone system theory [15, 13, 7]. In fact, according to [14], (1.5) has at least one asymptotically stable positive steady state.

If (U,V) is a positive solution to (1.6), by comparison principle, one can prove, in order, that $V < \theta, U > \underline{U}$, $V > \underline{V}$, and $U < \bar{U}$. This completes the proof. ☐

The following result follows from Theorem 1:

Lemma 7.5. *Suppose that all critical points of* m *are non-degenerate. Let* $\lambda(\alpha)$ *denote the principal eigenvalue of*

$$\nabla\cdot[\nabla\psi-\alpha\psi\nabla m]+(m-\theta)\psi=-\lambda(\alpha)\psi \quad \text{in } \Omega, \quad \partial_n\psi-\alpha\psi\partial_n m=0 \quad \text{on } \partial\Omega. \tag{7.6}$$

Then

$$\lim_{\alpha\to\infty}\lambda(\alpha)=\min_{\mathcal{M}}[\theta-m].$$

The first assertion of Theorem 4 follows from the following lemma.

Lemma 7.6. *Suppose that all critical points of* m *are non-degenerate. Then for any* $\epsilon>0$*, there exists* $\alpha_2>0$ *such that if* $\alpha\geq\alpha_2$*, for any positive solution* (U,V) *we have*

$$\max_{\bar\Omega}U\geq\max_{\mathcal{M}}[m-\theta]-\epsilon.$$

Proof. Let $\psi>0$ be the eigenfunction of $\lambda(\alpha)$ uniquely determined by $\|\psi\|_\infty=1$. For any $\delta\in(0,\max_{\mathcal{M}}[m-\theta])$, define $U_*=\delta\psi$. Choose $\alpha_2>0$ such that $-\lambda(\alpha)\geq\delta$ for all $\alpha\geq\alpha_2$. Therefore, for all $\alpha\geq\alpha_2$, U_* satisfies

$$\nabla\cdot[\nabla U_*-\alpha U_*\nabla m]+U_*(m-\theta-U_*)=\delta\psi[-\lambda(\alpha)-\delta\psi]\geq\delta\psi[-\lambda(\alpha)-\delta]\geq 0. \tag{7.7}$$

This implies that U_* is a sub-solution of $\underline{U}$. By the subsolution and supersolution method [16] we see that $\underline{U}\geq U_*=\delta\psi$. Since $U\geq\underline{\mathrm{U}}$, choosing $\delta=\max_{\mathcal{M}}[m-\theta]-\epsilon$ we have

$$\max_{\bar\Omega}U\geq\max_{\bar\Omega}\underline{\mathrm{U}}\geq\delta\max_{\bar\Omega}\psi=\max_{\mathcal{M}}[m-\theta]-\epsilon.$$

□

7.2. **Asymptotic Behavior** (U,V)**.** In the sequel, we denote a generic positive solution to (1.6) by $(U(\alpha,x),V(\alpha,x))$. Also we assume that $\partial_n m\leq 0$ on $\partial\Omega$, m has only one critical point, denote by x_0, and x_0 satisfies $x_0\in\Omega$ and $D^2m(x_0)<0$. Then there exist positive constants κ, κ_1, κ_2 such that

$$|\nabla m(x)|\geqslant\kappa|x-x_0|, \qquad \kappa_2|x-x_0|^2\geqslant m^*-m(x)\geqslant\kappa_1|x-x_0|^2 \qquad \forall\, x\in\bar\Omega. \tag{7.8}$$

This property, together with $\partial_n m\leqslant 0$ on $\partial\Omega$ enables us to prove the following.

Lemma 7.7. *For every* $\varepsilon\in(0,\alpha_1/2]$*, there exists* $R_\varepsilon>0$ *such that for every* $\alpha\geqslant\alpha_1$*, there are* $z_*=z_*(\varepsilon,\alpha)$ *and* $z^*=z^*(\varepsilon,\alpha)$ *satisfying*

$$|z_*-x_0|\leqslant\tfrac{R_\varepsilon}{\sqrt{\alpha}}, \qquad |z^*-x_0|\leqslant\tfrac{R_\varepsilon}{\sqrt{\alpha}}, \tag{7.9}$$

$$U(\alpha,z_*)e^{[\alpha+\varepsilon][m(x)-m(z_*)]}\leqslant U(\alpha,x)\leqslant U(\alpha,z^*)e^{[\alpha-\varepsilon][m(x)-m(z^*)]} \qquad \forall\, x\in\bar\Omega. \tag{7.10}$$

In particular, there exists a positive constant K *such that for every* $\alpha\geqslant\alpha_1$*,*

$$U(\alpha,x)\leqslant Ke^{\alpha[m(x)-m^*]} \quad \forall\, x\in\bar\Omega. \tag{7.11}$$

Proof. Let $\varsigma > 0$ be a constant. Consider the function

$$w(x) = w(\varsigma, \alpha, x) = e^{-\alpha\varsigma m(x)}U(\alpha, x).$$

Then in Ω, w satisfies

$$\Delta w + [2\varsigma - 1]\alpha\nabla m \cdot \nabla w - \Big\{\varsigma(1-\varsigma)\alpha^2|\nabla m|^2 + (1-\varsigma)\alpha\Delta m + V + U - m\Big\}w = 0.$$

Let $\varepsilon \in (0, \alpha_1/2]$. First set $\varsigma = 1 - \varepsilon/\alpha$. Let $z^* = z^*(\varepsilon, \alpha) \in \bar{\Omega}$ be a point such that $w(z^*) = \max_{\bar{\Omega}} w$. Then the second inequality in (7.10) follows from the definition of w and z^*. To estimate z^*, we notice that $\partial_n w = (1-\varsigma)\alpha w \partial_n m \leqslant 0$ on $\partial\Omega$. It follows that no matter $z^* \in \Omega$ or $z^* \in \partial\Omega$, we always have $\nabla w(z^*) = \mathbf{0}$ and $\Delta w(z^*) \leqslant 0$. It then follows that

$$\varepsilon(\alpha-\varepsilon)|\nabla m(z^*)|^2 + \varepsilon\Delta m(z^*) + U(\alpha, z^*) + V(\alpha, z^*) \leqslant m(z^*). \tag{7.12}$$

Hence,

$$|z^* - x_0|^2 \leqslant \frac{|\nabla m(z^*)|^2}{\kappa^2} \leqslant \frac{m^* - \varepsilon\Delta m(z^*)}{\varepsilon(\alpha-\varepsilon)\kappa^2} \leqslant \frac{R_\varepsilon^2}{\alpha} \tag{7.13}$$

where

$$R_\varepsilon = \sqrt{\frac{2(m^*/\varepsilon + \|\Delta m\|_{C^0(\bar{\Omega})})}{\kappa^2}}.$$

In addition, from (7.12) we have

$$U(\alpha, z^*) \leqslant m(z^*) - \varepsilon\Delta m(z^*).$$

Also, from (7.8) and (7.12) we have

$$(\alpha-\varepsilon)[m^* - m(z^*)] \leqslant \frac{\kappa_2(\alpha-\varepsilon)}{\kappa^2}|\nabla m(z^*)|^2 \leqslant \frac{\kappa_2[m^*/\varepsilon - \Delta m(z^*)]}{\kappa^2}.$$

Fix $\varepsilon = \varepsilon_0 := \alpha_1/2$. We then see from the second inequality in (7.10) that

$$\begin{aligned} e^{-\alpha[m(x)-m^*]}U(\alpha, x) &\leqslant e^{-\alpha[m(x)-m^*]}U(\alpha, z^*)e^{(\alpha-\varepsilon_0)[m(x)-m(z^*)]} \\ &= U(\alpha, z^*)e^{\varepsilon_0(m^*-m(x))+(\alpha-\varepsilon_0)[m^*-m(z^*)]} \\ &\leqslant \Big(m^* + \varepsilon_0\|\Delta m\|_\infty\Big)e^{2\varepsilon_0\|m\|_\infty + \kappa_2[m^*/\varepsilon_0 + \|\Delta m\|_\infty]/\kappa^2} =: K \quad \forall\, x \in \bar{\Omega}. \end{aligned}$$

This implies (7.11).

Finally, set $\varsigma = 1 + \varepsilon/\alpha$ ($\varepsilon > 0$) and let $z_* \in \bar{\Omega}$ be a point such that $w(z_*) = \min_{\bar{\Omega}} w$. Then the first inequality in (7.10) holds. The estimate of z_* follow from a similar argument for that of z^*, using the extra fact that $V(\alpha, z_*) + U(\alpha, z_*) \leqslant m^* + K$. □

Lemma 7.8.

$$\lim_{\alpha\to\infty} \|V(\alpha; x) - \theta(x)\|_{C^{1+\beta}(\bar{\Omega})} = 0 \quad \forall\, \beta \in (0,1),$$

$$\lim_{\alpha\to\infty} \Big\| \frac{U(\alpha, x)}{U(\alpha, x_0)} - e^{\alpha[m(x)-m^*]} \Big\|_{L^\infty(\Omega)} = 0.$$

Proof. From (7.11), we see that as $\alpha \to \infty$, $U \to 0$ in $L^p(\Omega)$ for every $p > 1$. It then follows that $V \to \theta$ in $W^{2,p}$ for every $p > 1$ so $V \to \theta$ in $C^{1+\beta}$ for every $\beta \in (0, 1)$.

Denote $M(\alpha) = \|U(\alpha, \cdot)\|$. From the second inequality in (7.10) (with $\varepsilon = 1$) and the last inequality in (7.8) we see that

$$\begin{aligned} U(\alpha, x) &\leqslant U(\alpha, z^*)e^{(\alpha-1)[m(x)-m(z^*)]} \\ &= U(\alpha, z^*)e^{\alpha[m(x)-m^*]}e^{m^*-m(x)+(\alpha-1)[m^*-m(z^*)]} \leqslant U(\alpha, z^*)K_1e^{-\kappa_1\alpha|x-x_0|^2}. \end{aligned}$$

Now consider the function

$$W(\alpha, y) = \frac{U(\alpha, x_0 + y/\sqrt{\alpha})}{M(\alpha)}.$$

Then

$$\sup W = 1, \qquad W(y) \leqslant K_1e^{-\kappa_1|y|^2}$$

for every $y \in \{y \in \mathbb{R}^N : x_0 + y/\sqrt{\alpha} \in \Omega\}$. Also,

$$\Delta_y W + \vec{P} \cdot \nabla_y W + QW = 0$$

where

$$\begin{aligned} \vec{P} = P(\alpha, y) &= -\sqrt{\alpha} \cdot \nabla m(x_0 + y/\sqrt{\alpha}), \\ Q(\alpha, y) &= -\Delta m(x_0 + y/\sqrt{\alpha}) - \frac{(U + V - m)(x_0 + y/\sqrt{\alpha})}{\alpha}. \end{aligned}$$

The boundedness of U and V implies that

$$\lim_{\alpha\to\infty} \vec{P}(\alpha, y) = -yD^2m(x_0), \qquad \lim_{\alpha\to\infty} Q(\alpha, y) = -\Delta m(x_0)$$

uniformly in any compact subset of $\mathbb{R}^N$. Hence by elliptic regularity, passing to a subsequence if necessary, W converges to some function W^* uniformly in any compact subset of $\mathbb{R}^N$, where W^* satisfies

$$\begin{aligned} &\Delta_y W^* - yD^2m(x_0)\nabla_y W^* - \Delta m(x_0)W^* = 0 \ \text{ in } \ \mathbb{R}^N, \\ &\sup_{\mathbb{R}^N} W^*(y) = 1, \qquad W^*(y) \leqslant K_1e^{-\kappa_1|y|^2} \quad \forall\, y \in \mathbb{R}^N. \end{aligned}$$

It is then easy to see that

$$W^*(y) = e^{\frac{1}{2}yD^2m(x_0)y^T}.$$

The uniqueness of the limit implies that the whole sequence $W(\alpha, y)$ approaches W^* uniformly in any compact subset of $\mathbb{R}^N$.

That W^* attains its strict maximum at the origin implies that

$$\lim_{\alpha\to\infty} \frac{U(\alpha, x_0)}{M(\alpha)} = W^*(0) = 1.$$

The assertion of the Lemma thus follows from the fact that

$$\lim_{\alpha\to\infty} e^{\alpha[m(x_0+y/\sqrt{\alpha})-m^*]} = e^{\frac{1}{2}yD^2m(x_0)y^T}$$

uniformly for $y \in \{y \in \mathbb{R}^N : x_0 + y/\sqrt{\alpha} \in \Omega\}$. □

Lemma 7.9.

$$\lim_{\alpha\to\infty}\left\|U(\alpha,x)e^{\alpha[m^*-m(x)]}-U(\alpha,x_0)\right\|_{C^0(\bar\Omega)}=0.$$

Proof. Fix small positive ε. It follows from the second inequality in (7.10) that

$$\frac{U(\alpha,x)e^{-\alpha[m(x)-m^*]}}{U(\alpha,x_0)}\leqslant\frac{U(\alpha,z^*)e^{-\alpha[m(z^*)-m^*]}}{U(\alpha,x_0)}e^{2\varepsilon\|m\|_\infty}\quad\forall\, x\in\bar\Omega. \tag{7.14}$$

Since $|z_*-x_0|\leqslant\frac{R_\varepsilon}{\sqrt{\alpha}}$, by previous lemma we have

$$\lim_{\alpha\to\infty}\frac{U(\alpha,z^*)e^{\alpha[m(z^*)-m^*]}}{U(\alpha,x_0)}=1.$$

Now sending $\alpha\to\infty$ we see from (7.14) that

$$\overline{\lim_{\alpha\to\infty}}\max_{x\in\bar\Omega}\frac{U(\alpha,x)e^{-\alpha[m(x)-m^*]}}{U(\alpha,x_0)}\leqslant e^{2\varepsilon\|m\|_\infty}\lim_{\alpha\to\infty}\frac{U(\alpha,z^*)e^{\alpha[m(z^*)-m^*]}}{U(\alpha,x_0)}=e^{2\varepsilon\|m\|_\infty}.$$

Sending $\varepsilon\to0$ we then have

$$\overline{\lim_{\alpha\to\infty}}\max_{x\in\bar\Omega}\frac{U(\alpha,x)e^{-\alpha[m(x)-m^*]}}{U(\alpha,x_0)}\leqslant1.$$

Similarly, using the first inequality in (7.10) we can derive that

$$\underline{\lim_{\alpha\to\infty}}\min_{x\in\bar\Omega}\frac{U(\alpha,x)e^{-\alpha[m(x)-m^*]}}{U(\alpha,x_0)}\geqslant1.$$

This completes the proof. ☐

7.3. Calculation of $U(\alpha,x_0)$.

Lemma 7.10. *Let (U,V) be a positive solution to* (1.6). *Then*

$$\frac{\int_\Omega U^3e^{-\alpha[m-m^*]}dx}{\int_\Omega[m-V]U^2e^{-\alpha[m-m^*]}dx}\leqslant1\leqslant\frac{\int_\Omega Ue^{\alpha[m-m^*]}dx}{\int_\Omega[m-V]e^{\alpha[m-m^*]}dx}. \tag{7.15}$$

Proof. Multiplying the equation $\nabla\cdot(\nabla U-\alpha U\nabla m)+(m-V-U)U=0$ by $e^{-\alpha m}U$ and integrating the resulting equation over Ω we obtained the identity

$$0=\int_\Omega\Big\{e^{\alpha m}|\nabla(e^{-\alpha m}U)|^2+[U+V-m]U^2e^{-\alpha m}\Big\}\geqslant\int_\Omega U^3e^{-\alpha m}-\int_\Omega[m-V]U^2e^{-\alpha m}.$$

The first inequality in (7.15) thus follows.

Similarly, dividing $(U+V-m)U=\nabla\cdot(\nabla U-\alpha U\nabla m)$ by $Ue^{-\alpha m}$ and integrating we obtain

$$\int_\Omega e^{\alpha m}[U+V-m]dx=\int_\Omega\frac{\nabla\cdot[e^{\alpha m}\nabla(e^{-\alpha m}U)]}{e^{-\alpha m}U}=\int_\Omega\frac{e^{\alpha m}|\nabla(e^{-\alpha m}U)^2|}{e^{-2\alpha m}U^2}>0.$$

This implies the second inequality in (7.15). ☐

Lemma 7.11.

$$\lim_{\alpha\to\infty}U(\alpha,x_0)=2^{N/2}\Big\{m(x_0)-\theta(x_0)\Big\}.$$

Proof. First of all notice that there exists a constant $K > 0$ such that

$$U(\alpha, x) \leqslant K U(\alpha, x_0) e^{\alpha[m(x)-m^*]} \qquad \forall\, x \in \bar{\Omega}, \quad \alpha \geqslant \alpha_1.$$

Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} & \lim_{\alpha\to\infty} \alpha^{N/2} \int_\Omega \frac{U^3(\alpha,x)}{U^3(\alpha,x_0)} e^{-\alpha[m(x)-m^*]} dx \\ = & \lim_{\alpha\to\infty} \int_{\{\Omega - x_0\}\sqrt{\alpha}} \frac{U^3(x_0 + y/\sqrt{\alpha})}{U^3(\alpha,x_0)} e^{-\alpha[m(x_0+y/\sqrt{\alpha})-m(x_0)]} dy \\ = & \int_{\mathbb{R}^N} e^{yD^2m(x_0)y^T} dy. \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha^{N/2} \int_\Omega [m-V] \frac{U^2(\alpha,x)}{U^2(\alpha,x_0)} e^{-\alpha[m-m^*]} dx &= [m(x_0) - \theta(x_0)] \int_{\mathbb{R}^N} e^{\frac{1}{2} yD^2m(x_0)y^T} dy, \\ \lim_{\alpha\to\infty} \alpha^{N/2} \int_\Omega \frac{U(\alpha,x)}{U(\alpha,x_0)} e^{\alpha[m-m^*]} dx &= \int_{\mathbb{R}^N} e^{yD^2m(x_0)y^T} dy, \\ \lim_{\alpha\to\infty} \alpha^{N/2} \int_\Omega [m-V] e^{\alpha[m-m^*]} dx &= [m(x_0) - \theta(x_0)] \int_{\mathbb{R}^N} e^{\frac{1}{2} yD^2m(x_0)y^T} dy. \end{aligned}$$

It then follows from the first inequality of (7.15) that

$$\frac{\int_{\mathbb{R}^N} e^{yD^2m(x_0)y^T} dy}{[m(x_0)-\theta(x_0)] \int_{\mathbb{R}^N} e^{\frac{1}{2} yD^2m(x_0)y^T} dy} \limsup_{\alpha\to\infty} U(\alpha, x_0) \leqslant 1.$$

From the second inequality of (7.15) we have

$$1 \leqslant \frac{\int_{\mathbb{R}^N} e^{yD^2m(x_0)y^T} dy}{[m(x_0)-\theta(x_0)] \int_{\mathbb{R}^N} e^{\frac{1}{2} yD^2m(x_0)y^T} dy} \liminf_{\alpha\to\infty} U(\alpha, x_0).$$

The assertion of the Lemma thus follows. ☐

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