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COXETER GROUPS ARE ALMOST CONVEX

ABSTRACT. In [C] Cannon introduced the notion of 'almost convexity' for the Cayley graph of a finitely generated group. In this paper, we observe that standard facts about Coxeter groups imply that the Cayley graph associated to any Coxeter system is almost convex.

ALMOST CONVEX GROUPS

Suppose G is a group and C is a finite set of generators such that $C = C^{-1}$. The *Cayley graph* of (G, C), denoted by $\Gamma(G, C)$, is the directed labelled graph with vertex set G and with a directed edge labelled c from the vertex g to the vertex gc, for each $g \in G$ and $c \in C$. Define a metric d on $\Gamma(G, C)$ by declaring each edge to be isometric to the unit interval and defining the distance between two points to be the length of the shortest path connecting them. A path of minimum length is a *geodesic*.

Given a directed edge path in $\Gamma(G, C)$ from the identity element to g, the labels on the edges, read in order, give a word for g in the generating set C. Conversely, each word for g corresponds to a path connecting 1 to g. For each g in G, put $l(g) = d(g, 1)$. The integer $l(g)$ is called the *word length* of g.

For each positive integer n, let $S(n)$ (respectively, $B(n)$) denote the sphere (respectively, ball) of radius n centered at 1 in $\Gamma(G, C)$, i.e. $S(n) = \{g \in G \mid l(g) = n\}$ and $B(n) = \{x \in \Gamma(G, C) \mid d(x, 1) \leqslant n\}$.

DEFINITION (Cannon [C, p. 198]). The graph $\Gamma(G, C)$ is (k) *almost convex*, written AC(k), if there is an integer $N(k)$ with the following property: any two elements g_1, g_2 in $S(n)$ with $d(g_1, g_2) \leqslant k$, can be joined by a path in $B(n)$ of length $\leqslant N(k)$. The pair (G, C) is AC(k) if $\Gamma(G, C)$ is AC(k); it is *almost convex*, written AC, if (G, C) is AC(k) for all k.

LEMMA 1 (Cannon [C, Th. 1.3, p. 198]). AC(2) $\Rightarrow$ AC.

COXETER GROUPS

Suppose that W is a group and that S is a finite set of generators each element of which is of order 2. Given s_1, $s_2 \in S$ denote the order of $s_1 s_2$ in W by $m(s_1, s_2)$.

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DEFINITION ([B, Ch IV, §1.3]). The pair (W, S) is a *Coxeter system* if W has a presentation:

$$\langle S | s^2, (s_1 s_2)^{m(s_1,s_2)} \rangle,$$

where s ranges over S and (s_1, s_2) ranges over pairs of distinct elements in S with $m(s_1, s_2) \neq \infty$. The group W is then called a *Coxeter group.*

LEMMA 2 ([B, Ch. IV, §1.2]). *Let m be an integer $\geqslant 2$ and let W be the dihedral group of order $2m$ with presentation $\langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^m \rangle$. (Then $(W, \{s_1, s_2\})$ is a Coxeter system.)*

(i) *Each element of W has length $\leqslant m$ and there is a unique element h of length m.*
(ii) *There are exactly two words of length m for h: one is $(s_1, s_2, \dots, s_{2-\varepsilon})$ and the other is $(s_2, s_1, \dots, s_{1+\varepsilon})$ where $\varepsilon = 0$ if m is even and $\varepsilon = 1$ if m is odd.*

The following lemma is also well known.

LEMMA 3. *Suppose that (W, S) is a Coxeter system and that w is an element of W with $l(w) = n + 1$. Suppose further that there are distinct elements w_1, w_2 in W of length n and elements s_1, s_2 in S such that $w_1 s_1 = w = w_2 s_2$. Then the following statements are true.*

(i) $m(s_1, s_2) \neq \infty$:
(ii) *Let h be the element of length m $(=m(s_1, s_2))$ in the dihedral group $\langle s_1, s_2 \rangle$* (cf. Lemma 2). *Then*

$$l(w) = l(wh^{-1}) + l(h).$$

Proof. A proof can be extracted from Exercise 3, p. 37 of [B]. Put $X = \{s_1, s_2\}$ and $W_X = \langle X \rangle$. According to this exercise, there is a unique element w' of shortest length in the coset wW_X. Thus, $w = w'h$ for some $h \in W_X$. Moreover, w' and h have the following two properties:

(a) $l(w) = l(w') + l(h)$
(b) $l(hs_i) < l(h)$ for $i = 1, 2$.

Property (b) implies that the group W_X is finite, i.e. $m \neq \infty$. Property (a) then yields (ii).

COROLLARY 1. *With the hypotheses of Lemma 3, there is a path from w_1 to w_2 in the ball $B(n)$ of length $2m - 2$ (where $m = m(s_1, s_2)$).*

Proof. Put $w' = wh^{-1}$. By Lemma 2, there are two geodesics from 1 to h. These can be translated by w' to yield two geodesics from w' to w; one ends in an edge labelled s_1 the other in an edge labelled s_2. Deleting these final edges, we obtain a geodesic of length $m - 1$ from w' to w_1 and a geodesic of length

$m - 1$ from w' to w_2. Both geodesics lie inside $B(n)$ (by Lemma 3(ii)). Concatenating the inverse of the first geodesic with the second we obtain a path in $B(n)$ from w_1 to w_2 of length $2m - 2$.

DEFINITION. Suppose (W, S) is a Coxeter system. Define an integer $m(W, S)$ by

$$m(W, S) = \max\{m(s_1, s_2) \mid (s_1, s_2) \in S \times S \text{ and } m(s_1, s_2) \neq \infty\}.$$

COROLLARY 2. *Let (W, S) be a Coxeter system. Then (W, S) is* AC(2) *with* $N(2) = 2m(W, S) - 2$.

Proof. We must consider elements w_1 and w_2 in $S(n)$ with $0 < d(w_1, w_2) \leqslant 2$. Since all relators are of even length, $d(w_1, w_2) \equiv l(w_1) + l(w_2) = 2n \pmod 2$. Hence, the case $d(w_1, w_2) = 1$ does not occur. The case $d(w_1, w_2) = 2$ follows immediately from Corollary 1.

Combining this with Lemma 1 yields the following.

THEOREM. *Any Coxeter system (W, S) is almost convex.*

REMARK. Poenaru [P] has recently proved that if a 3-manifold group is AC, then it is simply connected at infinity. It is proved in [D] that there are Coxeter groups W which (a) contain the fundamental group of a closed aspherical n-manifold, $n > 3$, as a subgroup of finite index and (b) are not simply connected at infinity. Hence, Poenaru's result is strictly 3-dimensional.

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