Exotic aspherical manifolds

Michael W. Davis
The Ohio State University
Department of Mathematics
Columbus, Ohio 43210–1174
United States
mdavis@math.ohio-state.edu

Contents

1 The geometric realization of a simplicial complex

A *simplicial complex* L consists of a set I (called the *vertex set*) and a collection of finite subsets $\mathcal{S}(L)$ of I such that

- $\emptyset \in \mathcal{S}(L)$
- for each $i \in I$, $\{i\} \in \mathcal{S}(L)$, and
- if $\sigma \in \mathcal{S}(L)$ and $\tau < \sigma$, then $\tau \in \mathcal{S}(L)$.

An element σ of $\mathcal{S}(L)$ is called a *simplex*; its *dimension* is defined by $\dim \sigma = \mathrm{Card}(\sigma) - 1$.

Let us assume that $I = \{1, 2, \ldots, m\}$. The *standard* $(m-1)$-*simplex* on I, denoted by Δ^{m-1}, is the convex polytope in $\mathbb{R}^m$ defined by intersecting the positive quadrant (defined by $x_i \geq 0$ for all $i \in I$) with the hyperplane $\Sigma x_i = 1$. A vertex of Δ^{m-1} is an element e_i of the standard basis for $\mathbb{R}^m$. The poset of faces of Δ^{m-1}, denoted $\mathcal{F}(\Delta^{m-1})$, is isomorphic to the poset of all nonempty subsets of I. This gives us a simplicial complex which we will also denote by Δ^{m-1}. (Usual practice is to blur the distinction between a simplicial complex and its geometric realization.)

If σ is a nonempty subset of I $(= \{1, \ldots m\})$, then let Δ_σ denote the face of Δ^{m-1} spanned by $\{e_i\}_{i \in \sigma}$.

If L is a simplicial complex with vertex set I, then its *geometric realization* is defined to be the union of all subspaces of Δ^{m-1} of the form Δ_σ for some $\sigma \in \mathcal{S}(L)$. The geometric realization will also be denoted L.

2 Cubical cell complexes

As before, $I = \{1, \ldots, m\}$. The *standard m-dimensional cube* is the convex polytope $[-1,1]^m \subset \mathbb{R}^m$. For each subset σ of I let $\mathbb{R}^\sigma$ denote the linear subspace spanned by $\{e_i\}_{i\in\sigma}$ and let $\Box_\sigma$ denote the standard cube in $\mathbb{R}^\sigma$. (If $\sigma = \emptyset$, then $\mathbb{R}^\emptyset = \Box_\emptyset = \{0\}$.) The faces of $[-1,1]^m$ which are parallel to $\Box_\sigma$ have the form $v + \Box_\sigma$ for some vertex v of $[-1,1]^m$.

Next we want to describe the poset of nonempty faces of $[-1,1]^m$. For each $i \in I$, let $r_i : [-1,1]^m \to [-1,1]^m$ denote the orthogonal reflection across the hyperplane $x_i = 0$. Let J be the group of symmetries of $[-1,1]^m$ generated by $\{r_i\}_{i\in I}$. Then J is isomorphic to $(\mathbb{Z}/2)^m$. The group J acts simply transitively on the vertex set of $[-1,1]^m$. The stabilizer of a face $v + \Box_\sigma$ is the subgroup J_σ generated by $\{r_i\}_{i\in\sigma}$. Hence, the poset of nonempty faces of $[-1,1]^m$ is isomorphic to the poset of cosets

$$\coprod_{\sigma\subset I} J/J_\sigma.$$

Roughly, a cubical cell complex P is a regular cell complex in which each cell is combinatorially isomorphic to a standard cube of some dimension. More precisely, P consists of a poset $\mathcal{F}(= \mathcal{F}(P))$ such that for each $c \in \mathcal{F}$ the subposet $\mathcal{F}_{\leq c}$ is isomorphic to the poset of nonempty faces of $[-1,1]^k$; k is the *dimension* of c. (Here $\mathcal{F}_{\leq c} = \{x \in \mathcal{F} | x \leq c\}$.) The elements of $\mathcal{F}$ are called *cells*. A *vertex* of P is synonymous with a 0-dimensional cell. By definition, the *link* of a vertex v in P, denoted by $Lk(v, P)$, is the subposet $\mathcal{F}_{>v}$ of all cells which are strictly greater than v (i.e., which have v as a vertex.) For example, if v is a vertex of $[-1,1]^m$, then $Lk(v, [-1,1]^m)$ is the simplicial complex Δ^{m-1}. It follows that the link of a vertex in any cubical cell complex is a simplicial complex.

The *geometric realization* of a cubical complex P can be defined by pasting together standard cubes, one for each element of $\mathcal{F}$. A neighborhood of a vertex v in (the geometric realization of) P is homeomorphic to the cone on $Lk(v, P)$.

3 The cubical complex P_L

Given a simplicial complex L with vertex set I, we shall now define a subcomplex P_L of [-1,1]^m. The vertex set of P_L will be the same as that of [-1,1]^m. The main property of P_L will be that the link of each of its vertices is isomorphic to L. The construction is very similar to the way in which we realized L as a subcomplex of Δ^{m-1}.

By definition, P_L is the union of all faces of [-1,1]^m which are parallel to $\square_\sigma$ for some $\sigma \in \mathcal{S}(L)$. Hence, the poset of cells of P_L can be identified with

$$\coprod_{\sigma \in \mathcal{S}(L)} J/J_\sigma.$$

(This construction is also described in [2] as well as in [12].)

Examples 3.1.

- If $L = \Delta^{m-1}$, then $P_L =$[-1,1]^m.
- If $L = \partial(\Delta^{m-1})$, then P_L is the boundary of an m-cube i.e., P_L is homeomorphic to S^{m-1}.
- If L is the disjoint union of m points, then P_L is the 1-skeleton of an m-cube.
- If $m = 3$ and L is the disjoint union of a 1-simplex and a point then P_L is the subcomplex of the 3-cube consisting of the top and bottom faces and the 4 vertical edges.

From the fact that a neighborhood of a vertex in P_L is homeomorphic to the cone on L we get the following.

Proposition 3.2. *If L is homeomorphic to S^{n-1}, then P_L is an n-manifold.*

Proof. The cone on S^{n-1} is homeomorphic to an n-disk. $\square$

4 The universal cover of P_L and the group W_L

Let $\tilde{P}_L$ denote the universal cover of P_L. The cubical cell structure on P_L lifts to a cubical structure on $\tilde{P}_L$. The group J ($\cong (\mathbb{Z}/2)^m$) acts on P_L. Let

W_L denote the group of all lifts of elements of J to $\tilde{P}_L$ and let $\varphi : W_L \to J$ be the homomorphism induced by the projection $\tilde{P}_L \to P_L$. We have a short exact sequence,

$$1 \to \pi_1(P_L) \to W_L \xrightarrow{\varphi} J \to 1.$$

We will use the notation: $\Gamma_L = \pi_1(P_L)$.

Since J acts simply transitively on the vertex set of P_L, the group W_L acts simply transitively on the vertex set of $\tilde{P}_L$. It follows that the 2-skeleton of $\tilde{P}_L$ is the Cayley 2-complex associated to a presentation of W_L. (In particular, the 1-skeleton is the Cayley graph associated to a set of generators.) Next, we use this observation to write down a presentation for W_L.

The vertex set of P_L can be identified with J. Fix a vertex v of P_L (corresponding to the identity element in J). Let $\tilde{v}$ be a lift of v in $\tilde{P}_L$. The 1-cells at v or at $\tilde{v}$ correspond to vertices of L, i.e., to elements of $\{1, \dots, m\}$. The reflection r_i stabilizes the i^{th} 1-cell at v. Let s_i denote the unique lift of r_i which stabilizes the i^{th} 1-cell at $\tilde{v}$. Since s_i^2 fixes $\tilde{v}$ and covers the identity on P_L, it follows that $s_i^2 = 1$. Suppose σ is a 1-simplex of L connecting vertices i and j. The corresponding 2-cell at $\tilde{v}$ is then a square with edges labelled successively by s_i, s_j, s_i, s_j. Hence, we get a relation $(s_i s_j)^2 = 1$ for each 1-simplex $\{i, j\}$ of L.

Let $\hat{W}_L$ denote the group defined by this presentation, i.e., a set of generators for $\hat{W}_L$ is $\{s_i\}_{i \in I}$ and the relations are given by: $s_i^2 = 1$, for all $i \in I$, and $(s_i s_j)^2 = 1$ whenever $\{i, j\}$ is a 1-simplex of L. The Cayley 2-complex associated to this presentation maps to the 2-skeleton of $\tilde{P}_L$ by a covering projection. Since $\tilde{P}_L$ is simply connected, this covering projection is a homeomorphism and the natural homomorphism $\hat{W}_L \to W_L$ is an isomorphism. Therefore, we have proved the following.

Proposition 4.1. *W_L has the presentation described above.*

Remarks 4.2.

- A group with a presentation of the above form is a Coxeter group, in fact, it is a "right-angled" Coxeter group. (In a general Coxeter group we allow relations of the form $(s_i s_j)^{m_{ij}} = 1$, where the integers m_{ij} can be > 2.)
- Examining the presentation, we see that the abelianization of W_L is J. Thus, Γ_L is the commutator subgroup of W_L.

For each subset σ of I, let W_σ denote the subgroup generated by $\{s_i\}_{i\in\sigma}$. If $\sigma \in \mathcal{S}(L)$, then W_σ is the stabilizer of the corresponding cube in $\tilde{P}_L$ which contains $\tilde{v}$ (and so, for $\sigma \in \mathcal{S}(L)$, $W_\sigma \cong (\mathbb{Z}/2)^{\mathrm{Card}(\sigma)}$). It follows that the poset of cells of $\tilde{P}_L$ is isomorphic to the poset of cosets,

$$\coprod_{\sigma\in\mathcal{S}(L)} W_L/W_\sigma.$$

5 When is $\tilde{P}_L$ contractible?

Definition 5.1. A simplicial complex L is a *flag complex* if any finite set of vertices which are pairwise connected by edges spans a simplex of L.

The following theorem is proved in [5]. Proofs can also be found in [8] or [12].

Theorem 5.2. *$\tilde{P}_L$ is contractible if and only if L is a flag complex.*

Definition 5.3. A path connected space X is *aspherical* if $\pi_i(X) = 0$, for all $i \geq 2$.

Suppose that a space X has the homotopy type of a CW complex. A basic fact of covering space theory is that the higher homotopy groups of X (that is, the groups $\pi_i(X)$, for $i \geq 2$) are isomorphic to those of its universal cover $\tilde{X}$. Thus, X is aspherical if and only if $\tilde{X}$ is contractible. Therefore, we have the following corollary to Theorem 5.2.

Corollary 5.4. *P_L is aspherical if and only if L is a flag complex.*

Before sketching the proof of Theorem 5.2 in Section 7, we make a few comments on the nature of flag complexes.

An *incidence relation* on a set I is a symmetric and reflexive relation R. A *flag* in I is defined to be a finite subset of I of pairwise incident elements. The poset *Flag*(R) of nonempty flags in I is then a simplicial complex with vertex set I. It is obviously a flag complex. Conversely, any flag complex arises from this construction. (Indeed, given a flag complex L define two vertices to be incident if they are connected by an edge; if R denotes this incidence relation, then $L = \mathit{Flag}(R)$.)

Given a poset, we can symmetrize the partial order relation to get an incidence relation. A flag is then a finite chain of elements in the poset. If $\mathcal{P}$

is a poset, then let $Flag(\mathcal{P})$ denote the flag complex of chains in $\mathcal{P}$. If $\mathcal{F}$ is the poset of nonempty cells in a regular convex cell complex P, then $Flag(\mathcal{F})$ can be identified the poset of simplices in the barycentric subdivision of P. It follows that the condition of being a flag complex does not restrict the topological type of L: it can be any polyhedron.

Example 5.5. A *polygon* is a simplicial complex homeomorphic to S^1. It is a *k-gon* if it has k edges. A k-gon is a flag complex if and only if $k > 3$.

Remarks 5.6.

- Gromov in [16] has used the terminology that a simplicial complex L satisfies the "no Δ condition" to mean that it is a flag complex. The idea is that L is a flag complex if and only if it has no "missing simplices".

- A flag complex is a simplicial complex which is, in a certain sense, "determined by its 1-skeleton". Indeed, suppose Λ is a 1-dimensional simplicial complex. Then Λ determines an incidence relation on its vertex set. The associated flag complex L is constructed by filling in the missing simplices corresponding to the complete subgraphs of Λ. (So Λ is the 1-skeleton of L.)

- The graph Λ also provides the data for a presentation of a right-angled Coxeter group W $(= W_\Lambda)$. In this case, the associated flag complex L is called the *nerve* of the Coxeter group.

6 Nonpositive curvature

The notion of "nonpositive curvature" makes sense for a more general class of metric spaces than Riemannian manifolds. A *geodesic* in a metric space X is a path $\gamma : [a, b] \to X$ which is an isometric embedding. X is called a *geodesic space* if any two points can be connected by a geodesic segment. A *triangle* in a geodesic space X is the image of three geodesic segments meeting at their endpoints. Given a triangle T in X, there is a triangle T^* in $\mathbb{R}^2$ with the same edge lengths. T^* is called a *comparison triangle* for T. To each point $x \in T$ there is a corresponding point $x^* \in T^*$. The triangle T is said to *satisfy the* $CAT(0)$-*inequality*, if given any two points $x, y \in T$ we have $d(x,y) \le d(x^*,y^*)$. The space X is *nonpositively* curved if the $CAT(0)$-inequality holds for all sufficiently small triangles. X is a $CAT(0)$-*space* (or a *Hadamard space*) if it is complete and if the $CAT(0)$-inequality holds for all triangles in X. It follows immediately from the definitions that there is a unique geodesic between any two points in a $CAT(0)$-space and from this that any $CAT(0)$-space is contractible. Gromov observed that the universal cover of a complete nonpositively curved geodesic space is $CAT(0)$. Hence, any such nonpositively curved space is aspherical. (An excellent reference for this material is [2].)

Next, suppose that P is a connected cubical cell complex. There is a natural piecewise Euclidean metric on P. Roughly speaking, it is defined by declaring each cell of P to be (locally) isometric to a standard Euclidean cube (of edge length 2). More precisely, the distance between two points $x, y \in P$ is defined to be the infimum of the lengths of all piecewise linear curves connecting x to y. It then can be shown that P is a complete geodesic space. Gromov [16, p. 122] proved the following.

Lemma 6.1 (Gromov's Lemma). *A cubical cell complex P is nonpositively curved if and only if the link of each of its vertices is a flag complex.*

A corollary is that the previously constructed cubical complex $\tilde{P}_L$ is $CAT(0)$ if and only if L is a flag complex. In particular, this gives a proof in one direction of Theorem 5.2: if L is a flag complex, then $\tilde{P}_L$ is contractible. (A proof of Gromov's Lemma can be found in [2], [8], or [12].)

7 Another construction of P_L and $\tilde{P}_L$

The group J $(= (\mathbb{Z}/2)^m)$ acts as a group generated by reflection on $[-1,1]^m$. The subspace $[0,1]^m$ is a fundamental domain. (Also, $[0,1]^m$ can be identified with the orbit space of the J-action.) Let e be the vertex $(1,\dots,1) \in [0,1]^m$ and for each subset σ of I $(= \{1,\dots,m\})$ let $\Box_\sigma^+ = (e + \Box_\sigma) \cap [0,1]^m$.

The corresponding fundamental domain K for the J-action on P_L is given by

$$K = P_L \bigcap [0,1]^m.$$

Thus,

$$K = \bigcup_{\sigma \in \mathcal{S}(L)} \Box_\sigma^+$$

For each $i \in I$ let $[0,1]^m_i$ denote the intersection of the fixed point set of r_i with $[0,1]^m$, i.e., $[0,1]^m_i = [0,1]^m \cap \{x_i = 0\}$. Set $K_i = K \cap [0,1]^m_i$ and call it a *mirror* of K. It is not difficult to see that $\bigcup K_i$ can be identified with the barycentric subdivision of L, that K_i is the closed star of the vertex i in this barycentric subdivision and that K is homeomorphic to the cone on L.

For each $x \in K$, let $\sigma(x) = \{i \in I | x \in K_i\}$. The space P_L can be constructed by pasting together copies of K, one for each element of J. More precisely, define an equivalence relation $\sim$ on $J \times K$ by $(g, x) \sim (g', x')$ if and only if $x = x'$ and $g^{-1}g' \in J_{\sigma(x)}$. (In other words, $\sim$ is the equivalence relation generated by identifying $g \times K_i$ with $gr_i \times K_i$, for all $i \in I$ and $g \in J$.) Then

$$P_L \cong (J \times K)/\sim .$$

Similarly,

$$\tilde{P}_L \cong (W_L \times K)/\sim$$

where the equivalence relation on $W_L \times K$ is defined in an analogous fashion.

Let $\hat{L}$ denote the flag complex determined by the 1-skeleton of L. Thus, $\hat{L}$ is the nerve of $W_L : \sigma \in \mathcal{S}(\hat{L})$ if and only if W_σ is finite. For each $\sigma \in \mathcal{S}(\hat{L})$, set

$$K^\sigma = \bigcup_{i \in \sigma} K_i$$

We are now in position to prove two lemmas which imply Theorem 5.2.

Lemma 7.1. *The following conditions are equivalent.*

(i) L is a flag complex.

(ii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is contractible.

(iii) For each $\sigma \in \mathcal{S}(\hat{L}), K^\sigma$ is acyclic.

Proof. If L is a flag complex, then $L = \hat{L}$ and K^σ can be identified with a closed regular neighborhood of σ in the barycentric subdivision of L. Hence, (i)$\Rightarrow$ (ii). The implication (ii)$\Rightarrow$(iii) is obvious. If $\sigma \in \mathcal{S}(\hat{L})$ is a k-simplex such that $\partial\sigma \subset L$ but $\sigma \not\subset L$, then K^σ has the homology of a $(k-1)$-sphere (since each K_i, $i \in \sigma$, as well as each proper family of intersections of the K_i is contractible). Hence, (iii)$\Rightarrow$(i). ∎

For each $w \in W$, let $\sigma(w) = \{i \in I | \ell(ws_i) < \ell(w)\}$. Here $\ell : W \to \mathbb{N}$ denotes word length. Thus, $\sigma(w)$ is the set of letters with which a minimal word for w can end. Geometrically, it indexes the set of mirrors of wK such that the adjacent chamber ws_iK is one chamber closer to the base chamber K. The following is a basic fact about Coxeter groups (its proof, which can be found in [5], is omitted).

Lemma 7.2. *For each $w \in W_L$, $\sigma(w) \in \mathcal{S}(\hat{L})$. (In other words, for any w, $\sigma(w)$ generates a finite subgroup of W_L.)*

Now we can sketch the proof of Theorem 5.2: that $\tilde{P}_L$ is contractible if and only if L is a flag complex.

Proof of Theorem 5.2. First note that $\tilde{P}_L$ is simply connected. Hence, it suffices to compute the homology of $\tilde{P}_L$. Order the elements of $W_L : w_1, w_2, \ldots,$ so that $\ell(w_{k+1}) \geq \ell(w_k)$. Set $Y_k = w_1K \cup \cdots \cup w_kK$. Then $w_kK \cap Y_{k-1} \cong K^{\sigma(w_k)}$. Hence, $H_*(Y_k, Y_{k-1}) \cong H_*(K, K^{\sigma(w_k)}) \cong \tilde{H}_{*-1}(K^{\sigma(w_k)})$, where the second isomorphism holds since K is contractible. The exact sequence of the pair (Y_k, Y_{k-1}) gives

$$\to H_*(Y_{k-1}) \to H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$$

The map $H_*(Y_k) \to H_*(K, K^{\sigma(w_k)})$ is a split surjection. Indeed, a splitting $\varphi_* : H_*(K, K^{\sigma(w_k)}) \to H_*(Y_k)$ can be defined by the formula:

$$\varphi(\alpha) = \sum_{u \in W_{\sigma(w_k)}} (-1)^{\ell(u)} w_k u \alpha$$

where α is a relative cycle in $C_*(K, K^{\sigma(w_k)})$. It is then easy to see that $\varphi(\alpha)$ is a cycle and that the induced map on homology is a splitting. Thus, $H_*(Y_k) \cong H_*(Y_{k-1}) \oplus H_*(K, K^{\sigma(w_k)})$, and consequently,

$$\tilde{H}_*(\tilde{P}_L) \cong \bigoplus_{i=1}^{\infty} \tilde{H}_{*-1}(K^{\sigma(w_k)}).$$

Thus, $\tilde{P}_L$ is acyclic if and only if each $K^{\sigma(w_k)}$ is acyclic. The theorem follows from Lemmas 7.1 and 7.2. □

Henceforth, we shall always assume that L is a flag complex. Moreover, we shall use the notation Σ_L instead of $\tilde{P}_L$.

8 The reflection group trick

Next we modify the construction of the previous section.

Suppose X is a space and that ∂X is a subspace which is homeomorphic to a polyhedron. Let L be a triangulation of ∂X as a flag complex with vertex set I. For each $i \in I$, put $X_i = K_i$ where K_i is the previously defined subcomplex of the barycentric subdivision of L. In other words, X_i is the closed star of the vertex i in the barycentric subdivision of L $(= \partial X)$. Let J, W_L and Γ_L be the groups defined previously. Set

$$P_L(X) = (J \times X)/\sim$$

and

$$\Sigma_L(X) = (W_L \times X)/\sim,$$

where the equivalence relations are defined exactly as before.

We record a few elementary properties of this construction. The orbit space of the J-action on $P_L(X)$ is X. Let $r : P_L(X) \to X$ be the orbit map. Since X can also be regarded as a subspace of $P_L(X)$ (namely as the image of $1 \times X$), we have the following theorem.

Theorem 8.1. *The map $r : P_L(X) \to X$ is a retraction.*

Corollary 8.2. *$\pi_1(X)$ is a retract of $\pi_1(P_L(X))$.*

Theorem 8.3. *If X is a compact n-dimensional manifold with boundary $(= \partial X)$, then $P_L(X)$ is a closed n-manifold.*

Proof. For each $x \in X$, let $\sigma(x) = \{i \in I | x \in X_I\}$. A neighborhood of x in ∂X has the form $\mathbb{R}^{n-k} \times \mathbb{R}_+^{\sigma(x)}$ where $k = \mathrm{Card}(\sigma(x))$, where $\mathbb{R}_+^{\sigma(x)}$ denotes the positive quadrant in $\mathbb{R}^{\sigma(x)}$ where all coordinates are nonnegative. It follows that a neighborhood of $(1, x)$ in $P_L(X)$ has the form $\mathbb{R}^{n-k} \times (J_{\sigma(x)} \times \mathbb{R}_+^{\sigma(x)})/\sim$ which is homeomorphic to $\mathbb{R}^n$. Thus, $P_L(X)$ is an n-manifold. $\square$

We also have that $\Sigma_L(X) \to P_L(X)$ is a regular covering with group of deck transformation Γ_L. The proof of the next proposition is the same as the proof of Theorem 5.2 given in Section 7.

Proposition 8.4. *(i) If X is simply connected, then $\Sigma_L(X)$ is the universal cover of $P_L(X)$ and hence, $\pi_1(P_L(X)) = \Gamma_L$.*

(ii) If X is contractible, then $\Sigma_L(X)$ is contractible.

Further discussion of the reflection group trick can be found in [5] and [7].

9 Aspherical manifolds not covered by Euclidean space

Suppose Y is a reasonable space (for example, suppose Y is a locally compact, locally path connected, second countable Hausdorff space). Also, suppose Y is not compact. A *neighborhood* of *infinity* in Y is the complement of a compact set. Y is *one-ended* if every neighborhood of infinity contains a connected neighborhood of infinity. A one-ended space Y is *simply connected at infinity* if for any compact subset $C \subset Y$ there is a larger compact subset $C' \supset C$ such that for any loop γ in $Y - C'$, γ is null-homotopic in $Y - C$.

For example, $\mathbb{R}^n$ is one-ended for $n \geq 2$ and simply connected at infinity for $n \geq 3$. The following characterization of Euclidean space was proved by Stallings for $n \geq 5$ and by Freedman for $n = 4$. For $n = 3$, the corresponding result is not known (it is a version of the 3-dimensional Poincaré Conjecture).

Theorem 9.1. (Stallings [26] and Freedman [15]) *Let M^n be a contractible n-manifold, $n \geq 4$. Then M^n is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity.*

In certain circumstances it is possible to define a "fundamental group at infinity" for a one-ended space Y. Suppose that $C_1 \subset C_2 \subset \dots$ is an exhaustive sequence of compact subsets (i.e., $Y = \bigcup C_i$). This gives an inverse system of fundamental groups, $\pi_1(Y - C_1) \leftarrow \pi_1, (Y - C_2) \leftarrow \dots$. In general, an inverse sequence of groups, $G_1 \leftarrow G_2 \leftarrow \dots$, is *Mittag-Leffler* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that the image of G_k in G_n is the same for all $k \geq f(n)$. The space Y is *semistable* if there is such an inverse system of fundamental groups which is Mittag-Leffler. If this condition holds for one such choice of inverse system, then it holds for all such choices and the resulting inverse limit is independent of the choice of base points. Hence, when Y is semistable, we can define its *fundamental group at infinity*, denoted by $\pi_1^\infty(Y)$, to be the inverse limit, $\varprojlim \pi_1(Y - C_k)$.

Definition 9.2. A closed manifold N^n is a *homology n-sphere* if $H_*(N^n) \cong H_*(S^n)$

Poincaré originally conjectured that any homology 3-sphere was homeomorphic to S^3. However, he soon discovered a counterexample. It was S^3/G, where S^3 is regarded as the Lie group $SU(2)$ and where G is the binary icosahedral group (a subgroup of order 120). (In order to prove that S^3/G

was not homeomorphic to the 3-sphere, Poincaré invented the concept of the fundamental group and proved that it was a topological invariant.) It turns out that, for $n \geq 3$, there are many other examples of homology spheres N^n which are not simply connected (however, note that $\pi_1(N^n)$ must be a perfect group).

For $n \geq 5$, the next result is an easy consequence of surgery theory. For $n = 4$, it was proved by Freedman [15]. (For $n = 4$, the result holds only in the category of topological manifolds.)

Theorem 9.3. *Let N^{n-1} be a homology $(n-1)$-sphere. Then there is a compact contractible n-manifold with boundary X such that $\partial X = N^{n-1}$.*

Now suppose that a flag complex L is a homology $(n-1)$-sphere and that X^n is a contractible n-manifold with $\partial X = L$. By Theorem 8.3 and Proposition 8.4, $P_L(X)$ is an aspherical n-manifold and its universal cover $\Sigma_L(X)$ is a contractible n-manifold.

Proposition 9.4. *Suppose, as above, that L is a homology $(n-1)$-sphere bounding a contractible n-manifold X. If L is not simply connected, then $\Sigma_L(X)$ is not simply connected at infinity.*

Proof. As in the proof of Theorem 5.2 at the end of Section 7, order the elements of W_L: $w_1, w_2, \ldots$ so that $\ell(w_{k+1}) \geq \ell(w_k)$ and put

$$Y_k = w_1 X \cup \cdots \cup w_k X.$$

Then Y_k is a contractible manifold with boundary. Let $\overline{Y}_k$ denote the complement of an open collared neighborhood of ∂Y_k in Y_k. Then it is easy to see that $\Sigma_L(X) - \overline{Y}_k$ is homotopy equivalent to ∂Y_k. Moreover, since $w_k X \cap Y_{k-1} \cong K^{\sigma(w_k)}$, which is an $(n-1)$-disk, it follows that ∂Y_k is the connected sum of k copies of ∂X $(= L)$. Hence, if $n \geq 3$, $\pi_1(\partial Y_k)$ is the free product of k copies of $\pi_1(L)$ and the inverse system of fundamental groups is,

$$\pi_1(L) \leftarrow \pi_1(L) * \pi_1(L) \leftarrow \ldots$$

Thus, if $\pi_1(L)$ is not trivial, then neither is $\pi_1^\infty(\Sigma_L(X))$. □

As a corollary of Proposition 9.4 and the fact that there exist homology $(n-1)$-spheres with nontrivial fundamental groups for all $n \geq 4$, we have proved the following result of [5].

Theorem 9.5. *For each $n \geq 4$, there is a closed aspherical n-manifold (of the form $P_L(X)$) such that its universal cover is not homeomorphic to $\mathbb{R}^n$.*

10 The reflection group trick, continued

As before, suppose $(X, \partial X)$ is a pair of spaces and L is a triangulation of ∂X as a flag complex.

Theorem 10.1. *If X is aspherical, then so is $P_L(X)$.*

Proof. It suffices to show that the covering space $\Sigma_L(X)$ of $P_L(X)$ is aspherical. Order the elements of W_L as before and set $Y_k = w_1X \cup \cdots \cup w_kX$. Then, Y_k is formed by gluing on a copy of X to Y_{k-1}. Since $w_kX \cap Y_{k-1}$ is contractible, Y_k is homotopy equivalent to $Y_{k-1} \vee X$ and hence, to $X \vee \cdots \vee X$ (k times). Since X is aspherical, so is Y_k. Since $\Sigma_L(X)$ is the increasing union of the Y_k, $\Sigma_L(X)$ is also aspherical. □

Definition 10.2. A group π is of *type F* if its classifying space $B\pi$ has the homotopy type of a finite complex. ($B\pi$ is also called the "$K(\pi, 1)$-complex.")

If B is a finite CW complex, then we can "thicken" it to a compact manifold with boundary. This means that we can find a compact manifold with boundary X which is homotopy equivalent to B. The proof goes as follows. First, up to homotopy, we can assume that B is a finite simplicial complex. The next step is to piecewise linearly embed B in some Euclidean space $\mathbb{R}^n$. So, we can assume B is a subcomplex of some triangulation of $\mathbb{R}^n$. Finally, possibly after taking barycentric subdivisions, we can replace B by a regular neighborhood X in $\mathbb{R}^n$.

The "reflection group trick" can then be summarized as follows. Start with a group π of type F. After thickening we may assume that $B\pi$ is a compact manifold with boundary X. Triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold which retracts onto $B\pi$.

11 The Assembly Map Conjecture

Conjecture 11.1 (The Borel Conjecture). *Let $(M, \partial M)$ and $(M', \partial M')$ be aspherical manifolds with boundary and $f : (M, \partial M) \to (M', \partial M')$ a homotopy equivalence such that $f|_{\partial M}$ is a homeomorphism. Then f is homotopic rel ∂M to a homeomorphism.*

In the case where $\partial M = \emptyset$, the Borel Conjecture asserts that two closed aspherical manifolds with the same fundamental group are homeomorphic.

Suppose $(X, \partial X)$ is a pair of finite complexes with $\pi_1(X) = \pi$. Then $(X, \partial X)$ is a *Poincaré pair* of dimension n if there is a $\mathbb{Z}\pi$-module D which is isomorphic to $\mathbb{Z}$ as an abelian group and a homology class $\mu \in H_n(X, \partial X; D)$ so that for any $\mathbb{Z}\pi$-module A, cap product with μ defines an isomorphism: $H^i(X; A) \cong H_{n-i}(X, \partial X; D \otimes A)$. If $\partial X = \emptyset$, then X is a *Poincaré complex.* A group π of type F is a *Poincaré duality group of dimension* n (or a PD^n-*group* for short) if $B\pi$ is a Poincaré complex.

Conjecture 11.2 (The PD^n-group Conjecture). *Suppose π is a group of type F and that $(X, \partial X)$ is a Poincaré pair with X homotopy equivalent to $B\pi$ and with ∂X a manifold. Then $(X, \partial X)$ is homotopy equivalent rel ∂X to a compact manifold with boundary.*

A weak version of this conjecture replaces the word manifold by "ANR homology manifold."

In the absolute case, where $\partial X = \emptyset$, the PD^n-group Conjecture asserts that for any PD^n-group π of type F, $B\pi$ is homotopy equivalent to a closed manifold.

As is explained elsewhere in this volume, there is a surgery exact sequence:

$$\dots L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(X, \partial X) \to [(X, \partial X), (G/TOP, *)] \to L_n(\mathbb{Z}\pi)$$

where $\mathcal{S}(X, \partial X)$ denotes the structure set, G/TOP is a certain space, $\pi = \pi_1(X)$ and where $L_n(\mathbb{Z}\pi)$ denotes Wall's surgery group for $\mathbb{Z}\pi$. The surgery groups are 4-periodic. The homotopy groups of G/TOP are also 4-periodic and are given by the formula:

$$\pi_i(G/TOP) = \begin{cases} \mathbb{Z}, & \text{if } i \equiv 0 \pmod 4; \\ \mathbb{Z}/2, & \text{if } i \equiv 2 \pmod 4; \\ 0, & \text{otherwise} \end{cases}$$

Moreover, the 4-fold loop space, $\Omega_4(\mathbb{Z} \times G/TOP)$, is homotopy equivalent to $\mathbb{Z} \times G/TOP$. It follows that $\mathbb{Z} \times G/TOP$ defines a spectrum $\mathbb{L}$ and a generalized homology theory $H_*(X; \mathbb{L})$. It is almost, but not quite, true that

$$H_n(X; \mathbb{L}) \cong \bigoplus_k H_{n-4k}(X; \mathbb{Z}) \oplus H_{n-4k-2}(X; \mathbb{Z}/2).$$

Quinn has defined an "assembly map" $A_n : H_n(X; \mathbb{L}) \to L_n(\mathbb{Z}\pi)$ so that if we use Poincaré duality to identify $H_n(X; \mathbb{L})$ with $[(X, \partial X), (G/TOP, *)] \cong H^0(X, \partial X; \mathbb{L})$, then A_n is identified with the surgery obstruction map.

Conjecture 11.3 (The Assembly Map Conjecture). *Suppose π is a group of type F. Then the assembly map $A_* : H_*(B\pi;\mathbb{L}) \to L_*(\mathbb{Z}\pi)$ is an isomorphism.*

In dimensions ≥ 5, for a given group π of type F, it follows from the surgery exact sequence that the truth the Assembly Map Conjecture and the conjecture that the Whitehead group of π vanishes is equivalent to the truth of both the Borel Conjecture and (the weak version of) the PD^n-group Conjecture.

Conceivably, the Assembly Map Conjecture could be true for any torsion-free group π.

Theorem 11.4. *The Assembly Map Conjecture is true for the fundamental groups of all closed aspherical manifolds if and only if it is true for all groups of type F.*

Proof. We use the reflection group trick. Suppose π is a group of type F. Thicken $B\pi$ to a manifold with boundary X and triangulate ∂X as a flag complex L. Then $P_L(X)$ is a closed aspherical manifold. Let $r : P_L(X) \to X$ be the retraction from Theorem 8.1 and let $G = \pi_1(P_L(X))$. We have the following commutative diagram:

$$\begin{array}{ccc} H_n(P_L(X);\mathbb{L}) & \to & L_n(\mathbb{Z}G) \\ i_* \uparrow\downarrow r_* & & i_* \uparrow\downarrow r_* \\ H_n(X;\mathbb{L}) & \to & L_n(\mathbb{Z}\pi). \end{array}$$

Hence, if the arrow on the top row is an isomorphism, so is the arrow on the bottom. □

Remark 11.5. A similar argument shows that the Whitehead group vanishes for the fundamental groups of all closed aspherical manifolds if and only if it vanishes for all groups of type F.

Remark 11.6. What these arguments show is that, in dimensions ≥ 5, if we have a counterexample to the relative version of the Borel Conjecture or to the relative version of the PD^n-group Conjecture, then the reflection group trick will provide us with a counterexample in the absolute case. For example, suppose $f : (M, \partial M) \to (M', \partial M')$ is a counterexample to the Borel Conjecture. We might as well assume that $\partial M = \partial M' = L$ and that

$f|_{\partial M} =$id. Then f induces a homotopy equivalence $P_L(M) \to P_L(M')$ which is not homotopy equivalent to a homeomorphism. Similarly, if $(X, \partial X)$ is a counterexample to the PD^n-group Conjecture and $L = \partial X$, then $G = \pi_1(P_L(X))$ is a PD^n-group which is not the fundamental group of a closed aspherical manifold.

12 Aspherical manifolds which cannot be smoothed

Suppose $(X, \partial X)$ is a compact aspherical n-manifold with boundary and that ∂X is triangulable. Suppose further that the Spivak normal fibration of X does not reduce to a linear vector bundle. (In other words, a certain map $X \to BG$ does not lift to BO.) Apply the reflection group trick with $L = \partial X$. Since X is codimension 0 in $P_L(X)$ and since X is a retract of $P_L(X)$, the Spivak normal fibration of $P_L(X)$ does not reduce to linear vector bundle. Hence, $P_L(X)$ cannot be homotopy equivalent to a smooth manifold. In [9] J-C. Hausmann and I showed that there exist examples of such X for each $n \geq 13$, thereby proving the following result. (We will prove a stronger result in Section 16.)

Theorem 12.1. ([9]) *In each dimension ≥ 13, there is a closed aspherical manifold not homotopy equivalent to a smooth manifold.*

13 Further applications of the reflection group trick

The next two results were proved by G. Mess.

Theorem 13.1. (Mess [21]). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which is not residually finite.*

(Recall that a group Γ is *residually finite* if given any two elements $\gamma_1, \gamma_2 \in \Gamma$ there is a homomorphism ϕ to some finite group F such that $\phi(\gamma_1) \neq \phi(\gamma_2)$.)

Theorem 13.2. (Mess [21]). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which contains an infinitely divisible abelian group.*

On the other hand, it is known that there are no such examples in dimension 3.

Remark 13.3. With regard to the first theorem, Tom Farrell pointed out to me that Raganuthan has observed that there are cocompact lattices in (nonlinear) Lie groups which are not virtually torsion-free and not residually finite. (See [23].) On the other hand, it follows from Selberg's Lemma that any finitely generated subgroup of a linear Lie group is residually finite.

The proofs of both theorems are similar. By a theorem of R. Lyndon [20], if π is a finitely generated 1-relator group and if the relation cannot be written as a proper power of another word, then the presentation 2-complex for π is aspherical. In particular, any such π is of type F with a 2-dimensional $B\pi$. This 2-complex can then be thickened to a compact 4-manifold. For Theorem 13.1 take π to be the Baumslag-Solitar group $< a, b | ab^2a^{-1} = b^3 >$. It is known that π is not residually finite; hence, neither is any group which contains it. For Theorem 13.2 take π to be the Baumslag-Solitar group $< a, b | aba^{-1} = b^2 >$. The centralizer of b in this group is isomorphic to a copy of the dyadic rationals.

S. Weinberger has noted that there are examples of finitely presented groups π with unsolvable word problem such that $B\pi$ is a finite 2-complex. (See [22, Theorem 4.12].) Since any group which retracts onto such a group also has unsolvable word problem, the reflection group trick gives the following result.

Theorem 13.4. (Weinberger). *For each $n \geq 4$, there is a closed aspherical n-manifold the fundamental group of which has unsolvable word problem.*

14 Hyperbolization

"Hyperbolization" refers to certain constructions, invented by Gromov [16], for converting any cell complex into an aspherical polyhedron (in fact, into a nonpositively curved polyhedron). One of the key features of these constructions is that they preserve local structure. Thus, hyperbolization will convert an n-manifold into an aspherical n-manifold. In this section we describe one such construction, Gromov's "Möbius band hyperbolization procedure," which given a cubical cell complex P produces an aspherical cubical cell complex $h(P)$. Before giving the definition, we list some properties of the construction:

1) The procedure is functorial in the following sense: if $f : P \to Q$ is a cellular embedding onto a subcomplex, then there is an induced embedding $h(f) : h(P) \to h(Q)$ onto a subcomplex.

2) The cubical complexes P and $h(P)$ have the same vertex set. Moreover, for each vertex v, $Lk(v, h(P))$ is the barycentric subdivision of $Lk(v, P)$.

In particular, it follows from 2) that if P is an n-manifold, then so is $h(P)$. Property 2) also shows that the link of each vertex in $h(P)$ is a flag complex. Hence, by Gromov's Lemma 6.1, the piecewise Euclidean metric on $h(P)$ is nonpositively curved. Consequently, we have the following property:

3) $h(P)$ is aspherical.

By functoriality, each cube $\square$ in P is converted into a subspace $h(\square)$ of $h(P)$ (called a "hyperbolized cell"). Furthermore, there is a map $c : h(P) \to P$, unique up to homotopy, which is the identity on the vertex set and which takes each hyperbolized cell to the corresponding cell of P. The map c has the following two properties:

4) c induces a surjection on homology groups with coefficients in $\mathbb{Z}/2$.

5) $c_* : \pi_1(h(P)) \to \pi_1(P)$ is surjective.

We also list one final property:

6) If P is an n-manifold, then there is an (unoriented) cobordism between P and $h(P)$.

A corollary of 6) is the following.

Theorem 14.1. *if two closed aspherical manifolds M_1 and M_2 were cobordant, then one could choose the cobordism to be an aspherical manifold with boundary. Although I don't know how to prove Gromov's assertion, the construction in Remark 14.2 does prove the following result. Every triangulable manifold is cobordant to an aspherical manifold.*

The definition of $h(P)$ is by induction on dim P. If dim $P \leq 1$, then $h(P) = P$. Suppose that the construction has been defined for all cubical complexes of dimension $< n$ and that properties 1) and 2) hold. Let $\square^n$

denote the standard n-cube and let $a : \square^n \to \square^n$ denote the antipodal map (a is also called the "central symmetry" of $\square^n$). By induction, $h(\partial\square^n)$ has been defined and by 1) the isomorphism a induces an involution $h(a) : h(\partial\square^n) \to h(\partial\square^n)$. The quotient space $h(\partial\square^n)/(\mathbb{Z}/2)$ is also a cubical complex. We define $h(\square^n)$ to be the canonical interval bundle over $h(\partial\square^n)/(\mathbb{Z}/2)$, i.e.,

$$h(\square^n) = [\text{-}1,1] \times_{\mathbb{Z}/2} h(\partial\square^n)$$

where $\mathbb{Z}/2$ acts on the first factor via $t \to -t$ and on the second via $h(a)$.

Since the restriction of this interval bundle to each cell of $h(\partial\square^n)/\mathbb{Z}/2)$ is a trivial bundle, $h(\square^n)$ naturally has the structure of a cubical complex: each new cell is the product of [-1,1] with a cell of $h(\partial\square^n)/(\mathbb{Z}/2)$. It follows that if v is a vertex of $\square^n$, then there is a $(k+1)$-simplex in $Lk(v, h(\square^n))$ for each k-simplex in $Lk(v, h(\partial\square^n))$. Thus, $Lk(v, h(\square^n))$ is the cone on $Lk(v, h(\partial\square^n))$. Using induction, this implies that $Lk(v, h(\square^n))$ is the barycentric subdivision of Δ^{n-1}.

We note that the boundary of $h(\square^n)$ is canonically identified with $h(\partial\square^n)$. (The identification is canonical because a lies in the center of the automorphism group of the cube.)

Let $P^{(k)}$ denote the k-skeleton of a cubical complex P. If dim $P = n$, then $h(P)$ is defined by attaching, for each n-cell $\square^n$ in P, a copy of $h(\square^n)$ to the subcomplex $h(\partial\square^n)$ of $h(P^{(n-1)})$ via the canonical identification.

Suppose that dim $P = n$, that $f : P \to Q$ is a cellular embedding, that $f^{(n-1)} : P^{(n-1)} \to Q^{(n-1)}$ denotes the restriction to the $(n-1)$-skeleton and that $h(f^{(n-1)})$ has been defined. The map $h(f) : h(P) \to h(Q)$ is induced by the map on each new cell which is the product of the identity map of [-1,1] with $h(f^{(n-1)})$. Property 1) follows. The new links are clearly the barycentric subdivisions of the old ones (so property 2) holds). The proof of properties 3) and 4) are straightforward and are left to the reader.

To check property 5), let $\tilde{P}$ denote the universal cover of P and let $\pi = \pi_1(P)$. By functorality, π acts freely on $h(\tilde{P})$ and $h(\tilde{P}) \to h(\tilde{P})/\pi$ is a covering projection. In fact, it is clear that $h(\tilde{P})/\pi \cong h(P)$. This defines an epimorphism $\varphi : \pi_1(h(P)) \to \pi$. It is not hard to see that φ is just the homomorphism induced by the canonical map $c : h(P) \to P$.

To check 6), suppose that M is triangulated manifold. Let CM denote the cone on M, i.e., $CM = (M \times [0,1])/\sim$ where $(x, 0) \sim (x', 0)$ for all $x, x' \in M$. Let c denote the cone point. The cone on the barycentric subdivision of a k-simplex is a standard subdivision of a $(k+1)$-cube. Regarding each simplex as the cone on its boundary, this gives a standard method for subdividing each simplex of CM into cubes and gives CM the structure of a cubical complex. The link of c in $h(CM)$ is isomorphic to the barycentric subdivision of M. Hence, removing the open star of c from $h(CM)$ we obtain a cobordism between M and $h(M)$ $(= h(M \times 1))$.

Remark 14.2. There is a slight variation of the above which provides a sort of "relative hyperbolization procedure." Suppose Q is a subcomplex of P. Let $p : \tilde{P} \to P$ be the universal cover and let $\hat{P}$ be the cubical complex formed by attaching a cone to each component of $p^{-1}(Q)$ in $\tilde{P}$. Let $h(\hat{P})_0$ denote the complement of small regular neighborhoods of the cone points in $h(\hat{P})$. By functorality, the fundamental group $\pi = \pi_1(P)$ acts on $h(\hat{P})_0$. Define the *hyperbolization of P relative to Q* by

$$h(P, Q) = h(\hat{P})_0/\pi,$$

In general, $h(P, Q)$ will not be aspherical. However, if Q is aspherical and if $\pi_1(Q) \to \pi_1(P)$ is injective, then the link of each cone point in $h(\hat{P})$ is contractible (since it is a copy of the universal cover of Q). Hence, $h(\hat{P})$ and $h(\hat{P})_0$ will be homotopy equivalent. So, in this case, $h(P, Q)$ is aspherical.

Not only did Gromov state Theorem 14.1, he also asserted that if an aspherical manifold M is a boundary, then it bounds an aspherical manifold N so that $\pi_1(M) \to \pi_1(N)$ is injective. We shall give a proof of this below. First we need the following result of [17].

Theorem 14.3. (Hausmann [17]). *Suppose that manifolds M_1 and M_2 are cobordant. Then they are cobordant by a cobordism N such that for $i = 1, 2$, $\pi_1(M_i) \to \pi_1(N)$ is injective.*

Shmuel Weinberger explained the following version of Hausmann's proof to me.

Proof. It is proved in [1, Theorem 5.5] that for any finitely generated group G one can find a finitely generated acyclic group A such that G injects into A. For $i = 1, 2$ choose an acyclic group A_i so that $\pi_1(M_i)$ injects into A_i. Then $\pi_1(M_i)$ injects into the acyclic group $A = A_1 \times A_2$. Let $\Omega_*(-)$ be any bordism theory. Since A is acyclic and since $\Omega_*(-)$ is a generalized homology theory, it follows from the Atiyah–Hirzebruch spectral sequence that $\Omega_*(BA) \to \Omega_*(\text{point})$ is an isomorphism. Hence, if M_1 and M_2 are bordant over a point, then they are bordant over BA. In other words, the fundamental group of the cobordism N maps to A by a map extending the maps on $\pi_1(M_1)$ and $\pi_1(M_2)$. In particular, both of these fundamental groups must inject into $\pi_1(N)$. □

Theorem 14.4. *Suppose that there is a triangulable cobordism between two closed aspherical manifolds M_1^n and M_2^n. Then there is an aspherical cobordism N^{n+1} between M_1^n and M_2^n so that for $i = 1, 2$, $\pi_1(M_i^n) \to \pi_1(N^{n+1})$ is injective.*

Proof. Let P be the cobordism between M_1 and M_2 provided by Theorem 14.3 and let $Q = M_1 \coprod M_2$. Set $N = h(P, Q)$. By Remark 14.2, N is the desired aspherical cobordism. □

Remark 14.5. Let $M^n = h(\partial\Box^{n+1})/(\mathbb{Z}/2)$. The manifolds M^n occur in nature. Indeed, consider real projective space $\mathbb{R}P^n$ and the collection of all its coordinate hyperplanes, defined by the equations $x_i = 0$, for $1 \leq i \leq n+1$. Then M^n is the manifold resulting from blowing up (in the sense of algebraic geometry) all projective subspaces which are intersections of such coordinate hyperplanes. It follows from this that M^n can also be described as the "closure of a generic torus orbit on a real flag manifold." More precisely, the flag manifold is the homogeneous space $SL(n+1, \mathbb{R})/B$ where B denotes the Borel subgroup of upper triangular matrices. The real "torus" H is the subgroup of diagonal matrices in $SL(n+1, \mathbb{R})$. Thus, $H \cong (\mathbb{R}^*)^n$. An H-orbit on the homogeneous space is "generic" if it is the orbit of a flag which is in general position with respect to the coordinate hyperplanes in $\mathbb{R}^{n+1}$ (in other words, it is generic if its stabilizer in H is trivial). It can then be shown that the closure of a generic H-orbit is homeomorphic to M^n, cf. [11].

15 An orientable hyperbolization procedure

The trouble with the Möbius band procedure is that the hyperbolized cells are not orientable. Gromov [16] gave a second construction which remedied this. We explain it below. (Further expositions of this construction can be found in [10] and [4].)

The rough idea behind any hyperbolization procedure is this. We first give some functorial procedure for hyperbolizing cells. Then, given a cell complex Λ, we define its hyperbolization by gluing together the hyperbolized cells in the same combinatorial pattern as the cells of Λ.

Gromov's second procedure can be applied to any finite dimensional simplicial complex Λ. The result will be an aspherical (in fact, nonpositively curved) cubical cell complex $h(\Lambda)$. The construction will have analogous properties to properties 1) through 6) in the previous section. In addition, it will have the following two properties:

7) The natural map $c : h(\Lambda) \to \Lambda$ induces on surjection on integral homology groups.

8) If Λ is a manifold, then c pulls back the stable tangent bundle of Λ to the stable tangent bundle of $h(\Lambda)$.

The definition of the construction again is by induction on dimension. In order to define the hyperbolization of an n-dimensional simplicial complex Λ, we first need to define the hyperbolization of an n-simplex, $h(\Delta^n)$. Then, to complete the definition of $h(\Lambda)$, we need to have some fixed identification of each n-simplex in Λ with the standard n-simplex. One way to insure this is to assume that Λ admits a "folding map" $p : \Lambda \to \Delta^n$, that is, a simplicial map p which restricts to an injection on each simplex. If we replace Λ with its barycentric subdivision Λ', then it admits such a folding map. Once we have such a p, $h(\Lambda)$ is defined to be the fiber product of $p : \Lambda \to \Delta^n$ and $c : h(\Delta^n) \to \Delta^n$. In other words, $h(\Lambda) = \{(x, y) \in \Lambda \times h(\Delta^n) | p(x) = c(y)\}$.

If dim $\Lambda \leq 1$, then, by definition, $h(\Lambda) = \Lambda$. Suppose that $h(\Lambda)$ has been defined for simplicial complexes of dimension $< n$.

We turn now to the definition on a hyperbolized n-simplex. Each transposition of two vertices of Δ^n induces a reflection on Δ^n. In order to make such a reflection into a simplicial isomorphism it is necessary to pass to the barycentric subdivision $(\Delta^n)'$. Choose a reflection $r : (\partial\Delta^n)' \to (\partial\Delta^n)'$. By functorality, we have an induced involution $h(r) : h((\partial\Delta^n)') \to h((\partial\Delta^n)')$. The involution $h(r)$ acts as a reflection on the $(n-1)$-manifold $h((\partial\Delta^n)')$ and its fixed point set separates $h((\partial\Delta^n)')$ into two "half-spaces" which we denote by H_+ and H_-. Then $h(\Delta^n)$ is defined by

$$h(\Delta^n) = (h((\partial\Delta)') \times \text{ [-1,1]})/\sim$$

where the equivalence relation $\sim$ identifies $H_- \times \{-1\}$ with $H_- \times \{+1\}$. The boundary of $h(\Delta^n)$ consists of two copies of H_+ glued together along the fixed point set of $h(r)$, i.e., it can be identified with $h((\partial\Delta^n)')$. Another way to describe this procedure is to form the manifold $h((\partial\Delta^n)') \times S^1$ and then cut it open along $H_+ \times \{1\}$. It follows from this that $h(\Delta^n)$ is an orientable manifold with boundary (assuming, by induction, that $h((\partial\Delta^n)')$ is orientable). Assuming further that $h((\partial\Delta^n)')$ is a cubical cell complex, we see that $h(\Delta^n)$ inherits the structure of a cubical cell complex (possibly after subdividing the [-1,1] factor). Moreover, the link of a vertex v in $h(\Delta^n)$ is the cone on $Lk(v, h((\partial\Delta^n)'))$. It follows that the link of any vertex in $h(\Delta^n)$ is a flag complex and hence, that $h(\Delta^n)$ is nonpositively curved. As was explained earlier, the hyperbolization $h(\Lambda)$ of an arbitrary n-dimensional simplicial complex Λ is then defined by the fiber product construction.

Using the fact that $H_n(h(\Delta^n), \partial h(\Delta^n)) \cong \mathbb{Z}$, it is not hard to verify property 7). Property 8) follows from the observation that when Λ is a manifold, $h(\Lambda)$ can be identified with a submanifold of $\Lambda \times \Delta^n$ with trivial normal bundle and from the fact that the stable tangent bundle of $h(\Delta^n)$ is trivial.

Remarks 15.1.

- One consequence of 8) is that for any (triangulable) manifold M, the map $c : h(M) \to M$ pulls back the stable characteristic classes of M (e.g., its Pontryagin classes and Stiefel-Whitney classes) to those of $h(M)$. Thus, the characteristic numbers of $h(M)$ are the same as those of M. This shows that the condition of being aspherical does not impose any restrictions on the characteristic numbers of a manifold.

- Property 8) can be rephrased as saying that $c : h(M) \to M$ is covered by a normal map.

- The proof of Theorem 14.1 shows that $c : h(M) \to M$ to normally cobordant to the identity map, $\mathrm{id} : M \to M$.

- The proofs of Theorems 14.3 and 14.4 show that if aspherical manifolds M_1 and M_2 are cobordant in any bordism theory, then they are cobordant, in the same theory, by an aspherical cobordism.

16 A nontriangulable aspherical 4-manifold

The E_8-form is a certain positive definite, unimodular, even, symmetric bilinear form on $\mathbb{Z}^8$ which is associated to the Dynkin diagram E_8. It is represented by the following matrix:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

(*Unimodular* means that the determinant of this matrix is 1; *even* means that the diagonal entries are even integers.) Since the E_8 is positive definite, its signature is 8.

The plumbing construction of Kervaire–Milnor gives a smooth, simply connected 4-manifold with boundary, N^4, with intersection form the E_8-form. (One "plumbs" together 8 copies of the tangent disk bundle of S^2 according to the diagram E_8.) It follows from the exact sequence of the pair $(N^4, \partial N^4)$ and from the fact that the E_8-form is unimodular that the boundary of N^4 is a homology 3-sphere. It turns out that it is not simply connected. In fact, ∂N^4 is Poincaré's homology 3-sphere. (The same construction was used in higher dimensions, by Kervaire–Milnor, to construct exotic spheres of dimension $4k-1$, $k \geq 2$.)

Let Λ^4 be a simplicial complex formed by triangulating N^4 and then attaching the cone on ∂N^4. Of course, Λ^4 is not a 4-manifold since there is no Euclidean neighborhood of the cone point. On the other hand, it is a polyhedral homology manifold in the sense that the link of each vertex is a homology sphere. It is called the E_8 *homology* 4-*manifold*. Stiefel-Whitney classes make sense for homology manifolds and it follows from the fact that N^4 is stably parallelizable that the Stiefel-Whitney classes of Λ^4 all vanish. Its signature is 8.

It is a theorem from algebra that any unimodular, even, symmetric bilinear form over the integers has signature divisible by 8. For a closed orientable 4-manifold, the condition that its intersection form is even is equivalent to its second Stiefel–Whitney class, w_2, being 0. The following is a famous theorem of Rohlin [24].

Theorem 16.1 (Rohlin's Theorem). *For any smooth or PL closed 4-manifold M^4 with $w_1(M^4) = 0 = w_2(M^4)$, the signature of its intersection form must be divisible by 16.*

It follows that Λ^4 is not homotopy equivalent to a smooth or PL 4-manifold. On the other hand, by Freedman's result, Theorem 9.3, there is a contractible manifold N' with $\partial N' = \partial N$. Hence, Λ^4 is homotopy equivalent to the topological manifold $M^4 = N \cup N'$. (We have replaced the cone on a homology 3-sphere by the contractible manifold N'.) By Rohlin's Theorem, M^4 cannot be homotopy equivalent to a PL 4-manifold.

The condition that a manifold admit a PL structure is stronger than the condition that it be homeorphic to a simplicial complex: to have a PL structure it must, in addition, be locally piecewise linearly (PL) homeomorphic to Euclidean space (with its standard PL structure). In particular, the link of any vertex in a PL manifold must be PL homeomorphic to a sphere. It is easy to see that in a general triangulated n-manifold the link of any vertex must be a simply connected (provided $n \geq 3$) homology manifold with the same homology as S^{n-1}, but there is no reason for it to be PL homeomorhic to S^{n-1}. When $n = 4$, since any polyhedral homology 3-manifold is a 3-manifold, each such link must be a homotopy 3-sphere.

Thus, for a few years after Freedman's result [15] was proved, it still seemed possible that Λ^4 could be homotopy equivalent to a triangulated manifold M^4 (since the 3-dimensional Poincaré Conjecture is not known). However, it follows from Casson's work on the Casson invariant that this is also not the case. Indeed, the link of any vertex in a triangulation of M^4 must be a homotopy 3-sphere (and at least one such link must be a fake 3-sphere since the triangulation cannot be PL). One can then arrange that the connected sum of all fake 3-spheres which arise as links bounds a PL submanifold of M^4 of signature 8. This implies that the Casson invariant of such a fake 3-sphere must be an odd integer. On the other hand, Casson showed that the Casson invariant of homology 3-sphere depends only on its fundamental group. Hence, this integer is 0. This contradiction shows that M^4 is not triangulable.

The hyperbolization technique of the previous section allows us to promote this result to aspherical 4-manifolds. Consider $h(\Lambda^4)$, the result of applying Gromov's oriented hyperbolization procedure to Λ^4. It is a polyhedral homology manifold with only one non-manifold point (namely, the hyperbolization of the cone point). By property 8) of the previous section, its Stiefel-Whitney classes vanish. Since the complement of a regular neighborhood of the cone point is oriented cobordant rel ∂N to N^4 and since the signature is an oriented cobordism invariant, it follows that the signature of $h(\Lambda^4)$ is 8. Now let M^4 denote the result of replacing the regular neighborhood of the cone point in $h(\Lambda^4)$ by the contractible manifold N'. Since M^4 is homotopy equivalent to $h(\Lambda^4)$, it is aspherical. The previous arguments now prove the following result of [10].

Theorem 16.2. *There is an aspherical 4-manifold M^4 which is not homotopy equivalent to any triangulable 4-manifold.*

In particular, M^4 is not homotopy equivalent to a PL manifold. Standard arguments show that this property is preserved when we take the product with a k-torus; hence, we also have the following result of [10].

Theorem 16.3. *For each $n \geq 4$, there is a closed aspherical n-manifold which is not homotopy equivalent to a PL manifold.*

Remark 16.4. For another application of this orientable hyperbolization procedure, see [25].

17 Relative hyperbolization

As is explained elsewhere in this volume, Farrell and Jones have proved the Assembly Map Conjecture (cf. Section 11) for the fundamental group of any nonpositively curved, closed Riemannian manifold. It seems likely (or at least plausible) that the Farrell-Jones program can be adapted to prove the Assembly Map Conjecture for the fundamental group of any closed PL manifold M^n equipped with a nonpositively curved, piecewise Euclidean metric (cf. Section 6). (Here we want to assume that the piecewise Euclidean structure on M^n is compatible with its structure of a PL manifold. The reason for this hypothesis is that a theorem of [10] then asserts that the compactification of its universal cover is homeomorphic to an n-disk.)

In this section we describe a variant of the reflection group trick (it is also a variant of hyperbolization) which can be used to prove the following result.

Theorem 17.1. *Suppose that the Assembly Map Conjecture is true for the fundamental group of any closed PL manifold with a nonpositively curved, piecewise Euclidean metric. Then it is also true for the fundamental group of any finite polyhedron with a nonpositively curved, piecewise Euclidean metric.*

This program has been partially carried out by B. Hu in [18] and [19]. Hu [18] first showed that the Farrell–Jones arguments work to prove the vanishing the Whitehead group of the fundamental group of any closed PL manifold with a nonpositively curved, polyhedral metric. He then used a construction similar to the one described below to derive the following theorem.

Theorem 17.2. (Hu [18]) *For any finite polyhedron with a nonpositively curved piecewise Euclidean metric, the Whitehead group of its fundamental group vanishes.*

As Farrell [13] pointed out in his lectures, Hu [19, p. 146] also observed that the Farrell–Hsiang [14] proof of the Novikov Conjecture [14] works for closed PL manifolds with nonpositively curved polyhedral metrics. (This uses the fact that the compactification of its universal cover is homeomorphic to a disk.) Hence, the construction described below yields the following theorem.

Theorem 17.3. (Hu [19]) *The Novikov Conjecture holds for the fundamental group of any finite polyhedron with a nonpositively curved piecewise Euclidean metric.*

Suppose B is a cell complex equipped with a piecewise Euclidean metric. Subdividing if necessary, we may assume that B is a simplicial complex. Suppose further that B is a subcomplex of another simplicial complex X. Possibly after another subdivision, we may assume that B is a full subcomplex of X. This means that if a simplex σ of X has nonempty intersection with B, then the intersection is a simplex of B (and a face of σ). We note that while each simplex of B is given a Euclidean metric, no metric is assumed on the simplices of X which are not in B. In practice, X will be a PL manifold.

We will define a new cell complex $D(X,B)$ equipped with a polyhedral metric. We will also define a finite-sheeted covering space $\tilde{D}(X,B)$ of $D(X,B)$. Each cell of $D(X,B)$ (or of $\tilde{D}(X,B)$) will have the form $\alpha\times[-1,1]^k$, for some integer $k \geq 0$, where α is a simplex of B and where $\alpha\times[-1,1]^k$ is equipped with the product metric. (Usually we will use α to stand for a simplex of B and σ for a simplex in X which is not in B.) Define $\dim(X,B)$ to be the maximum dimension of any simplex of X which intersects B but which is not contained in B. Here are some properties which the construction will have:

i) For $n = \dim(X, B)$, there will be 2^n disjoint copies of B in $D(X, B)$.

ii) For each such copy and for each vertex v in B, the link of v in $D(X, B)$ will be isomorphic to a subdivision of $Lk(v, X)$. In particular, if X is a manifold, then $D(X, B)$ will be a manifold.

iii) If the metric on B is nonpositively curved, then the metric on $D(X, B)$ will be nonpositively curved and each copy of B will be a totally geodesic subspace of $D(X, B)$.

iv) The group $(\mathbb{Z}/2)^n$ will act as a reflection group on $D(X, B)$. A fundamental chamber for this action will be denoted by $K(X, B)$. It will be homeomorphic to a regular neighborhood of B in X. Thus, $K(X, B)$ will be a retract of $D(X, B)$ and B will be a deformation retract of $K(X, B)$.

In fact, the entire construction depends only on a regular neighborhood of B in X. More precisely, it depends only on the set of simplices of X which intersect B.

Let $\mathcal{P}$ denote the poset of simplices σ in X such that $\sigma \cap B \neq \emptyset$ and such that σ is not a simplex of B. For each simplex α of B, let $\mathcal{P}_{>\alpha}$ denote the subposet of $\mathcal{P}$ consisting of all σ which have α as a face. Let $\mathcal{F}$ =Flag($\mathcal{P}$) denote the poset of chains in $\mathcal{P}$ (an element of $\mathcal{F}$ is a nonempty, finite, totally ordered subset of $\mathcal{P}$).

Given a chain $f = \{\sigma_0 < \cdots < \sigma_k\} \in \mathcal{F}$, let σ_f denote its least element, i.e., $\sigma_f = \sigma_0$. Given a simplex α of B, let $\mathcal{F}_{>\{\alpha\}}$ denote the set of chains f with $\sigma_f > \alpha$.

We begin by defining the fundamental chamber K $(= K(X, B))$. Each cell of K will have the form $\alpha\times$ $[0,1]^f$, for some $f \in \mathcal{F}_{>\{\alpha\}}$. Thus, the number of interval factors of $\alpha\times[0,1]^f$ is the number of elements of f. If $f \leq f'$, then we identify $[0,1]^f$ with the face of $[0,1]^{f'}$ defined by setting the coordinates $x_\sigma = 1$, for all $\sigma \in f' - f$.

We define an incidence relation on the set of such cells as follows: $\alpha\times[0,1]^f \leq \alpha'\times[0,1]^{f'}$ if and only if $\alpha \leq \alpha'$ and $f \leq f'$. (Notice that if $\alpha \leq \alpha'$, then $\mathcal{F}_{>\{\alpha'\}} \subset \mathcal{F}_{>\{\alpha\}}$.) K is defined to be the cell complex formed from the disjoint union $\coprod \alpha\times[0,1]^f$ by gluing together two such cells whenever they are incident. It is clear that K is homeomorphic to a regular neighborhood of B in X.

Next we define the mirrors of K. For each $\sigma \in \mathcal{P}_{>\alpha}$ and for each chain f with $\sigma_f = \sigma$ define $\delta_\sigma(\alpha\times[0,1]^f)$ to be the face of $\alpha\times[0,1]^f$ defined by setting $x_\sigma = 0$, i.e.,

$$\delta_\sigma(\alpha \times [0,1]^f) = \alpha \times 0 \times [0,1]^{f-\{\sigma\}}.$$

The mirror $\delta_\sigma K$ is the subcomplex of K consisting of all such cells. In other words, $\delta_\sigma K = \alpha \times S_\sigma$, where $\alpha = B \cap \sigma$ and where S_σ denotes the star of the barycenter of σ in the simplicial complex $\mathcal{F}$ (=Flag($\mathcal{P}$)).

Next we apply the reflection group trick. Set $J = (\mathbb{Z}/2)^{\mathcal{P}}$. Define $\tilde{D}$ ($= \tilde{D}(X, B)$) by

$$\tilde{D} = (J \times K)/\sim$$

where the equivalence relation $\sim$ is defined as in Section 7.

Remark 17.4. Suppose X is the cone on a simplicial complex ∂X and that B is the cone point. Then $\tilde{D}(X, B)$ coincides with the cubical complex P_L defined in Section 3 (where L is the barycentric subdivision of ∂X).

The definition of the space D ($= D(X, B)$) is similar to that of $\tilde{D}$ only one uses the smaller group $(\mathbb{Z}/2)^n$, $n = \dim(X, B)$, instead of J. If $\{r_1, \ldots, r_n\}$ is the standard set of generators for $(\mathbb{Z}/2)^n$, then we identify the points (gr_i, x) and (g, x) whenever x belongs to a mirror $\delta_\sigma K$ with $i = \dim \sigma$.

There is another definition of $D(X, B)$ which is similar to the definitions of the hyperbolization constructions given in the previous two sections. We shall define a space $D^{(k)}(X, B)$ for any pair (X, B) with $\dim(X, B) \leq k$. The definition is by induction on $\dim(X, B)$. First of all, $D^{(0)}(X, B)$ is defined to be B. Assume that $D^{(n-1)}$ has been defined and that $\dim(X, B) = n$. Set

$$D^{(n)}(X^{(n-1)} \cup B, B) = D^{(n-1)}(X^{(n-1)} \cup B, B) \times \{-1, 1\}.$$

If σ is an n-simplex such that $\sigma \cap B \neq \emptyset$ and σ is not in B, then define

$$D^{(n)}(\sigma, \sigma \cap B) = D^{(n-1)}(\partial\sigma, \partial\sigma \cap B) \times [-1, 1].$$

We note that the boundary of $D^{(n)}(\sigma, \sigma \cap B)$ is naturally a subcomplex of $D^{(n)}(X^{(n-1)} \cup B, B)$. Hence, we can glue in each hyperbolized simplex $D^{(n)}(\sigma, \sigma \cap B)$ to obtain $D^{(n)}(X, B) = D(X, B)$.

The advantage of this definition is that it makes it easier to prove property iii) (that if B is nonpositively curved, then so is $D(X, B)$.) The proof is based on a Gluing Lemma of Reshetnyak [3, p. 316] or [16, p. 124]. This lemma asserts that if we glue together two nonpositively curved spaces via an isometry of a common totally geodesic subspace, then the new metric space is nonpositively curved. The inductive hypothesis gives that the spaces $D^{(n-1)}(X^{(n-1)} \cup B, B)$ and $D^{(n-1)}(\partial\sigma, \partial\sigma \cap B)$ are nonpositively curved. Using the Gluing Lemma, we get that $D^{(n)}(X, B)$ is also nonpositively curved.

Remark 17.5. The above construction of $D(X, B)$ was explained to me about eight years ago by Lowell Jones. It is a variation of the "cross with interval" hyperbolization procedure which had been described previously by Tadeusz Januszkiewicz and me in [10]. Relative versions of this were described by Hu [19] and by Ruth Charney and me in [4]. In these earlier versions of relative hyperbolization the 1-skeleton of X was not changed. Jones realized that the construction is nicer if, as in this section, we also hyperbolize the 1-simplices.

18 Comments on the references

My first work on the complexes P_L and Σ_L was in my paper [5] on reflection groups. This paper was inspired by lectures of Thurston on Andreev's Theorem. This theorem concerned the realization of convex polytopes in hyperbolic 3-space with prescribed dihedral angles, thereby producing many examples of reflection groups on hyperbolic 3-space. (Andreev's Theorem provided the first large class of manifolds for which Thurston's Geometrization Conjecture was verified.) In the right-angled case, one of Andreev's conditions was the flag complex condition of Section 5. Thurston's explanation of the relationship of this condition to the asphericity of the corresponding 3-manifolds led me to guess (and then prove in [5]) that a similar result held in all dimensions.

In [5], I defined the complexes P_L and Σ_L as in Section 7, I gave the examples in Section 9 of aspherical manifolds not covered by Euclidean space and I discussed the reflection group trick. In [6], a version of Theorem 11.4 appeared for the first time. (However, since I was using $[(X, \partial X), (G/TOP, *)]$ instead of $H_n(X; \mathbb{L})$, I only made a statement about the injectivity of the assembly map.)

Subsequently, in a seminal paper, Gromov [16] described the cubical structure on P_L and proved that the resulting piecewise Euclidean metric was nonpositively curved. (For an exposition of Gromov's ideas on nonpositive curvature, the reader is referred to the excellent book of Bridson and Haefliger [2] as well as to the new textbook [3].) In the same paper Gromov introduced the hyperbolization techniques, described in Sections 14 and 15, for producing nonpositively curved polyhedra. (At least in the case of the Möbius band procedure, he also claimed that the hyperbolized space admitted a metric of strict negative curvature. This stronger assertion was in error, but later Charney and I in [4] showed how Gromov's constructions could be modified so that it would be true.)

In [10] Januszkiewicz and I used Gromov's hyperbolization construction to produce the example in Section 16 of a nontriangulable aspherical 4-manifold.

I have written three other survey papers [7], [8] and [12] which the reader is referred to for further details on the material discussed here.

References

[1] G. Baumslag, E. Dyer, A. Heller, *The topology of discrete groups*, J. of Pure and Applied Algebra **16** (1980), 1–47.

[2] M. Bridson and A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Springer–Verlag, Berlin, Heidelberg and New York, 1999.

[3] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Grad. Studies in Math. **33**, American Math. Society, Providence, 2001.

[4] R. Charney and M.W. Davis, *Strict hyperbolization*, Topology **34** (1995), 329–350.

[5] M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117**(1983), 293–325.

[6] M.W. Davis, *Coxeter groups and aspherical manifolds* Algebraic Topology Aarhus 1982, Lecture Notes in Math. **1051**, Springer–Verlag, Berlin, Heidelberg and New York, 1984, pp. 197–221.

[7] M.W. Davis, *Poincaré duality groups*, Surveys in Surgery Theory, Vol. 1 (eds. S. Cappell, A. Ranicki, J. Rosenberg), Annals of Math. Studies **145**, Princeton Univ. Press, Princeton, 2000, pp. 167–193.

[8] M.W. Davis, *Nonpositive curvature and reflection groups*, Handbook of Geometric Topology (eds. R. Daverman and R. Sher), Elsevier, Amsterdam, 2001.

[9] M.W. Davis and J-C. Hausmann, *Aspherical manifolds without smooth or PL structures*, Algebraic Topology, Lecture Notes in Math. **1370**, Springer–Verlag, Berlin, Heidelberg and New York, 1989, pp. 135–142.

[10] M.W. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geometry **34** (1991), 347–388.

[11] M.W. Davis, T. Januszkiewicz and R. Scott, *Nonpositive curvature of blow-ups*, Selecta. Math. (N.S.) **4** (1998), 491–547.

[12] M.W. Davis and G. Moussong, *Notes on nonpositively curved polyhedra*, Low Dimensional Topology (eds. K. Boroczsky, W. Neumann, A. Stipicz), Bolyai Soc. Math. Studies **8**, Budapest, 1999, pp. 11–94.

[13] F.T. Farrell, this volume.

[14] F.T. Farrell and W.C. Hsiang, *On Novikov's Conjecture for nonpositively curved manifolds*, Ann. of Math. (2) **113** (1981), 199–209.

[15] M. Freedman, *The topology of four–dimensional manifolds*, J. Differential Geometry **17** (1982), 357–453.

[16] M. Gromov, *Hyperbolic groups*, Essays in Group Theory (ed. S.M. Gersten) M.S.R.I. Publ. **8** Springer–Verlag, Berlin, Heidelberg and New York, 1987.

[17] J-C. Hausmann, *On the homotopy of non-nilpotent spaces*, Math. Z. **178** (1981), 115–123.

[18] B. Hu, *Whitehead groups of finite polyhedra with nonpositive curvature*, J. Differential Geometry **38** (1993), 501–517.

[19] B. Hu, *Retractions of closed manifolds with nonpositive curvature*, Geometric Group Theory (eds. R. Charney, M.W. Davis, M. Shapiro), Ohio State Univ. Math. Research Inst. Publ. **3**, de Gruyter, Berlin and New York, 1995, pp. 135–148.

[20] R. Lyndon and P. Schupp, *Combinatorial Group Theory*, Springer–Verlag, Berlin, Heidelberg and New York, 1978.

[21] G. Mess, *Examples of Poincaré duality groups*, Proc. Amer. Math. Soc. **110** (1990), 1145–1146.

[22] C.F. Miller, *Decision problems for groups — survey and reflections*, Algorithms and Classification in Combinatorial Group Theory (eds. G. Baumslag, C.F. Miller) M.S.R.I. Publ. **23** Springer–Verlag, Berlin, Heidelberg and New York, 1991, pp. 1–60.

[23] M.S. Raghunathan, *Torsion in cocompact lattices in coverings of* $Spin(2,n)$, Math. Ann. **226** (1984), 403–419.

[24] V.A. Rohlin, *A new result in the theory of* 4*-dimensional manifolds*, Soviet Math. Doklady **8** (1952), 221–224 (in Russian).

[25] R. Roy, *A counterexample to questions on the integrality property of virtual signatures*, Topology Appl. **100** (2000),177–185

[26] J.R. Stallings, *The piecewise–linear structure of Euclidean space*, Proc. Cambridge Phil. Soc. **58** (1962), 481–488.