Problems for
"Groups Associated to Knots and 3-Manifolds, Part I"

1. Let G be a finitely presented group.

   (a) (Medium) Prove that, for all $n \geq 4$, there is a compact, connected, orientable n-dimensional manifold X with $\pi_1(X) \cong G$.

   (b) (Hard) Prove that, for all $n \geq 4$, there is a **closed (compact without boundary)**, connected, orientable n-dimensional manifold X with $\pi_1(X) \cong G$.

2. (Medium) Suppose M is a **prime**, closed, orientable, connected 3-manifold. Prove that $M \cong S^1 \times S^2$ or M is irreducible (every embedded 2-sphere in M bounds a 3-dimensional ball embedded in M).

3. (Easy except part c) Show that there are 3-manifolds (non-prime) with the same fundamental group (which is of infinite order) but that are not homeomorphic by completing the following problems.

   (a) Let M_i be an oriented 3-manifold for $i = 1, \ldots, n$. Prove that

$$\pi_1(M_1 \# \cdots \# M_n) \cong \pi_1(M_1) * \cdots * \pi_1(M_n).$$

   (b) Let P be a prime, closed, **oriented**, connected 3-manifold with no orientation reversing self-homeomorphism. Prove that $P \# P$ is not homeomorphic to $P \# -P$ (by $-P$, we mean P with the opposite orientation).

   (c) (Hard) Show that the Lens space $L(3, 1)$ has no orientation reversing self-homeomorphism.

4. (Easy) Use the following steps to complete the proof that $\pi_1(S^1 \times S^2 \# \cdots \# S^1 \times S^2)$ is residually finite. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

   (a) Define regions in the plane R_α, R_β, $R_{\overline{\alpha}}$, $R_{\overline{\beta}}$ such that $\alpha(R_\alpha \cup R_\beta \cup R_{\overline{\beta}}) \subset R_\alpha$, $\alpha^{-1}(R_{\overline{\alpha}} \cup R_\beta \cup R_{\overline{\beta}}) \subset R_{\overline{\alpha}}$, $\beta(R_\alpha \cup R_\beta \cup R_{\overline{\alpha}}) \subset R_\beta$ and $\beta^{-1}(R_{\overline{\alpha}} \cup R_\alpha \cup R_{\overline{\beta}}) \subset R_{\overline{\beta}}$.

   (b) Let w be a reduced word in α and β. Show that $w(R_c) \not\subset R_c$ for some $c \in \{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$.

   (c) Show that the free group on two generators is residually finite by mapping the above matrices to matrices with entries in $\mathbb{Z}/p\mathbb{Z}$.

   (d) Prove that $\pi_1(S^1 \times S^2 \# \cdots \# S^1 \times S^2)$ is residually finite where we are taking the connected sum of n copies of $S^1 \times S^2$ (you may also have to use 3a).

5. (Medium/Hard) Let M be a closed, orientable, connected prime 3-manifold with $|\pi_1(M)| = \infty$. Show that $\pi_1(M)$ is torsion free (i.e. there are no elements of finite order).