PROBLEMS FOR THE NILPOTENT GROUPS CLASS

(1) (**E**) For a Γ be a finitely generated nilpotent group, show that $\Gamma^{(i)} \lhd \Gamma^{(i-1)}$, that $\Gamma^{(i)} \lhd \Gamma$. Also show that Γ is polycyclic (and hence solvable).

(2) (a) (**E**) Show that every element of H_3 can be written as $a^l b^m c^n$ for some $l, m, n \in \mathbb{Z}$.
(b) (**E**) Show that the two descriptions of H_3 are in fact isomorphic.

(3) (**M**) Let Γ be nilpotent of class k. Then $\Gamma^{(k)}$ is in the center of Γ. Is $\Gamma^{(k)}$ equal to the center?

(4) (a) (**E**) Describe how to construct the Cayley graph of H_3 starting with the integer points in $\mathbb{R}^3$.
(b) (**E**) Complete the argument that the Dehn function of H_3 is bounded below by a function $\simeq n^3$
(c) (**E**) Show that every word of length n in the generators which represents the identity in H_3 can be expressed as a product of conjugates of at most n^3 relators. (This is just a more careful version of 2a)

Note that (b) and (c) show that the Dehn function of H_3 is $\simeq n^3$

(5) (**E/M**) Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Show that the map $\varphi : \mathbb{R}^n \to G$ defined by

$$\varphi(s_1, \ldots, s_n) = \exp(s_1 X_1) \cdots \exp(s_n X_n) \qquad \text{(i.e. the product in } G \text{ of the } s_i X_i\text{)}$$

is a polynomial coordinate map.

Use the following:
Given any Lie group G with $\exp : \mathfrak{g} \to G$, define $X * Y := \log(\exp X \cdot \exp Y)$. (It is well defined near $X = Y = 0$). The Baker-Campbell-Hausdorff formula says:

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \text{comms in } \geq 4 \text{ terms}$$

For nilpotent Lie groups exp is a diffeomorphism and the Baker-Campbell-Hausdorff formula holds for all X, Y in $\mathfrak{g}$.

(continued)

(6) (a) (**E**) Use the Ball-Box Theorem to show that for a simply connected nilpotent Lie group with a polynomial coordinate map, we have

$$||(s_1, ..., s_n)||_G \sim \max\{|s_i|^{1/w_i}\},$$

where w_i is the weight of the ith basis vector. (Say $f \sim g$ if there exists C such that $(1/C)f(n) \leq g(n) \leq Cf(n)$ for all n.)

(b) (**H**) Show that if H is a Lie subgroup of G, then

$$\text{distortion}_H^G(r) \simeq r^d, \text{ where } d = \max_{h \in H} \frac{w_G(h)}{w_H(h)}$$

(Here w_G and w_H denote the weights of h in G and H respectively.)
Remark: This is a result of Osin, but he doesn't use the Ball-Box Theorem to prove it.

(7) (a) (**E**) Let F be the free group on n generators. Show that $F_{n,k} = F/F^{(k+1)}$ is nilpotent of class k. This is the *free nilpotent group of class k with n generators.*
(b) (**E**) Show that every nilpotent group is a quotient of some free nilpotent group.
(c) (**M**) Show that H_3 is a free nilpotent group.
(d) (**?**) Embed $F_{n,k}$ in T_m for some m.

(8) (**O**) Is the Dehn function of a finitely generated nilpotent group $\simeq$ a polynomial?