

# THE $\ell^2$ -COHOMOLOGY OF ARTIN GROUPS

M. W. DAVIS AND I. J. LEARY

ABSTRACT. For each Artin group we compute the reduced  $\ell^2$ -cohomology of (the universal cover of) its “Salvetti complex”. This is a CW-complex which is conjectured to be a model for the classifying space of the Artin group. In the many cases when this conjecture is known to hold our calculation describes the reduced  $\ell^2$ -cohomology of the Artin group.

## 1. INTRODUCTION

Suppose that  $G$  is a countable discrete group acting properly and cellularly on a CW-complex  $E$  with compact quotient. As in [14] or [16] one can define the reduced  $\ell^2$ -cohomology group  $\mathcal{H}^i(E)$  for each non-negative integer  $i$ . (We will give the definition in Section 3.) For each  $i$ ,  $\mathcal{H}^i(E)$  is a Hilbert space with an orthogonal  $G$ -action. When  $G$  is infinite, these Hilbert spaces tend to be infinite-dimensional unless they are zero. However, each  $\mathcal{H}^i(E)$  is a “Hilbert  $G$ -module”, and to any Hilbert  $G$ -module  $V$  one may associate a real number, called its von Neumann dimension and denoted by  $\dim_G(V)$ . One can then define the  $\ell^2$ -Betti numbers of  $E$  with respect to  $G$  by

$$\ell^2 b_i(E; G) := \dim_G \mathcal{H}^i(E).$$

If  $E$  is contractible and  $G$  acts freely on  $E$  (so that  $E/G$  is a classifying space for  $G$ ), these  $\ell^2$ -cohomology groups (and their von Neumann dimensions) are independent of the choice of  $E$  and so are invariants of  $G$ . In this case we will use the notation  $\mathcal{H}^i(G)$  and  $\ell^2 b_i(G)$  to stand for  $\mathcal{H}^i(E)$  and  $\ell^2 b_i(E; G)$  respectively.

$\ell^2$ -cohomology groups have proved difficult to compute directly. To date, most of the calculations have been based on two principles: (i) certain vanishing theorems, and (ii) Atiyah’s formula. A typical vanishing

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*Date:* February 19, 2002.

2000 *Mathematics Subject Classification.* Primary: 57M07; Secondary: 20F36, 20J05.

The first author was partially supported by NSF grant DMS-0104026. The second author acknowledges support from EPSRC (via grant GR/R07813), and from the Ohio State Mathematical Research Institute.

theorem will assert that for a certain class of groups  $G$ ,  $\mathcal{H}^i(G)$  is zero for  $i$  within a certain range. Atiyah's formula states that in the case when  $G$  acts freely on  $E$ , the alternating sum of the  $\ell^2$ -Betti numbers of  $G$  is equal to the ordinary Euler characteristic of  $E/G$ . In cases where  $\mathcal{H}^*(E)$  is known to vanish except in one dimension, this formula provides an exact calculation of the unique non-zero  $\ell^2$ -Betti number. In this paper we compute the  $\ell^2$ -Betti numbers  $\ell^2 b_i(G)$  when  $G$  is an Artin group. Surprisingly, there is a completely clean answer which is non-trivial in the sense that many of the  $\ell^2 b_i(G)$  are non-zero.

Associated to any Coxeter matrix  $(m_{ij})$  on a finite set  $I$ , there is a Coxeter group  $W$ , an Artin group  $A$  and a simplicial complex  $L$  with vertex set  $I$ , called the “nerve” of the Coxeter matrix. (The definitions will be given in Section 5.) Coxeter groups are intimately connected to groups generated by reflections. For example, if  $n = \text{Card}(I)$ , there is a standard representation of  $W$  as a linear reflection group on  $\mathbb{R}^n$ , so that there is a certain convex cone in  $\mathbb{R}^n$  with  $W$  acting properly on its interior  $\Omega$ .

Artin groups have a related geometric interpretation. Consider the complexification of the  $W$ -representation described above. Then  $\mathbb{R}^n + i\Omega$  is a  $W$ -stable open convex subset of  $\mathbb{C}^n$  on which  $W$  acts properly. Moreover,  $W$  acts freely on the hyperplane complement

$$Y = (\mathbb{R}^n + i\Omega) - \bigcup_r H_r,$$

where  $r$  ranges over all reflections in  $W$  and  $H_r$  denotes the intersection of the complex hyperplane fixed by  $r$  and  $(\mathbb{R}^n + i\Omega)$ . It is proved in [7] that there is a finite CW-complex  $X$ , called the “Salvetti complex” which is homotopy equivalent to  $Y/W$ . The complex  $X$  has one vertex and, for each  $k > 0$ , one  $k$ -cell for each  $(k - 1)$ -simplex of  $L$ . The 2-skeleton of  $X$  is the presentation complex associated to the standard presentation of  $A$ . It follows that  $\pi_1(Y/W) = \pi_1(X) = A$ . It is conjectured that the universal cover  $\tilde{X}$  of  $X$  is contractible, or equivalently that  $X$  is a classifying space  $BA$  for  $A$ . Deligne [12] proved this conjecture in the case when  $W$  is finite. The conjecture is also known to be true when  $W$  is “right-angled”, when  $\dim L \leq 1$ , and in many other cases (see [6, 7]). Our actual calculation is of  $\mathcal{H}^i(\tilde{X})$ . So our title is somewhat misleading: we only have a computation of the reduced  $\ell^2$ -cohomology of  $A$  modulo the conjecture that  $X = BA$ .

**Theorem 1.** *For any Artin group  $A$  and any  $i$ , there is an isomorphism of Hilbert  $A$ -modules:*

$$\mathcal{H}^i(\tilde{X}) \cong \overline{H}^{i-1}(L) \otimes \ell^2(A).$$

**Corollary 2.**  $\ell^2 b_i(\tilde{X}; A) = \bar{b}_{i-1}(L)$ .

Here  $\bar{b}_j(L)$  denotes the ordinary “reduced Betti number” of  $L$ , i.e.,  $\bar{b}_j(L)$  is the dimension of the reduced homology group  $\bar{H}_j(L; \mathbb{R})$ .

When  $W$  is finite, the Artin group  $A$  is said to be “spherical”. If  $A$  is spherical, then its centre always has an infinite cyclic subgroup (coming from the  $\mathbb{C}^*$ -action on the hyperplane complement). For such groups the reduced  $\ell^2$ -cohomology is known to vanish (see Section 4). Hence, in the spherical case, Theorem 1 is well-known. This vanishing result in the spherical case is the key ingredient for our proof in the general case.

In fact, a stronger result than Theorem 1 is true. In the following statement, the “standard homomorphism” from an Artin group to  $\mathbb{Z}$  is the homomorphism that sends each Artin generator to  $1 \in \mathbb{Z}$ .

**Theorem 3.** *Suppose that  $A'$  is a normal subgroup of the Artin group  $A$  that is contained in the kernel of the standard homomorphism  $A \rightarrow \mathbb{Z}$ . Let  $X' \rightarrow X$  be the corresponding covering space of  $X$  and let  $G = A/A'$ . Then  $\mathcal{H}^i(X')$  and  $\bar{H}^{i-1}(L) \otimes \ell^2(G)$  are isomorphic as Hilbert  $G$ -modules. In particular,  $\ell^2 b_i(X'; G) = \bar{b}_{i-1}(L)$ .*

Theorem 3 was suggested by a result of [15].

*Acknowledgement.* The authors wish to thank Tadeusz Januszkiewicz, Wolfgang Lück and Thomas Schick for numerous helpful conversations concerning the work in this paper.

## 2. HILBERT $G$ -MODULES

Let  $G$  be a countable discrete group. The Hilbert space of square-summable real-valued functions on  $G$  is denoted by  $\ell^2(G)$ . There is an orthogonal (right) action of  $G$  on  $\ell^2(G)$ , and a diagonal action on any direct sum of copies of  $\ell^2(G)$ . A Hilbert space  $V$  equipped with an orthogonal(right)  $G$ -action is called a (finitely generated) *Hilbert  $G$ -module* if it is  $G$ -equivariantly isomorphic to a  $G$ -stable closed subspace of the direct sum of finitely many copies of  $\ell^2(G)$ . A map  $f : V \rightarrow V'$  of Hilbert  $G$ -modules is by definition a  $G$ -equivariant bounded linear operator. The kernel of  $f : V \rightarrow V'$ , denoted  $\text{Ker}(f)$ , is a closed subspace of  $V$ , but the image,  $\text{Im}(f)$ , need not be closed. The map  $f$  is *weakly surjective* if  $\overline{\text{Im}(f)} = V$ ;  $f$  is a *weak isomorphism* if it is both injective and weakly surjective.

To each Hilbert  $G$ -module  $V$  it is possible to associate a non-negative real number  $\dim_G(V)$ , called its *von Neumann dimension*. The three most important properties of  $\dim_G(-)$  are:

- (1)  $\dim_G(V) = 0$  if and only if  $V = 0$ ;
- (2)  $\dim_G(V \oplus V') = \dim_G(V) + \dim_G(V')$ ;
- (3)  $\dim_G(\ell^2(G)) = 1$ .

It follows from (2) that for any map  $f : V \rightarrow V'$ ,

$$\dim_G(V) = \dim_G(\text{Ker}(f)) + \dim_G(\overline{\text{Im}(f)}).$$

Using property (1), one can then deduce the following.

**Lemma 4.** *Suppose that  $V$  and  $V'$  are Hilbert  $G$ -modules and that  $\dim_G(V) = \dim_G(V')$ . The following are equivalent:*

- (a)  $f$  is injective;
- (b)  $f$  is weakly surjective;
- (c)  $f$  is a weak isomorphism.

*Proof.* By definition, (c) is equivalent to the conjunction of (a) and (b). Thus it suffices to show that (a) and (b) are equivalent. But we have that  $\dim_G(\overline{\text{Im}(f)}) = \dim_G(V) - \dim_G(\text{Ker}(f))$ , and so  $\dim_G(\text{Ker}(f)) = 0$  if and only if  $\dim_G(\overline{\text{Im}(f)}) = \dim_G(V) = \dim_G(V')$ .  $\square$

Next suppose that  $C^* = \{(C^n, \delta_n)\}$  is a cochain complex of Hilbert  $G$ -modules, i.e., the coboundary maps  $\delta_i : C^i \rightarrow C^{i+1}$  are  $G$ -equivariant bounded linear operators. The cohomology group

$$H^i(C^*) := \text{Ker}(\delta_i) / \text{Im}(\delta_{i-1})$$

need not be a Hilbert space since  $\text{Im}(\delta_{i-1})$  is not necessarily closed in  $\text{Ker}(\delta_i)$ . However, the *reduced cohomology groups* defined as

$$\mathcal{H}^i(C^*) := \text{Ker}(\delta_i) / \overline{\text{Im}(\delta_{i-1})}$$

are Hilbert  $G$ -modules.

**Lemma 5.** *Suppose that  $(C^*, \delta_*)$  and  $(C'^*, \delta'_*)$  are cochain complexes of Hilbert  $G$ -modules and that  $f : C^* \rightarrow C'^*$  is a weak isomorphism of cochain complexes (i.e.,  $f$  is a cochain map and for each  $i$ ,  $f_i : C^i \rightarrow C'^i$  is a weak isomorphism). Then the induced map  $\mathcal{H}(f) : \mathcal{H}^i(C^*) \rightarrow \mathcal{H}^i(C'^*)$  is also a weak isomorphism. In particular,  $\mathcal{H}^i(C^*)$  and  $\mathcal{H}^i(C'^*)$  are isometric Hilbert  $G$ -modules.*

*Proof.* Let  $Z^i$  and  $B^i$  (resp.  $Z'^i$  and  $B'^i$ ) denote the cocycles and coboundaries in  $C^*$  (resp.  $C'^*$ ). Since  $f$  is a cochain map,  $f(Z^i) \subseteq Z'^i$  and  $f(B^i) \subseteq B'^i$ . Moreover, since  $f$  is continuous,  $f(\overline{B^i}) \subseteq \overline{B'^i}$ . Since  $f$  is injective this gives

$$\dim_G(Z'^i) \geq \dim_G(Z^i) \quad \text{and} \quad \dim_G(B'^i) \geq \dim_G(B^i). \quad (1)$$

From the short exact sequences

$$0 \rightarrow Z^i \rightarrow C^i \rightarrow B^{i+1} \rightarrow 0$$

$$0 \rightarrow Z^i \rightarrow C^i \rightarrow B^{i+1} \rightarrow 0$$

one obtains

$$\dim_G(C^i) = \dim_G(Z^i) + \dim_G(B^{i+1}), \quad \text{and}$$

$$\dim_G(C'^i) = \dim_G(Z'^i) + \dim_G(B'^{i+1}) \quad (2).$$

Since  $f$  is a weak isomorphism, one has that  $\dim_G(C'^i) = \dim_G(C^i)$ . Hence

$$\begin{aligned} \dim_G(C^i) &= \dim_G(C'^i) = \dim_G(Z'^i) + \dim_G(B'^{i+1}) \\ &\geq \dim_G(Z^i) + \dim_G(B^{i+1}) = \dim_G(C^i), \end{aligned}$$

which together with (1) implies that

$$\dim_G(Z'^i) = \dim_G(Z^i) \quad \text{and} \quad \dim_G(B'^i) = \dim_G(B^i). \quad (3)$$

It follows from Lemma 4 that the maps  $f : Z^i \rightarrow Z'^i$  and  $f : B^i \rightarrow B'^i$  are weak isomorphisms. This implies that  $\mathcal{H}(f) : \mathcal{H}^i(C^*) \rightarrow \mathcal{H}^i(C'^*)$  is weakly surjective.

To show that  $\mathcal{H}(f)$  is injective, let  $H$  be the orthogonal complement to  $\overline{B^i}$  in  $Z^i$ . Note that  $H$  is closed and  $G$ -stable. Now

$$\begin{aligned} \dim_G(Z'^i) &= \dim_G(\overline{f(H + \overline{B^i})}) \\ &= \dim_G(\overline{f(H)}) + \dim_G(\overline{B'^i}) - \dim_G(\overline{f(H)} \cap \overline{B'^i}) \\ &= \dim_G(Z'^i) - \dim_G(\overline{f(H)} \cap \overline{B'^i}), \end{aligned}$$

where the third equality follows from (3). Hence  $\dim_G(\overline{f(H)} \cap \overline{B'^i}) = 0$ , and therefore  $\overline{f(H)}$  is a complementary subspace for  $\overline{B'^i}$ . Since  $f$  is injective when restricted to  $H$ , it follows that so is  $\mathcal{H}(f)$ . (This follows from the fact that any two weakly isomorphic Hilbert  $G$ -modules are in fact equivariantly isometric [14].)  $\square$

### 3. COHOMOLOGY WITH LOCAL COEFFICIENTS AND $\ell^2$ -COHOMOLOGY

Suppose that  $Y$  is a connected CW-complex with fundamental group  $\pi$  and universal cover  $\tilde{Y}$ . Suppose also that  $Y$  is of finite type (i.e., has finite skeleta.) Let  $C_*(\tilde{Y})$  denote the chain complex of  $\tilde{Y}$ , which is a chain complex of finitely generated free (left)  $\mathbb{Z}\pi$ -modules. By a *local*

*coefficient system on  $Y$*  we shall mean a (left)  $\mathbb{Z}\pi$ -module. The cochain complex for  $Y$  with coefficients  $M$  is defined by

$$C^*(Y; M) := \text{Hom}_{\mathbb{Z}\pi}(C_*(\tilde{Y}), M).$$

Let  $Y' \rightarrow Y$  be a regular covering space corresponding to a normal subgroup  $\pi'$  of  $\pi$ , and let  $G$  denote the quotient group  $\pi/\pi'$ . Then  $\ell^2(G)$  may be viewed as a  $\pi$ - $G$ -bimodule, where the left action of  $\pi$  is via the action of  $G$ . The  $\ell^2$ -cochains on  $Y'$  are defined by

$$\begin{aligned} \ell^2 C^*(Y') &:= \text{Hom}_{\mathbb{Z}G}(C_*(Y'), \ell^2(G)) \\ &= \text{Hom}_{\mathbb{Z}\pi}(C_*(\tilde{Y}), \ell^2(G)) \\ &= C^*(Y; \ell^2(G)). \end{aligned}$$

Note that the orthogonal right action of  $G$  on  $\ell^2(G)$  makes  $\ell^2 C^*(Y')$  a chain complex of Hilbert  $G$ -modules. The unreduced and reduced cohomology groups of  $\ell^2 C^*(Y')$  will be denoted  $\ell^2 H^*(Y')$  and  $\mathcal{H}^*(Y')$  respectively. The  $\ell^2$ -Betti numbers of  $Y'$  with respect to  $G$  are then defined by  $\ell^2 b_i(Y'; G) := \dim_G \mathcal{H}^i(Y')$ .

In the case when  $Y'$  is acyclic,  $C_*(Y')$  is a free resolution for  $\mathbb{Z}$  over  $\mathbb{Z}G$ . Any two such resolutions are chain homotopy equivalent. Similarly, if  $Y'$  is only rationally acyclic, the rational chain complex  $C_*(Y', \mathbb{Q})$  is a free resolution for  $\mathbb{Q}$  over  $\mathbb{Q}G$ , and is unique up to chain homotopy equivalence. It follows that in the case when  $Y'$  is rationally acyclic, the  $\ell^2$ -cohomology groups and  $\ell^2$ -Betti numbers depend only on  $G$  (and not on the particular choice of  $Y'$ ). In this case we shall use the notation

$$\begin{aligned} \ell^2 H^i(G) &:= \ell^2 H^i(Y'), \\ \mathcal{H}^i(G) &:= \mathcal{H}^i(Y'), \\ \ell^2 b_i(G) &:= \dim_G \mathcal{H}^i(Y'). \end{aligned}$$

(Of course, we have in mind the case when  $G = \pi$  and  $Y' = \tilde{Y}$  is contractible.)

#### 4. A VANISHING THEOREM

Let  $G$  be a group and  $Z$  a central subgroup of  $G$ . Any (left)  $G$ -module  $M$  may be viewed as a  $G$ - $Z$ -bimodule where the right action of  $z \in Z$  is defined by  $mz = zm$ . This induces a right action of  $Z$  on  $H^*(G; M)$ . However one has:

**Proposition 6.** *With notation as above, the action of  $Z$  on  $H^*(G; M)$  is trivial.*

*Proof.* Let  $P_*$  be a free resolution for  $\mathbb{Z}$  over  $\mathbb{Z}G$ , so that  $H^*(G; M)$  may be computed using the cochain complex  $\text{Hom}_G(P_*, M)$ . In terms of this cochain complex the right action of  $z \in Z$  on  $H^*(G; M)$  sends  $f \in \text{Hom}_G(P_i, M)$  to  $zf$ , defined by  $zf(p) = z(f(p))$  for all  $p \in P_i$ . It suffices to show that this map is chain-homotopic to the identity map. Since  $z$  is central, the map  $P_* \rightarrow P_*$  given by  $p \mapsto zp$  is a chain map, and this chain map lifts the identity map from  $\mathbb{Z}$  to  $\mathbb{Z}$ , so is chain-homotopic to the identity map of  $P_*$ . But for all  $f \in \text{Hom}_G(P_*, M)$  and all  $p \in P_*$ ,  $zf(p) = f(zp)$ , and so the two actions of  $Z$  (on  $M$  and on  $P_*$ ) induce the same action on  $\text{Hom}_G(P_*, M)$ .  $\square$

**Corollary 7.** *Let  $\pi$  be a group admitting a finite classifying space  $Y$ , let  $\pi'$  be a normal subgroup of  $\pi$ , let  $G = \pi/\pi'$ , and suppose that the centre of  $\pi$  contains an element whose image in  $G$  has infinite order. Then  $\mathcal{H}^*(Y') = \{0\}$ , where  $Y'$  denotes the regular cover of  $Y$  corresponding to  $\pi'$ .*

*Proof.* Let  $Z$  denote an infinite cyclic subgroup of the centre of  $\pi$  with  $Z \cap \pi' = \{1\}$ . The usual right action of  $Z$  on  $\ell^2(G)$  and the action used in Proposition 6 agree, and hence the right action of  $Z$  on  $\ell^2 H^*(Y')$  is trivial. It follows that the action of  $Z$  on  $\mathcal{H}^*(Y')$  is trivial. Since the image of  $Z$  in  $G$  is infinite, no non-zero element of  $\ell^2(G)$  is fixed by the right action of  $Z$ . However, each  $\mathcal{H}^i(Y')$  is a Hilbert  $G$ -module, and so one obtains a contradiction unless each  $\mathcal{H}^i(Y') = \{0\}$ .  $\square$

**Corollary 8.** *Suppose that  $G$  has a finite classifying space and that the centre of  $G$  contains an element of infinite order. Then the  $\ell^2$  cohomology of  $G$  vanishes.*

*Proof.* This is just the case  $G = \pi$  of Corollary 7.  $\square$

Other more general vanishing theorems for  $\ell^2$ -cohomology may be found in [8, 16, 19, 20].

## 5. COXETER GROUPS AND ARTIN GROUPS

Let  $I$  be a finite set. A *Coxeter matrix*  $M = (m_{ij})$  on  $I$  is an  $I$ -by- $I$  symmetric matrix with entries in  $\mathbb{N} \cup \{\infty\}$  such that each  $m_{ii} = 1$  and such that whenever  $i \neq j$ ,  $m_{ij} \geq 2$ . Associated to  $M$  there is a *Coxeter group* denoted by  $W_I$  or  $W$  with generating set  $\{s_i : i \in I\}$ , with relations

$$(s_i s_j)^{m_{ij}} = 1 \quad \text{for all } (i, j) \in I \times I.$$

There is also an *Artin group*  $A_I$  (or  $A$ ) with generators  $\{a_i : i \in I\}$  and with a presentation given by the relations

$$a_i a_j \cdots = a_j a_i \cdots$$

for all  $i \neq j$ , where there are  $m_{ij}$  terms on each side of the equation. In both the Coxeter and Artin cases, the relation is interpreted as being vacuous if  $m_{ij} = \infty$ .

Since each of the Artin relators contains the same number of copies of a generator on each side, there is a homomorphism from any Artin group to the infinite cyclic group  $\mathbb{Z}$ , defined by sending each Artin generator to 1. Call this the *standard homomorphism* from  $A$  to  $\mathbb{Z}$ .

Let  $q : A \rightarrow W$  denote the natural homomorphism sending  $a_i$  to  $s_i$ . There is a set-theoretic section for  $q$  denoted  $w \mapsto a_w$  which may be defined as follows: if  $s_{i_1} \cdots s_{i_n}$  is any word of minimal length for  $w$  in terms of the  $s_i$ , then  $a_w := a_{i_1} \cdots a_{i_n}$ . As explained in [6, p. 602], it follows from Tits' solution to the word problem for Coxeter groups that  $w \mapsto a_w$  is well-defined.

Given a subset  $J$  of  $I$ , let  $M_J$  denote the minor of  $M$  whose rows and columns are indexed by  $J$ , and let  $W_J$  (resp.  $A_J$ ) be the corresponding Coxeter group (resp. Artin group). It is known (cf. [2]) that the natural map  $W_J \rightarrow W_I$  (resp.  $A_J \rightarrow A_I$ ) is injective, and hence  $W_J$  (resp.  $A_J$ ) can be identified with the subgroup of  $W_I$  (resp.  $A_I$ ) generated by  $\{s_i : i \in J\}$  (resp.  $\{a_i : i \in J\}$ ).

Say that the subset  $J \subseteq I$  is *spherical* if  $W_J$  is finite. If this is the case, then the groups  $W_J$  and  $A_J$  are both called *spherical*. (Note that this differs from the terminology of [6], where spherical Artin groups were said to be "of finite type".) The poset of spherical subsets of  $I$  is denoted by  $\mathcal{S}(I)$  or simply  $\mathcal{S}$ . The subposet of non-empty spherical subsets is an abstract simplicial complex which will be denoted  $L$ , and called the *nerve* of  $M$  (or the nerve of  $(W, \{s_i : i \in I\})$ ). Thus the vertex set of  $L$  is  $I$  and a subset  $\sigma$  of  $I$  spans a simplex of  $L$  if and only if  $\sigma \in \mathcal{S}$  if and only if  $W_\sigma$  is finite. (Greek letters such as  $\sigma$  or  $\tau$  will be used to denote spherical subsets of  $I$  when viewed as simplices of  $L$ .)

*Example 1.* If  $W$  is finite, then  $L$  is a simplex of dimension  $\text{Card}(I) - 1$ .

Suppose now that  $m_{ij} = \infty$  for all  $i \neq j$ , and let  $n = \text{Card}(I)$ . In this case  $W$  is the free product of  $n$  copies of  $\mathbb{Z}/2$ ,  $A$  is a free group of rank  $n$ , and  $L$  is a 0-dimensional complex consisting of  $n$  points.

A Coxeter matrix is *right-angled* if each off-diagonal entry is either 2 or  $\infty$ . The associated Coxeter group and Artin group are also said to be right-angled.

For any subset  $J$  of  $I$  and  $w \in W_I$ , the following two conditions are equivalent:

- (1)  $w$  is the (unique) element of shortest word length in  $wW_J \in W_I/W_J$ ;
- (2) for each  $j \in J$ ,  $l(ws_j) = l(w) + 1$  (where  $l(w)$  denotes the word length of  $w$ ).

If  $w$  satisfies either condition, then it is called *J-reduced*. Moreover, if  $w$  is  $J$ -reduced and  $v \in W_J$ , then  $l(wv) = l(w) + l(v)$  (cf. [2, Ex. 3, p. 37]). For each pair  $J, J'$  of subsets of  $I$  with  $J \subseteq J'$ , let  $W_{J'}^J$  denote the set of  $J$ -reduced elements of  $W_{J'}$ .

We recall now a formula from [22, 11]. Given a pair of spherical subsets  $\tau, \sigma$  with  $\tau \leq \sigma$ , define an element  $T_\sigma^\tau \in \mathbb{Z}A_\sigma$  by the formula

$$T_\sigma^\tau := \sum_{w \in W_\sigma^\tau} (-1)^{l(w)} a_w.$$

**Lemma 9.** *For any three spherical subsets,  $\tau, \sigma, \rho \in \mathcal{S}$  with  $\tau \leq \sigma \leq \rho$ , we have*

$$T_\rho^\sigma \cdot T_\sigma^\tau = T_\rho^\tau.$$

*Proof.* If  $u \in W_\rho^\sigma$  and  $v \in W_\sigma^\tau$ , then  $l(uv) = l(u) + l(v)$ . It follows that  $a_u a_v$  equals  $a_{uv}$ . Since, for each  $i \in \tau$ ,  $l(uvs_i) = l(u) + l(vs_i) = l(u) + l(v) + 1 = l(uv) + 1$ , the element  $uv$  is  $\tau$ -reduced. Hence, as  $u$  ranges over  $W_\rho^\sigma$  and  $v$  ranges over  $W_\sigma^\tau$ ,  $uv$  ranges over  $W_\rho^\tau$ . Therefore,

$$\begin{aligned} T_\rho^\sigma \cdot T_\sigma^\tau &= \left( \sum_{u \in W_\rho^\sigma} (-1)^{l(u)} a_u \right) \left( \sum_{v \in W_\sigma^\tau} (-1)^{l(v)} a_v \right) \\ &= \sum_{w \in W_\rho^\tau} (-1)^{l(w)} a_w = T_\rho^\tau. \end{aligned}$$

□

## 6. SOME HOMOLOGICAL ALGEBRA

Let  $L$  be a finite abstract simplicial complex. Later we shall assume that  $L$  is ordered, in the sense that there is a given linear ordering of its vertex set. The *face category*  $\mathcal{F}(L)$  of  $L$  is defined to be the category whose objects are the simplices of  $L$ , with one morphism from  $\tau$  to  $\sigma$  whenever  $\tau$  is a face of  $\sigma$ . The *augmented face category*  $\mathcal{F}^+(L)$  has the same objects and morphisms as  $\mathcal{F}(L)$ , but also has one extra object (the “ $-1$ -simplex”, denoted  $\emptyset$ ) equipped with one morphism to each object. If  $L$  has vertex set  $I$ , each of  $\mathcal{F}(L)$  and  $\mathcal{F}^+(L)$  is isomorphic to a full subcategory of the category of subsets of  $I$ . A *cosheaf* on  $L$

with values in a category  $\mathcal{C}$  is a covariant functor  $F$  from  $\mathcal{F}^+(L)$  to  $\mathcal{C}$ . In the cases of interest to us,  $\mathcal{C}$  will be either abelian groups or Hilbert  $G$ -modules. A homomorphism of cosheaves on  $L$  is given by  $\phi : F \rightarrow F'$ , a natural transformation from  $F$  to  $F'$ .

If  $\tau \leq \sigma$  are faces of  $L$  (possibly including the  $-1$ -simplex), let  $\iota_\sigma^\tau$  denote the morphism in  $\mathcal{F}(L)$  from  $\tau$  to  $\sigma$ . The morphisms of  $\mathcal{F}(L)$  are generated by those  $\iota_\sigma^\tau$  such that  $\tau$  is a codimension one face of  $\sigma$ , and all relations between the morphisms are consequences of the relations

$$\iota_{\tau'}^{\sigma'} \iota_\sigma^{\tau'} = \iota_\tau^{\sigma'} \iota_\sigma^\tau, \quad (4)$$

where  $\sigma'$  is a codimension two face of  $\sigma$ , and  $\tau, \tau'$  are the two codimension one faces of  $\sigma$  containing  $\sigma'$ . To define a cosheaf  $F$  on  $L$  it suffices to specify the objects  $F(\sigma)$ , and morphisms  $f_\sigma^\tau = F(\iota_\sigma^\tau) : F(\tau) \rightarrow F(\sigma)$  for all codimension one pairs, so that relations (4) are satisfied for all codimension two pairs.

Now suppose that  $L$  is ordered. Then for any  $n \geq 0$  the vertices of an  $n$ -simplex form an ordered set isomorphic to  $0 < 1 < \dots < n$ . For any  $0 \leq i \leq n$  and any  $n$ -simplex  $\sigma$ , the  $i$ th face of  $\sigma$  is defined to be the  $(n-1)$ -simplex spanned by all vertices of  $\sigma$  except the  $i$ th. If one writes  $\partial_i = \iota_\sigma^\tau$ , where  $\tau$  is the  $i$ th face of  $\sigma$ , then the relations between the morphisms become the familiar ‘‘cosimplicial identities’’ as in [24] 8.1.

A cosheaf  $F$  of abelian groups on an ordered simplicial complex  $L$  gives rise to a cochain complex  $C^*(L; F)$  defined as follows:  $C^n = 0$  for  $n < -1$ , and for  $n \geq -1$ ,

$$C^n = \bigoplus_{\tau \in L^n} F(\tau),$$

where the indexing set is the set of  $n$ -simplices of  $L$ . Under the natural isomorphism

$$\mathrm{Hom}(C^n, C^{n+1}) \cong \bigoplus_{\tau \in L^n, \sigma \in L^{n+1}} \mathrm{Hom}(F(\tau), F(\sigma)),$$

the coboundary map  $d : C^n \rightarrow C^{n+1}$  corresponds to the matrix  $(d_{\tau\sigma})$ , where  $d_{\tau\sigma} = 0$  unless  $\tau$  is a face of  $\sigma$ , and is equal to  $(-1)^i F(\iota_\sigma^\tau)$  if  $\tau$  is the  $i$ th face of  $\sigma$ .

*Example 2.* For any abelian group  $N$ , and any simplicial complex  $L$ , there is a constant cosheaf on  $L$  denoted by  $\underline{N}$ , which sends each object of  $\mathcal{F}^+(L)$  to  $N$  and each morphism to the identity map. The cohomology of the corresponding cochain complex is isomorphic to the reduced cohomology of  $L$  with coefficients in  $N$ :

$$H^*C^*(L; \underline{N}) = \overline{H}^*(L; N).$$

Now suppose that  $M$  is a Coxeter matrix with nerve  $L$  and Artin group  $A$ , and let  $N$  be a  $\mathbb{Z}A$ -module. Define a cosheaf  $\mathcal{N}$  on  $L$  by sending each object to  $N$ , and for each  $\tau < \sigma$  sending the morphism  $\iota_\sigma^\tau$  to left multiplication by  $T_\sigma^\tau$ . (Lemma 9 implies that this does define a cosheaf.)

One of the key observations of this paper is:

**Lemma 10.** *For any Coxeter matrix,  $M$ , with  $L$ ,  $A$  and  $N$  as above, there is a cosheaf homomorphism  $\phi : \underline{N} \rightarrow \mathcal{N}$  defined by*

$$\phi(\sigma) : n \mapsto T_\sigma^0 \cdot n.$$

*Proof.* Immediate from Lemma 9. □

**Corollary 11.** *Let  $N$  be a  $\mathbb{Z}A$ -module upon which  $T_\sigma^0$  acts isomorphically for each  $\sigma \in L$ . Then*

$$H^*C^*(L; \mathcal{N}) \cong \overline{H}^*(L; N)$$

*Proof.* In this case the map described in Lemma 10 is an isomorphism of cosheaves, and so induces an isomorphism of cochain complexes  $C^*(L; \underline{N}) \rightarrow C^*(L; \mathcal{N})$ . □

## 7. THE SALVETTI COMPLEX

Let  $M$  be a spherical Coxeter matrix on a set  $I$  of cardinality  $n$ . By definition the associated Coxeter group  $W$  is finite, and so admits a standard representation as an orthogonal reflection group on  $\mathbb{R}^n$ . In this case, Salvetti [22, 23] defined a regular CW-complex  $X'$  which is equivariantly homotopy equivalent to  $Y$ , the complexification of  $\mathbb{R}^n$  minus the hyperplanes fixed by reflections in  $W$ . Each closed cell of  $X'$  is naturally identified with a face of a certain convex polytope associated to  $W$  (namely the convex hull of a generic  $W$ -orbit in  $\mathbb{R}^n$ ). Such a polytope is called a ‘‘Coxeter cell’’. The group  $W$  acts freely on  $X'$ . The quotient space, denoted by  $X_I$  or  $X$ , is called the *Salvetti complex*, and has fundamental group  $A$ .

As mentioned in the introduction, the definition of  $X$  was extended to arbitrary Coxeter matrices in [7]. As before, there is a regular  $W$ -CW-complex  $X'$  such that each closed cell is a Coxeter cell and such that  $X'$  is homotopy equivalent to a suitably defined hyperplane complement  $Y$ . The quotient space  $X = X'/W$  is the Salvetti complex, and has fundamental group  $A$ . The space  $X$  has one open cell  $e_\sigma$  for each spherical subset  $\sigma$  of  $I$ . Moreover the closure of  $e_\sigma$  in  $X$  is the subcomplex  $X_\sigma$  corresponding to the spherical Artin group  $A_\sigma$ .

*Example 3.* Let  $\mathbb{T}^n$  denote the usual cubical structure on the  $n$ -torus, with one  $k$ -dimensional cell for each subset of  $I = \{1, \dots, n\}$  of cardinality  $k$ . In the case when  $W = (\mathbb{Z}/2)^n$ , then  $A = \mathbb{Z}^n$ , and  $X = \mathbb{T}^n$ .

More generally, if  $A$  is right-angled with generating set  $I$  of cardinality  $n$ , then  $X$  is a subcomplex of  $\mathbb{T}^n$ , consisting of those cells of  $\mathbb{T}^n$  that correspond to spherical subsets of  $I$ .

It is proved in [7] that the poset of cells in  $X'$  can be identified with  $W \times \mathcal{S}$ , where the partial order given by incidence of faces is given by  $(w, \tau) < (v, \sigma)$  if and only if  $\tau < \sigma$  and  $v^{-1}w \in W_\sigma^\tau$ . Henceforth we identify a cell with the corresponding element of  $W \times \mathcal{S}$ . The cell corresponding to  $(v, \sigma)$  is a Coxeter cell associated to  $W_\sigma$ ; its vertex set is the set  $\{(vu, \emptyset) : u \in W_\sigma^\emptyset\}$  and its dimension is  $\text{Card}(\sigma)$ .

An edge or 1-cell of  $X'$  has the form  $(w, \{i\})$  for some  $w \in W$  and  $i \in I$ . Orient this edge by declaring  $(w, \emptyset)$  to be its initial vertex and  $(ws_i, \emptyset)$  to be its terminal vertex. Since the  $W$ -action preserves the edge orientations, this gives rise to an orientation of the edges of  $X = X'/W$ . Let  $\{e_i : i \in I\}$  denote the set of edges of  $X$  with their induced orientation. Since  $X$  has only one vertex, each  $e_i$  gives rise to an oriented loop. By definition the generator  $a_i$  of the Artin group  $A = \pi_1(X)$  is the homotopy class of the oriented loop around  $e_i$ .

A vertex  $x$  of a cell  $C$  in  $X'$  is called a *top vertex* (resp. *bottom vertex*) of  $C$  if each edge of  $C$  that contains  $x$  points away from  $x$  (resp. towards  $x$ ). Each cell  $(v, \sigma)$  of  $X'$  has a unique top vertex,  $(v, \emptyset)$ , and a unique bottom vertex  $(vu_\sigma, \emptyset)$ , where  $u_\sigma$  denotes the element of longest length in the finite Coxeter group  $W_\sigma$ .

Let  $\tilde{X}$  denote the universal cover of  $X$  (or equivalently of  $X'$ ). The orientations of the edges of  $X'$  lift to orientations on the edges of  $\tilde{X}$ . Each cell of  $\tilde{X}$  then also has a unique top vertex and a unique bottom vertex. The 1-skeleton of  $\tilde{X}$  is an oriented graph and each edge is labelled by an element of  $\{a_i : i \in I\}$  so that if  $e$  is an oriented edge labelled by  $a_i$  whose initial vertex is  $x$ , then the terminal vertex of  $e$  is  $x \cdot a_i$ .

Now fix a lift  $\tilde{x} \in \tilde{X}$  of the vertex  $(1, \emptyset)$  of  $X'$ . This gives an identification of the vertex set of  $\tilde{X}$  with  $A$  and of the 1-skeleton of  $\tilde{X}$  with the Cayley graph of the standard presentation of  $A$ .

For each  $\sigma \in \mathcal{S}$ , let  $\tilde{e}_\sigma$  be the lift of the cell  $(1, \sigma)$  whose top vertex is  $\tilde{x}$ . Using the  $A$ -action we get an identification of the poset of cells in  $\tilde{X}$  with  $A \times \mathcal{S}$ . The partial order on  $A \times \mathcal{S}$  corresponding to “is a face of” is defined by  $(b, \tau) < (a, \sigma)$  if and only if  $b = aa_u$  for some

$u \in W_\sigma^\tau$ . Thus for a given cell  $(a, \sigma)$  of  $\tilde{X}$ , its set of faces of type  $\tau$  is  $\{(aa_u, \tau) : u \in W_\sigma^\tau\}$ .

Once we have fixed an ordering on  $I$  there is an induced orientation for each cell of  $\tilde{X}$ : a cell  $C$  with top vertex  $x$  is oriented so that the edges at  $x$  in their natural order give an oriented basis for the tangent space of  $C$  at  $x$  (recall that  $C$  may be viewed as a convex polytope in  $\mathbb{R}^n$  for some  $n$ ). Suppose that  $C$  is of type  $\sigma$ ,  $\sigma = \{i_1 < \cdots < i_k\}$  and that  $D$  is a codimension one face of type  $\tau = \sigma - \{i_j\}$ . Suppose also that  $D$  has the same top vertex as  $C$ . In this case the natural orientation of  $D$  and its orientation induced by the orientation on  $C$  (by taking an outward pointing normal) differ by the sign  $\epsilon(\sigma, \tau)$  defined by  $\epsilon(\sigma, \tau) = (-1)^{j-1}$ .

If  $(v, \sigma)$  is a cell of  $X'$ , then the (right) action of  $W_\sigma$  on the cell (induced by the action on its vertex set) is such that the action of  $u \in W_\sigma$  changes the sign of the orientation by  $(-1)^{l(u)}$ . Hence, we have a formula for the boundary maps in  $C_*(\tilde{X})$ :

$$\begin{aligned} \partial(a\tilde{e}_\sigma) = \partial(a, \sigma) &= \sum_{\tau} \epsilon(\sigma, \tau) \sum_{u \in W_\sigma^\tau} (-1)^{l(u)} (aa_u, \tau) \\ &= a \sum_{\tau} \tilde{e}_\tau T_\sigma^\tau \cdot \epsilon(\tau, \sigma). \end{aligned}$$

where the summation is over all codimension one faces  $\tau$  of  $\sigma$ .

*Remark.* Note the similarity with the formula occurring in Lemma 9. An alternative proof of Lemma 9 can be given by considering the oriented incidence relation between cells in  $\tilde{X}$  of types  $\tau$ ,  $\sigma$  and  $\rho$ , where  $\tau \subseteq \sigma \subseteq \rho$ .

The cells  $\tilde{e}_\sigma$  for  $\sigma \in \mathcal{S}$  form a  $\mathbb{Z}A$ -basis for the cellular chain groups of  $\tilde{X}$ . In matrix terms, the  $i$ th cellular chain group of  $\tilde{X}$  may be viewed as consisting of row vectors over  $\mathbb{Z}A$ , with the standard basis elements corresponding to the cells  $\tilde{e}_\sigma$ , where  $\sigma$  ranges over the subsets of  $\mathcal{S}$  of size  $i$ . The  $\mathbb{Z}A$ -action is the standard left action, and the boundary map in the chain complex is given by right multiplication by a matrix whose entry in the  $(\sigma, \tau)$ -position is  $\epsilon(\tau, \sigma)T_\sigma^\tau$  if  $\tau \subseteq \sigma$  and is zero otherwise.

Now suppose that  $N$  is any  $\mathbb{Z}A$ -module. After identifying  $C_i(\tilde{X})$  with a space of row vectors with entries in  $\mathbb{Z}A$  as above, it is natural to identify  $\text{Hom}_A(C_i(\tilde{X}), N)$  with a space of column vectors (of the same length) with entries in  $N$ , so that the evaluation map corresponds to multiplying a row vector and a column vector to produce a single

element of  $N$ . In these terms, the coboundary map in the cochain complex  $\text{Hom}_A(C_*(\tilde{X}), N)$  is described as *left* multiplication by the same matrix over  $\mathbb{Z}A$  as used previously to describe the boundary map in  $C_*(\tilde{X})$ . This gives the following theorem.

**Theorem 12.** *For any  $A$ -module  $N$ , there is an isomorphism between  $C^*(X; N) = \text{Hom}_A(C_*(\tilde{X}), N)$  as defined at the beginning of Section 3 and  $C^{*-1}(L; N)$  as defined in Section 6.*

In the spherical case, a statement equivalent to the above theorem is contained in [23, 11], and it is used to compute some cohomology.

*Remark.* There is a simple description of the CW-complex  $X$ . Firstly, for each  $\sigma \in \mathcal{S}$  we describe a CW-complex  $X_\sigma$ . Start with a Coxeter cell  $C_\sigma$  of type  $W_\sigma$ . For each  $\tau < \sigma$  identify  $C_\tau$  with the face of  $C_\sigma$  of type  $\tau$  which has the same top vertex as  $C_\sigma$ . Now for each  $\tau < \sigma$  and each  $u \in W_\sigma^\tau$  glue together  $C_\tau$  and  $uC_\tau$  via the homeomorphism induced by  $u$ . The result is  $X_\sigma$ . To construct  $X$ , start with the disjoint union of the  $X_\sigma$  for  $\sigma \in \mathcal{S}$ , and then use the natural maps to identify  $X_\tau$  with a subcomplex of  $X_\sigma$  whenever  $\tau < \sigma$ .

*Remark.* In section 3 of [6] it is shown how to associate a “simple complex of groups”  $\mathcal{A}$  to a Coxeter matrix. The underlying poset is  $\mathcal{S}$ . To each  $\sigma \in \mathcal{S}$  one associates the Artin group  $A_\sigma$ , and for each  $\tau < \sigma$ , the associated homomorphism  $A_\tau \rightarrow A_\sigma$  is the natural inclusion. There is a natural projection from (the barycentric subdivision of)  $X$  to the geometric realization of  $\mathcal{S}$  such that the inverse image of the vertex corresponding to  $\sigma$  is a copy of  $X_\sigma$ . Since it follows from [12] that  $X_\sigma \simeq BA_\sigma$ , it follows that  $X$  is an aspherical realization of the complex of groups  $\mathcal{A}$  in the sense of [17]. Thus  $X$  is homotopy equivalent to the classifying space  $B\mathcal{A}$  of  $\mathcal{A}$  (which is defined as a homotopy colimit of  $BA_\sigma$  over the category  $\mathcal{S}$ ). The main conjecture of [6] is that  $B\mathcal{A}$  (or equivalently  $X$ ) is aspherical. This has been proved in many cases: for example, it holds if the simplicial complex  $L$  is a flag complex (e.g., if the Coxeter matrix is either spherical or right-angled) and it holds whenever the dimension of  $L$  is at most 1.

## 8. COHOMOLOGY WITH GENERIC COEFFICIENTS

This section represents a slight digression, although the methods used are similar to those that will be applied to  $\ell^2$ -cohomology in later sections.

Let  $A$  be an Artin group with Artin generators  $\{a_i : i \in I\}$ , let  $G$  be the abelianization of  $A$ , and let  $\alpha_i$  be the image of  $a_i$  in  $G$ . In the

case when  $m_{ij}$  is even, the abelianization of the Artin relator between  $a_i$  and  $a_j$  is trivial, and in the case when  $m_{ij}$  is odd the abelianization of the Artin relator is  $\alpha_i = \alpha_j$ . Hence one sees that  $G$  is free abelian, of rank equal to the number of components of the graph with vertex set  $I$  and an edge joining  $i$  to  $j$  if and only if  $m_{ij}$  is odd. In particular,  $G$  is never trivial. Let  $m$  denote the rank of  $G$ , and (after changing the ordering on  $I$ ), suppose that  $\alpha_1, \dots, \alpha_m$  freely generate  $G$ .

Now suppose that  $k$  is a field of characteristic zero, and that  $N$  is a  $kA$ -module whose underlying  $k$ -vector space is 1-dimensional. The action of  $A$  on  $N$  factors through  $G$ , a free abelian group of rank  $m$ , and so  $N$  is classified up to isomorphism by an  $m$ -tuple

$$\Lambda = (\lambda_1, \dots, \lambda_m) \in (k^*)^m,$$

where  $k^* = k - \{0\}$  and for  $1 \leq i \leq m$ , the generator  $\alpha_i$  acts on  $N$  as multiplication by  $\lambda_i$ . As an illustration of the methods we shall apply to  $\ell^2$ -cohomology, we shall compute the cohomology of the Salvetti complex  $X$  with local coefficients  $N$  for “generic  $N$ ”, i.e., for  $N$  corresponding to elements of a dense Zariski open subset of  $(k^*)^m$ .

**Theorem 13.** *For a generic 1-dimensional  $kA$ -module  $N$ , there is an isomorphism*

$$\overline{H}^{*-1}(L; k) \cong H^*(X; N)$$

*Proof.* Each side of the supposed isomorphism is isomorphic to the cohomology of  $L$  with coefficients in a certain cosheaf. The left hand side is:

$$\overline{H}^*(L; k) \cong \overline{H}^*(L; N) \cong H^*C^*(L; \underline{N})$$

and by Theorem 12, the cochain complex for the right hand side is:

$$C^*(X; N) \cong C^{*-1}(L; \mathcal{N}).$$

It therefore suffices to show that for generic  $N$ , the cosheaf homomorphism  $\phi : \underline{N} \rightarrow \mathcal{N}$  described in Lemma 10 is an isomorphism. Equivalently, it suffices to show that for each simplex  $\sigma$  of  $L$ , multiplication by  $T_\sigma^\emptyset$  is an isomorphism for generic  $N$ .

For  $\Lambda = (\lambda_1, \dots, \lambda_m) \in (k^*)^m$ , let  $\psi'_\Lambda : \mathbb{Z}G \rightarrow k$  be defined by  $\psi'_\Lambda(a_i) = \lambda_i$ , and let  $\psi_\Lambda : \mathbb{Z}A \rightarrow k$  be the composite of the projection  $\mathbb{Z}A \rightarrow \mathbb{Z}G$  and  $\psi'$ . For each  $\sigma \in L$ , the set of  $\Lambda$  such that  $\psi_\Lambda(T_\sigma^\emptyset) = 0$  is a closed subset of  $(k^*)^m$ . The union over all  $\sigma$  of these closed sets is not the whole of  $(k^*)^m$ , since in the case when  $\Lambda = (-1, -1, \dots, -1)$ ,  $\psi_\Lambda(T_\sigma^\emptyset) = \text{Card}(W_\sigma)$ . Since  $(k^*)^m$  is connected, it follows that the set of  $\Lambda$  for which each  $\psi_\Lambda(T_\sigma^\emptyset) \neq 0$  is open and dense in  $(k^*)^m$  as required.  $\square$

*Remark.* The theorem does not hold for arbitrary 1-dimensional coefficients, as can be seen by comparing the case of “generic” 1-dimensional coefficients  $N$  with the trivial 1-dimensional module  $k$ . As a first example, consider  $H^1$ . For any  $A$ ,  $H^1(A; k)$  is naturally isomorphic to  $\text{Hom}(A, k)$ , which has dimension equal to the rank of the abelianization of  $A$ . In particular it is non-zero for every Artin group, whereas for  $A$  of spherical type and generic  $N$ , we have seen that  $H^*(A; N) = \{0\}$ . (Note that in the case when  $A$  is of spherical type,  $H^*(A; k)$  is computed in [4].)

As a further example, consider the case when  $A$  is right-angled. In this case it may be shown that  $H^*(A; k)$  is isomorphic to the “exterior face ring of  $L$ ”, defined by taking the exterior algebra over  $k$  with generators  $\{x_i : i \in I\}$  with each  $x_i$  in degree 1, and adding in the relation  $x_{i_1} \cdots x_{i_m} = 0$  whenever  $i_1, \dots, i_m$  do not form the vertex set of a simplex of  $L$  [13, 18]. In particular, the rank of  $H^i(X; k) = H^i(A; k)$  is equal to the number of  $(i - 1)$ -simplices in  $L$  for each  $i > 0$ .

## 9. REDUCED $\ell^2$ -COHOMOLOGY

Each Artin group of spherical type has a non-trivial centre. If  $A$  is a spherical Artin group and  $w$  is the longest element of the corresponding Coxeter group, then the element  $(a_w)^2$  is in the centre of  $A$  (c.f. [12]). Note that the expression for this element contains only positive powers of the Artin generators, and so in particular its image under the standard homomorphism  $A \rightarrow \mathbb{Z}$  is non-zero.

**Lemma 14.** *Suppose that  $A$  is an Artin group of spherical type, and that  $A'$  is a normal subgroup of  $A$  contained in the kernel of the standard homomorphism. Let  $X' \rightarrow X$  be the corresponding covering space of  $X$  and let  $G = A/A'$ . For each  $i$ , the Hilbert  $G$ -module  $\mathcal{H}^i(X')$  is the zero module.*

*Proof.* In view of the remarks preceding the statement of the lemma, this is a special case of Corollary 7.  $\square$

**Proposition 15.** *Let  $H \leq G$ , and let  $\xi \in \mathbb{Z}H$ . Then left multiplication by  $\xi$  induces a weak isomorphism of  $\ell^2(H)$  if and only if it induces a weak isomorphism of  $\ell^2(G)$ .*

*Proof.* Consider  $\mathcal{H}$ , defined to be the set of all functions on  $G$  that are square-summable on each right coset  $Hg$  of  $H$ . As a left  $\mathbb{Z}H$ -module,  $\mathcal{H}$  is isomorphic to a direct product of copies of  $\ell^2(H)$  indexed by the coset space  $H \backslash G$ . One has that  $\ell^2(H) \leq \ell^2(G) \leq \mathcal{H}$ , and so multiplication by  $\xi$  is injective as a self-map of  $\ell^2(H)$  if and only if it

is injective as a self-map of  $\ell^2(G)$ . The corresponding assertion with “weak isomorphism” in place of “injective” follows from Lemma 4.  $\square$

**Lemma 16.** *Suppose that  $A = A_\sigma$  is an Artin group of spherical type, and that  $A'$  is a normal subgroup of  $A$  contained in the kernel of the standard homomorphism  $A \rightarrow \mathbb{Z}$ . Let  $G = G_\sigma = A/A'$ , and let  $\phi_\sigma : \ell^2(G) \rightarrow \ell^2(G)$  denote left multiplication by the element  $T_\sigma^\emptyset \in \mathbb{Z}A_\sigma$ . Then  $\phi_\sigma$  is a weak isomorphism.*

*Proof.* Let  $\pi$  denote the quotient map from  $A_\sigma$  to  $G_\sigma$ . For any  $\tau < \sigma$ , let  $i_\tau$  denote the inclusion of  $A_\tau$  in  $A_\sigma$ , and define a subgroup  $G_\tau \leq G_\sigma$  by  $G_\tau = \pi \circ i_\tau(A_\tau)$ . The composite of  $i_\tau$  and the standard homomorphism for  $A_\sigma$  is equal to the standard homomorphism for  $A_\tau$ , and so the pair  $(A_\tau, G_\tau)$  satisfy the hypotheses of the Lemma.

The lemma is trivially satisfied if  $\text{Card}(\sigma) = 0$ , since  $T_\emptyset^\emptyset = 1$ . Hence by induction on  $\text{Card}(\sigma)$ , we may assume that for any  $\tau < \sigma$ , the map  $\phi_\tau : \ell^2(G_\tau) \rightarrow \ell^2(G_\tau)$  is a weak isomorphism. Here  $\phi_\tau$  is defined to be left multiplication by the element  $T_\tau^\emptyset \in \mathbb{Z}A_\tau$ . By Proposition 15, we may assume that  $\phi_\tau : \ell^2(G_\sigma) \rightarrow \ell^2(G_\sigma)$  is also a weak isomorphism.

Now let  $L$  be the simplex with vertex set  $\sigma$ , where  $n = \text{Card}(\sigma)$ . Let  $N$  denote  $\ell^2(G_\sigma) = \ell^2(G)$ , and consider the cosheaves of Hilbert  $G$ -modules on  $L$  that were denoted by  $\underline{N}$  and  $\mathcal{N}$  in Section 6. Note that in this case,  $C^*(L; \underline{N})$  and  $C^*(L; \mathcal{N})$  are cochain complexes of Hilbert  $G$ -modules. Since the reduced cohomology of  $L$  is trivial, the cochain complex  $C^*(L; \underline{N})$  is exact. By Theorem 12 the cochain complex  $C^*(L; \mathcal{N})$  (with a shift of degree) may be used to compute the cohomology of  $A$  with coefficients in  $\ell^2(G)$ , and so by Lemma 14, it follows that  $C^*(L; \mathcal{N})$  is weakly exact.

Now consider the cosheaf homomorphism  $\phi : \underline{N} \rightarrow \mathcal{N}$  defined in Lemma 10, and the induced map

$$\phi^* : C^*(L; \underline{N}) \rightarrow C^*(L; \mathcal{N}).$$

In degree  $i$ , each of  $C^i(L; \underline{N})$  and  $C^i(L; \mathcal{N})$  is isomorphic to a direct sum of copies of  $\ell^2(G)$  indexed by the subsets  $\tau \leq \sigma$  with  $\text{Card}(\tau) = i$ . With respect to these bases, the map  $\phi^i$  is given by a diagonal matrix, whose  $(\tau, \tau)$ -entry is the map  $\phi_\tau$ . By induction, this map is a weak isomorphism in each degree except possibly  $n$ , the top degree. Consider the following commutative diagram, consisting of the right-hand ends of the two cochain complexes.

$$\begin{array}{ccc} \longrightarrow & C^{n-1}(L; \underline{N}) & \xrightarrow{\delta'} C^n(L; \underline{N}) \\ & \downarrow \phi_{n-1} & \downarrow \phi_n \\ \longrightarrow & C^{n-1}(L; \mathcal{N}) & \xrightarrow{\delta} C^n(L; \mathcal{N}) \end{array}$$

This can be rewritten as follows, where  $\tau$  runs over the subsets of  $\sigma$  of size  $n - 1$ .

$$\begin{array}{ccc} \longrightarrow & \bigoplus_{\tau} \ell^2(G) & \xrightarrow{\delta'} \ell^2(G) \\ & \downarrow \bigoplus_{\tau} \phi_{\tau} & \downarrow \phi_{\sigma} \\ \longrightarrow & \bigoplus_{\tau} \ell^2(G) & \xrightarrow{\delta} \ell^2(G) \end{array}$$

The coboundary map  $\delta'$  is surjective, the coboundary map  $\delta$  is weakly surjective, and by induction the left-hand vertical map is a weak isomorphism. It follows that the right-hand vertical map,  $\phi_{\sigma} : \ell^2(G) \rightarrow \ell^2(G)$  is weakly surjective. The claim now follows from Lemma 4.  $\square$

**Corollary 17.** *Let  $A$  be any Artin group, let  $A'$  be a normal subgroup of  $A$  contained in the kernel of the standard homomorphism  $A \rightarrow \mathbb{Z}$ , let  $G = A/A'$ , and let  $N = \ell^2(G)$ . Then the map  $\phi : \underline{N} \rightarrow \mathcal{N}$  defined in Lemma 10 is a weak isomorphism of cosheaves of Hilbert  $G$ -modules on the simplicial complex  $L$ .*

*Proof.* On each object  $\sigma \in \mathcal{S}$ , the map  $\phi(\sigma) : \underline{N}(\sigma) \rightarrow \mathcal{N}(\sigma)$  is equal to left multiplication by  $T_{\sigma}^{\emptyset}$ . By Lemma 16 and Proposition 15, each of these maps is a weak isomorphism.  $\square$

The next Corollary is immediate from the above.

**Corollary 18.** *With notation as in Corollary 17, the induced map of cochain complexes*

$$\phi^* : C^*(L; \underline{N}) \rightarrow C^*(L; \mathcal{N})$$

*is a weak isomorphism.*

The proofs of Theorems 1 and 3 are now completed by combining the description of  $\mathcal{H}(X')$  as the reduced cohomology of  $C^*(L; \mathcal{N})$ , given in Theorem 12, with Corollary 18 and Lemma 5.

## 10. CLOSING REMARKS

In this section we make some remarks concerning  $\ell^p$ -cohomology for  $p \neq 2$ , and describe an alternative method for proving Theorems 1 and 3, using a spectral sequence argument. In a subsequent paper with T. Januszkiewicz we plan to use this technique to calculate the  $\ell^2$ -cohomology of the universal cover of the Salvetti complex associated to an arbitrary real hyperplane arrangement. To simplify the notation, we shall discuss this proof for Theorem 1 but not for Theorem 3.

**10.1. A spectral sequence argument.** Recall from Section 7 that for a general Artin group  $A$ , the Salvetti complex  $X$  may be expressed as a union of subcomplexes  $X_\sigma$ , where  $\sigma$  ranges over the spherical subsets of the Artin generating set, and  $X_\sigma$  is a copy of the Salvetti complex for  $A_\sigma$ . Let  $Y_\sigma$  be the subcomplex of  $\tilde{X}$ , the universal cover of  $X$ , consisting of lifts of cells of  $X_\sigma$ . Each  $Y_\sigma$  is a free  $A$ -CW-complex. By Corollary 8 and Proposition 15, the reduced  $\ell^2$ -cohomology of  $Y_\sigma$  vanishes for  $\sigma \neq \emptyset$ . In contrast,  $Y_\emptyset$  is just the 0-skeleton of  $\tilde{X}$ , which consists of a single free  $G$ -orbit of cells. Hence the reduced  $\ell^2$ -cohomology of  $Y_\emptyset$  is a copy of  $\ell^2(G)$  in degree zero.

One may construct a ‘‘Mayer-Vietoris double chain complex’’  $C_{*,*}$  for  $\tilde{X}$  expressed as the union of the  $Y_\sigma$ . Let  $\mathcal{L}$  denote the set of maximal simplices in  $L$ , define  $Y(\emptyset) = \tilde{X}$ , and for  $\emptyset \neq S \subseteq \mathcal{L}$ , define  $Y(S) = \bigcap_{\sigma \in S} Y_\sigma$ . Define  $C_{*,0}$  to be the cellular chain complex for  $\tilde{X}$ , and for  $j > 0$  define  $C_{*,j}$  to be the direct sum over the subsets  $S \subseteq \mathcal{L}$  of size  $j$  of the cellular chain complex for  $Y(S)$ . The boundary maps in  $C_{*,*}$  of degree  $(-1, 0)$  are the boundary maps in the chain complexes  $C_*(Y(S))$ , and the boundary maps of degree  $(0, -1)$  are given by the matrices whose  $(S, T)$ -entry is  $\epsilon(S, T)$  times the map induced by the inclusion of  $Y(S)$  in  $Y(T)$ , where  $\epsilon(S, T) = (-1)^i$  if  $T$  is obtained from  $S$  by omitting the  $i$ th element of  $S$  (for some fixed ordering of  $\mathcal{L}$ ). By construction, this double complex has trivial homology, since the boundary map of degree  $(0, -1)$  is exact. Since each  $C_{i,j}$  is free, it follows that for each  $i$ , the chain complex  $C_{i,*}$  is split exact.

Now suppose that  $N$  is a  $\mathbb{Z}A$ -module such that for each  $\sigma \in L$ ,  $H^*(A_\sigma; N) = \{0\}$ . Define a double cochain complex by

$$E_0^{*,*} = \text{Hom}_A(C_{*,*}, N),$$

and let  $E_*^{*,*}$  denote the spectral sequence in which the differential  $d_0$  is induced by the boundary map of degree  $(-1, 0)$  on  $C_{*,*}$ . The boundary map on  $E_0^{*,*}$  of degree  $(0, 1)$  is exact, and so the homology of the total complex for  $E_0^{*,*}$  is zero. It follows that the  $E_\infty$ -page of the spectral sequence is identically zero. The  $E_1$ -page has  $E_1^{i,0} \cong H^i(X; N)$ ,  $E_1^{i,j} = \{0\}$  if both  $i > 0$  and  $j > 0$ , and  $E_1^{0,j}$  is isomorphic to a direct sum of copies of  $N$ , indexed by those  $j$ -element subsets of  $\mathcal{L}$  such that the intersection of the corresponding simplices of  $L$  is empty.

The structure of the  $E_1$ -page implies that  $E_2^{i,j} = E_1^{i,j}$  for  $i > 0$ , and it may be shown that  $E_2^{0,j} \cong \overline{H}^j(L; N)$ . To see this, note that the cochain complex  $E_1^{0,*}$  embeds as a subcomplex in an exact complex  $C^*$ , where  $C^j$  is isomorphic to a direct sum of copies of  $N$  indexed by all  $j$ -element subsets of  $\mathcal{L}$ , and the quotient complex  $C^*/E_1^{0,*}$  is

isomorphic to the augmented cochain complex for the nerve of the covering of  $L$  by the elements of  $\mathcal{L}$ , shifted in degree by 1. The long exact sequence in cohomology coming from this short exact sequence of cochain complexes gives the claimed isomorphism.

Since the  $E_\infty$ -page is identically zero, and the only non-zero groups on the  $E_2$ -page are those  $E_2^{i,j}$  for which either  $i = 0$  or  $j = 0$ , it follows that the differential  $d_r$  must be an isomorphism from  $E_r^{0,r-1} = E_2^{0,r-1}$  to  $E_r^{r,0} = E_1^{r,0}$ . Thus we obtain

**Theorem 19.** *For any  $\mathbb{Z}A$ -module  $N$  such that  $H^*(A_\sigma; N) = \{0\}$  for all non-trivial spherical subgroups  $A_\sigma \leq A$ , there is an isomorphism*

$$H^*(X; N) \cong \overline{H}^{*-1}(L; N).$$

One corollary of this is a version of Theorem 13

**Corollary 20.** *Let  $k$  be a field of characteristic zero, and let  $N$  be a 1-dimensional  $kA$ -module, such that for each non-trivial spherical subgroup  $A_\sigma$ , the centre of  $A_\sigma$  acts non-trivially on  $N$ . Then  $H^*(X; N) \cong \overline{H}^{*-1}(L; N)$ .*

*Proof.* By Proposition 6, the action of the centre  $Z$  of  $A_\sigma$  on  $H^*(A_\sigma : N)$  is trivial. But as a  $kZ$ -module  $H^i(A_\sigma; N)$  is isomorphic to a submodule of a direct sum of copies of  $N$ . The only possibility is that  $H^*(A_\sigma; N) = \{0\}$ . Hence Theorem 19 may be applied.  $\square$

Unfortunately, Theorem 19 does not apply directly to the case when  $N = \ell^2(A)$ , since we have only that the *reduced*  $\ell^2$ -cohomology groups  $\mathcal{H}^*(A_\sigma)$  are equal to  $\{0\}$ . One way around this difficulty is to use W. Lück's equivalence of categories (for any discrete group  $G$ ) between the category of Hilbert  $G$ -modules and the category of projective modules for the group von Neumann algebra  $\mathcal{N}(G)$ , as explained in chapter 6 of [20]. The von Neumann dimension is defined for projective  $\mathcal{N}(G)$ -modules in such a way that it is preserved by the above equivalence of categories. Furthermore, von Neumann dimension extends to a dimension function on all (finitely generated)  $\mathcal{N}(G)$ -modules so that it is additive on extensions, and such that  $\dim_G(M) = 0$  if and only if  $\text{Hom}_G(M, \mathcal{N}(G)) = \{0\}$ . For any CW-complex  $Y$  with fundamental group  $G$ , the von Neumann dimension of the ordinary homology group  $H_i(Y; \mathcal{N}(G))$  is equal to the von Neumann dimension of  $\mathcal{H}^i(Y)$ . Hence the following statement is a weak version of Theorem 1.

**Theorem 21.** *For any Artin group  $A$  and any  $i$ , the von Neumann dimensions of  $H_i(X; \mathcal{N}(A))$  and  $\overline{H}_{i-1}(L; \mathcal{N}(A)) \cong \mathcal{N}(A) \otimes \overline{H}_{i-1}(L)$  are equal.*

*Proof.* Use the homology version of the Mayer-Vietoris spectral sequence described above. In more detail, define a double chain complex

$$E_{*,*}^0 = \mathcal{N}(A) \otimes_{\mathbb{Z}A} C_{*,*}.$$

The corresponding spectral sequence has the following properties:

- (1)  $E_{*,*}^\infty = \{0\}$ ;
- (2)  $E_{0,j}^2 \cong \overline{H}_j(L; N)$ ;
- (3)  $E_{i,0}^1 \cong H_i(A; \mathcal{N}(A))$ ;
- (4) For  $i, j > 0$ ,  $\dim_G(E_{i,j}^1) = 0$ .

(The fourth of these properties follows from Corollary 7.) It may be shown that for  $1 < s \leq r$ ,

$$\dim_G(E_{0,r-1}^s) = \dim_G(E_{0,r-1}^2) \quad \text{and} \quad \dim_G(E_{r,0}^s) = \dim_G(E_{r,0}^1).$$

Since the  $E^\infty$ -page is identically zero, the map  $d^r : E_{r,0}^r \rightarrow E_{0,r-1}^r$  is an isomorphism, and hence  $\dim_G(E_{r,0}^r) = \dim_G(E_{0,r-1}^r)$ .  $\square$

*Remark.* In the case when the Artin group  $A$  contains no finite conjugacy classes, the von Neumann dimension of a (finitely generated) Hilbert  $A$ -module determines the module up to isomorphism. Hence in this case, Theorem 21 is equivalent to Theorem 1.

**10.2.  $\ell^p$ -cohomology.** For  $1 \leq p < \infty$ , let  $\ell^p(G)$  denote the space of real-valued functions  $f$  on  $G$  for which  $\sum_{g \in G} |f(g)|^p$  is finite. Each  $\ell^p(G)$  is a Banach space with respect to the obvious norm. If  $Y'$  is a free  $G$ -CW-complex with only finitely many orbits of cells of each dimension, one may define reduced  $\ell^p$ -cohomology groups  $\mathcal{H}_{(p)}^*(Y')$  in the same way as the reduced  $\ell^2$ -cohomology groups were defined in Section 3. For  $1 < p < \infty$ , Gromov has proved vanishing theorems that imply the  $\ell^p$ -analogues of Corollaries 7 and 8 in [16].

The main problem with attempting to generalize Theorems 1 and 3 to  $\ell^p$ -cohomology is that there is no analogue of von Neumann dimension in this generality, and we do not know whether the analogue of Lemma 5 holds. Nevertheless, we conjecture that Theorems 1 and 3 remain valid for reduced  $\ell^p$ -cohomology, for  $1 < p < \infty$ .

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W.  
18TH AVENUE, COLUMBUS, OHIO 43210, UNITED STATES  
*E-mail address:* mdavis@math.ohio-state.edu

FACULTY OF MATHEMATICAL STUDIES, UNIVERSITY OF SOUTHAMPTON,  
SOUTHAMPTON, SO17 1BJ, UNITED KINGDOM  
*E-mail address:* I.J.Leary@maths.soton.ac.uk