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MANIFOLDS, VARIETIES, AND KNOTS

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Regular Convex Cell Complexes

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INTRODUCTION

As $\varepsilon = +1$, 0, or -1, let Y_ε^n stand for the n-sphere, Euclidean n-space, or hyperbolic n-space. The study of regular tessellations of Y_ε^n by convex cells is a classical topic. Such tessellations have been completely classified (e.g., see [2] and [3]). The theory of regular tessellations of the n-sphere is essentially identical with the theory of regular convex polyhedra of dimension n + 1. In the case of hyperbolic space, regular tessellations exist only in dimensions 2, 3, and 4 (cf. [2]).

There is a close connection between the theory of regular tessellations of Y_ε^n and the theory of Coxeter groups: the group of isometric symmetries of such a tessellation is a group generated by the reflections across the faces of an n-simplex in the barycentric subdivision of the tessellation; such reflection groups are Coxeter groups. To a large extent this relationship is of a purely combinatorial nature. This paper is a systematic exposition of the combinatorial aspects of this relationship. Most of this material is classical; however, some new results do emerge.

Suppose that the geometric realization of a convex cell complex K is a PL-manifold of dimension n. We shall say that K is <u>symmetrically regular</u> if its group of combinatorial symmetries acts transitively on the set of n-simplices in its derived complex K'. More generally, K is said to be <u>regular</u> if there is an n-tuple $(m_1, \cdots, m_n)$ of integers ≥ 3 such that (a) the boundary of each 2-cell in K is an m_1-gon, (b) the link of each (n-2)-cell in K is an m_n-gon, and (c) for $2 \leq i \leq n-1$, for each (i+1)-cell F_{i+1} in K, and for each (i-2)-face F_{i-2} of F_{i+1}, the link of F_{i-2} in

∂F_{i+1} is an m_i-gon. The n-tuple $(m_1, \cdots, m_n)$ is called the Schläfli symbol of K. (It is easy to see that symmetric regularity implies regularity. We give a proof in (2.7).) We note that any covering space of a regular convex cell complex naturally has the structure of a regular convex cell complex.

We shall prove the following result in section 3.

THEOREM. Suppose that K is a regular convex cell complex and that K is a connected PL n-manifold. Then the universal cover of K is combinatorially equivalent to a classical regular tessellation of Y^n_ε by convex cells, for some $\varepsilon \in \{+1, 0, -1\}$. The fundamental group π of K is then identified with a subgroup of the group of isometric symmetries of this tessellation of Y^n_ε and K is combinatorially equivalent to the induced tessellation of Y^n_ε/π.

In particular, this result implies that if a manifold admits the structure of a regular convex cell complex, then it must be PL-homeomorphic to a complete Riemannian manifold of constant sectional curvature.

In dimension 2, the above theorem was proved by Edmonds, Ewing, and Kulkarni in [6]. In the special case where K is the boundary complex of a convex (n+1)-cell and where K is symmetrically regular, it is due to McMullen [9]. The theorem was proved in full generality by Kato in [7].

Actually, we shall carry out the whole theory in the broader context where K is a connected n-dimensional pseudo-manifold and where the link of each i-cell, $i \leq n-2$, in K is connected. Regular convex cell complexes are classified in this generality; there are some further possibilities besides the classical tessellations.

Here is a sketch of the main argument. The derived complex of K is a simplicial complex with a natural projection p to the standard n-simplex Δ^n. One associates to $(m_1, \cdots, m_n)$ a Coxeter group W, the diagram of which is a connected line segment. To an n-simplex Σ in K', one associates a subgroup $\pi(K', \Sigma)$ of W. There is a close analogy with the theory of covering spaces: the projection $p : K' \to \Delta^n$ plays the role of a covering projection, the Coxeter group W plays the role of the fundamental group of the base, and $\pi(K', \Sigma)$ plays the role of the fundamental group of K' (when K is a PL-manifold it actually is the fundamental group). It turns out that the role of the universal cover of the base is played by the so-called "Coxeter complex" of W. The theorem is proved by showing that K' is a PL-manifold only in the cases where the Coxeter complex is naturally identified with Y_ε^n.

The preceding paragraph suggests that we generalize the situation by studying simplical complexes over Δ^n to which an arbitrary Coxeter group can be associated (rather than restricting ourselves to Coxeter groups with diagrams connected line segments). This is done in section 2.

As we have already mentioned, the theorem stated above was proved in [7] (by somewhat different methods than those of this paper). Tits' paper [12] is concerned with a generalization of the material discussed here to the theory of buildings; the methods of [12] are very similar to those of this paper.

1. CONVEX CELL COMPLEXES

1.1. Suppose that E is a convex cell in some finite-dimensional real vector space V. Let V_E denote the linear subspace of V consisting of all vectors of the form $t(x-y)$, where $x, y \in E$ and $t \in \mathbb{R}$. In other words, V_E is the linear subspace parallel to the affine subspace supported by E. For $x \in E$, denote by $C_{E,x}$ the set of v in V_E such that $x + tv$ lies in E for some t in $[0,\varepsilon)$ and $\varepsilon > 0$. Suppose that F is a proper face of E (written as $F < E$). Let $\mathring{F}$ denote the relative interior of F. If $x \in \mathring{F}$ and $y \in F$, then $C_{E,y} \subset C_{E,x}$, with equality if and only if $y \in \mathring{F}$. If $x, y \in \mathring{F}$, then $C_{E,x}$ and $C_{E,y}$ have the same image in V_E/V_F. This common image is denoted by Cone(F,E); it is a convex polyhedral cone in V_E/V_F.

The <u>unit sphere</u> S(V) in a real vector space V is the quotient space $(V - \{0\})/\mathbb{R}_+$.

1.2. Suppose that E is a convex cell and that $F < E$. The <u>link of F in E</u>, denoted Link(F,E), is the image of (Cone(F,E) - {0}) in $S(V_E/V_F)$; it is a convex cell in $S(V_E/V_F)$. (The link of a simplex in a simplicial complex is usually defined in another way; our definition is especially for use in convex cell complexes.) If $F_1 < F_2 < E$, then the inclusions $V_{F_2} \subset V_E$, $V_{F_2}/V_{F_1} \subset V_E/V_{F_1}$, and $C_{F_2,y} \subset C_{E,y}$ induce a natural identification of $\mathrm{Link}(F_1,F_2)$ with a face of $\mathrm{Link}(F_1,E)$. Thus, the set of faces of Link(F,E) is in bijective correspondence with the set of faces of E which properly contain F.

1.3. Classically a convex cell complex K is a set of convex cells in some finite dimensional vector space satisfying the following two conditions:

(i) If $E \in K$ and F is a face of E, then $F \in K$.

(ii) If E, $F \in K$, then either $E \cap F = \phi$ or $E \cap F$ is a common face of E and of F.

A convex cell complex, in the above sense, has the structure of a poset: the partial ordering is given by inclusion of faces.

1.3.1. Let K be any poset. If $E \in K$, then let $K_{\leq E}$ denote the subposet $\{F \in K \mid F \leq E\}$. Generalizing the definition in (1.3), we shall say that a poset K is a convex cell complex if the following two conditions are satisfied:

(i') If $E \in K$, then $K_{\leq E}$ is isomorphic to the set of faces of some convex cell.

(ii') If E and F are elements of K, then either $K_{\leq E} \cap K_{\leq F} = \phi$ or else there exists an element F' in this intersection such that $K_{\leq E} \cap K_{\leq F} = K_{\leq F'}$.

For example, an abstract simplicial complex is a convex cell complex in this sense. An element of K is called a cell. A convex cell complex K is n-dimensional if it contains cells of dimension n but none of dimension n+1.

1.4. Associated to a convex cell complex K, there is a topological space called its geometric realization: this is the polyhedron formed by pasting together convex cells, one for each element of K, in the obvious fashion. As is common practice, we shall use the same symbol K to stand for a convex cell complex and its geometric realization.

1.5. The derived complex of K, denoted by K', is the poset of all finite chains in K. (A chain in a poset is a totally ordered nonempty subset.) An element of K' is a simplex; it is a k-simplex if it consists of k + 1 elements of K. The poset K' is an abstract simplicial complex; its vertex set can be identified with K. The geometric realization of K' is naturally identified with the barcentric subdivision of K.

1.6. If K is a convex cell complex and if F is a cell in K, then the link of F in K, denoted Link(F,K), is the convex cell complex consisting of all cells of the form Link(F,E), where $F < E \in K$, and where, of course, whenever $F_1 < F_2 < F_3 \in K$, we identify $\text{Link}(F_1,F_2)$ with the corresponding face of $\text{Link}(F_1,F_3)$.

The poset Link(F,K) is in bijective correspondence with the poset of cells in K which properly contain F. It follows that the simplicial complex Link(F,K)' can be identified with the subcomplex of K' consisting of all simplices $\sigma = \{F_0, F_1, \cdots, F_k\}$, such that $F < F_0 < F_1 \cdots < F_k \in K$.

1.7. A <u>combinatorial equivalence</u> f from a convex cell complex K to another one L is an isomorphism of posets f : K --> L. Such an equivalence f induces a simplicial isomorphism f' : K' --> L'. Hence, a combinatorial equivalence induces a PL-homeomorphism of geometric realizations. A combinatorial self-equivalence of K is called a <u>combinatorial symmetry</u> of K (or sometimes simply a "symmetry"). The group of combinatorial symmetries of K will be denoted by Aut(K).

1.8. Suppose that K is connected and that $p : \tilde{K} \to K$ is a covering projection. Since cells are simply connected, each cell of K is evenly covered by p. Thus, $\tilde{K}$ inherits the structure of a convex cell complex. Let Γ denote the group of covering transformations. Then Γ is a subgroup of Aut($\tilde{K}$) and Γ freely permutes the cells of $\tilde{K}$. If $p : \tilde{K} \to K$ is a regular covering (i.e., if $K \cong \tilde{K}/\Gamma$), then Γ is a normal subgroup of Aut($\tilde{K}$) and Aut($\tilde{K}$)/Γ can be identified with the group of combinatorial symmetries of K.

1.9. In a similar vein, suppose that $\tilde{K}$ is a convex cell complex and that Γ is a subgroup of Aut($\tilde{K}$) which freely permutes the cells of $\tilde{K}$. By an abuse of language, we shall also call the quotient space $\tilde{K}/\Gamma$ a "convex cell complex." (Strictly speaking, $\tilde{K}/\Gamma$ might not be a convex cell complex as defined in (1.3), since distinct faces of a cell in $\tilde{K}$ might be identified by an element of Γ.) However, by passing back to $\tilde{K}$, the notion of the link of a cell in $\tilde{K}/\Gamma$ still makes sense.

1.10. In the remaining sections of this paper we shall often impose the following conditions on an n-dimensional convex cell complex K (or an appropriate cover).

1.10.1. Each cell in K is a face of an n-cell. Each (n-1)-cell in K is a face of precisely two n-cells.

1.10.2. K is connected and for each cell F in K of dimension $\leq$ n-2, Link(F,K) is connected.

1.11. Condition (1.10.1) means that K is an n-dimensional pseudo-manifold. It follows from (1.10.1) that for each k-cell F in K, $0 \leq k \leq n-1$, Link(F,K) is also a pseudo-manifold of dimension n - k - 1. The complex K is a <u>PL-manifold</u> if for each k-cell F, $0 \leq k \leq n-1$, Link(F,K) is PL-homeomorphic to S^{n-k-1} with its standard PL structure. Conditions (1.10.1) and (1.10.2) imply that the link of any i-cell in K has the same number of path components as the link of an i-cell in a PL n-manifold. We shall sometimes also want to impose the following condition.

1.11.1. The link of each (n-2)-cell in K is a circle and the link of each cell of dimension $\leq$ n-3 is simply connected.

This condition means that the link of any i-cell in K has the same fundamental group as the link of an i-cell in a PL n-manifold.

Next, suppose that L is an n-dimensional simplicial complex.

1.12. An n-simplex in L is called a chamber. Let Chamb(L) denote the set of chambers in L. Also, for each simplex σ in L, let $\mathrm{Chamb}_\sigma(L)$ denote the set of chambers in L which have σ as a face.

Two distinct chambers are adjacent if their intersection is an (n-1)-simplex. A gallery in L is a sequence of adjacent chambers. The gallery $(\Sigma_0, \cdots, \Sigma_m)$ is said to begin at Σ_0, to end at Σ_m, and to connect Σ_0 to Σ_m.

1.12.1. If L satisfies (1.10.1), then given any $\Sigma \in \mathrm{Chamb}(L)$ and any (n-1)-dimensional face σ of Σ, there is a unique chamber Σ' adjacent to Σ with $\Sigma \cap \Sigma' = \sigma$.

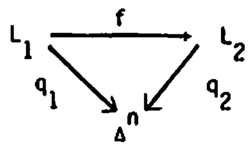
1.12.2. If, in addition, L satisfies the connectivity conditions in (1.10.2), then any two chambers in L can be connected by a gallery; moreover, if $\sigma \in L$, then any two chambers in $\mathrm{Chamb}_\sigma(L)$ can be connected by a gallery of chambers in $\mathrm{Chamb}_\sigma(L)$.

1.13. The standard n-simplex Δ^n is the poset of all nonempty subsets of $\{0, 1, \cdots, n\}$. It is an abstract simplicial complex. If $\sigma \in \Delta^n$, then put $\mathrm{Type}(\sigma) = \{0,1, \cdots, n\} - \sigma$. Thus, if σ is a k-simplex, Type(σ) is a proper subset of $\{0,1, \cdots, n\}$ of cardinality n - k.

1.14. A projection from L onto Δ^n is a simplicial map $q : L \to \Delta^n$ such that the restriction of q to each simplex is injective. The pair (L,q) is called a simplicial complex over Δ^n. If (L,q) is a simplicial complex over Δ^n, then for any simplex σ in L, put $\mathrm{Type}(\sigma) = \mathrm{Type}(q(\sigma))$.

Note that if Σ is a chamber in L and I is a proper subset of $\{0,1, \cdots, n\}$, then, since $q|\Sigma : \Sigma \to \Delta^n$ is an isomorphism, Σ has a unique face of type I.

Suppose that (L_1,q_1) and (L_2,q_2) are simplicial complexes over Δ^n. A simplicial map $f : L_1 \to L_2$ is called a <u>map over</u> Δ^n if the following diagram commutes.



1.15. If $f : L_1 \to L_2$ is a map over Δ^n, then it follows easily from the definitions, that f preserves the dimension of simplices. In particular, there is an induced map $\bar{f} : \mathrm{Chamb}(L_1) \to \mathrm{Chamb}(L_2)$.

LEMMA 1.16. Suppose that (L_1,q_1) and (L_2,q_2) are simplicial complexes over Δ^n, that L_1 satisfies (1.10), and that f and g are two simplicial maps over Δ^n from L_1 to L_2. If there exists a chamber Σ in $\mathrm{Chamb}(L_1)$ such that $\bar{f}(\Sigma) = \bar{g}(\Sigma)$, then $f = g$.

Proof. Suppose that $\bar{f}(\Sigma) = \bar{g}(\Sigma)$. Let v_i denote the vertex of Σ which projects to the vertex $\{i\} \in \Delta^n$, so that $\Sigma = \{v_0, \cdots, v_n\}$. Since f and g preserve type, we must have $f(v_i) = g(v_i)$, $0 \le i \le n$. Thus, f and g agree on the geometric realization of Σ. In particular, they agree on every (n-1)-dimensional face of Σ. It follows from (1.12.1) that f and g agree on every chamber adjacent to Σ. By (1.12.2), this implies that f and g agree on every chamber of L_1 and hence, that $f = g$.

COROLLARY 1.17. Suppose that (L,q) is a simplicial complex over Δ^n and that L satisfies (1.10). Then the group of automorphisms over Δ^n of L, denoted by Aut(L,q), acts freely on Chamb(L).

1.18. Suppose that (L,q) is a simplicial complex over Δ^n and that L satisfies (1.10). Let $\Sigma \in \mathrm{Chamb}(L)$. For each pair of integers i and j, with $0 \le i, j \le n$ and $i \ne j$, let $\Sigma_{\{i,j\}}$ be the (n-2)-face of Σ of type $\{i,j\}$ and put

1.18.1. $L_{ij}(\Sigma) = \mathrm{Link}(\Sigma_{\{i,j\}}, L)$.

It follows from (1.10) that $L_{ij}(\Sigma)$ is a connected 1-manifold; hence, it is either a polygon with a finite number of sides or the tessellation of $\mathbb{R}$ by intervals. The map q induces a simplicial map

$$q_{ij} : L_{ij}(\Sigma) \dashrightarrow \mathrm{Link}(\Delta^n_{\{i,j\}}, \Delta^n) \cong \Delta^1.$$

Thus, $L_{ij}(\Sigma)$ is a complex over Δ^1. The vertices of $L_{ij}(\Sigma)$ correspond to (n-1)-simplices which contain $\Sigma_{\{i,j\}}$. The existence of the map q_{ij} shows that the vertices of $L_{ij}(\Sigma)$ alternate between type {i} and type {j} (i.e., the corresponding (n-1)-simplices alternate between these types). Hence, if $L_{ij}(\Sigma)$ is a finite polygon, it must have an even number of sides. If $L_{ij}(\Sigma)$ is finite, then let $m_{ij}(\Sigma)$ denote 1/2 the number of its sides; if $L_{ij}(\Sigma)$ is infinite, then put $m_{ij}(\Sigma) = \infty$. Also, for $0 \leq i \leq n$, put $m_{ii}(\Sigma) = 1$. In this way, we obtain an (n+1) by (n+1) matrix $(m_{ij}(\Sigma))_{0\leq i,j\leq n}$. Its diagonal entries are all equal to 1; its off-diagonal entries are integers ≥ 2 or ∞.

1.19. Let $X = \{x_0, \cdots, x_n\}$ be a set of n + 1 symbols and let G(X) (resp. $G_+(X)$) denote the free group (resp. free monoid) on X. Let (L,q) be as above. For each chamber Σ in L and for $0 \leq i \leq n$, let Σx_i denote the unique chamber in L such that Σ and Σx_i are adjacent and such that the (n-1)-simplex $\Sigma \cap \Sigma x_i$ is of type {i}. This defines an injective function from X into the group of permutations of Chamb(L). By the universal property of G(X), this function extends to a homomorphism defined on G(X). Hence, we get an action of G(X) (from the right) on Chamb(L). (N.B. This action is <u>not</u> induced from a simplicial action on L.)

1.19.1. Suppose $(\Sigma_0, \cdots, \Sigma_m)$ is a gallery in L. Then $\Sigma_i = \Sigma_{i-1}y_i$ for some $y_i \in X$. Hence, the gallery can be rewritten as

$$(\Sigma_0, \Sigma_0 y_1, \Sigma_0 y_1 y_2, \cdots, \Sigma_0 y_1 \cdots y_m)$$

where the element $g = y_1 \cdots y_m$ is in $G_+(X)$. Conversely, any element $g \in G_+(X)$ yields a gallery from Σ_0 to $\Sigma_0 g$. Since any two chambers can be connected by a gallery, the group G(X) acts transitively on Chamb(L).

1.19.2. Suppose that σ is a simplex in L of type I for some proper subset I of $\{0,1, \cdots, n\}$. Put $X_I = \{x_i \in X \mid i \in I\}$. For any $\Sigma \in \mathrm{Chamb}_\sigma(L)$ and any $x \in X_I$, we have $\Sigma x \in \mathrm{Chamb}_\sigma(L)$. It follows that for $g \in G_+(X_I)$ the corresponding gallery from Σ to Σg is a gallery of chambers in $\mathrm{Chamb}_\sigma(L)$. Thus, $G(X_I)$ stabilizes $\mathrm{Chamb}_\sigma(L)$. It follows from (1.12.2) that $G(X_I)$ acts transitively on $\mathrm{Chamb}_\sigma(L)$.

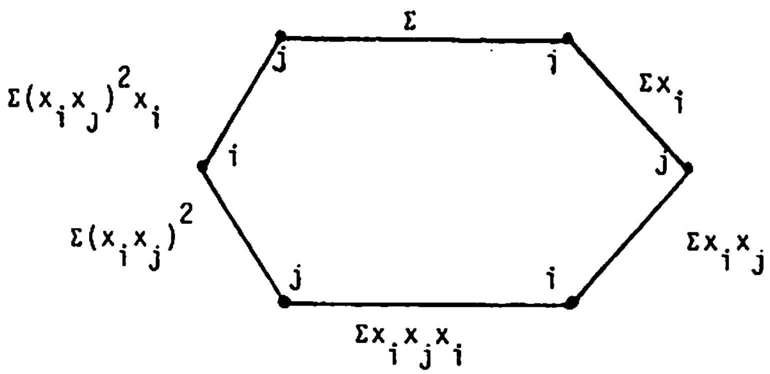


FIGURE 1

Moreover, with σ and Σ as above, we clearly have that

$$\sigma = \Sigma \cap \bigcap_{i \in I} \Sigma x_i; \tag{1.19.3}$$

hence,

$$\sigma = \bigcap_{g \in G_+(X_I)} \Sigma g. \tag{1.19.4}$$

1.20. With notation as above the following formula holds:

$$\Sigma(x_i x_j)^{m_{ij}(\Sigma)} = \Sigma, \tag{1.20.1}$$

for $0 \leq i, j \leq n$ and $m_{ij}(\Sigma) < \infty$. Moreover, if $m_{ij}(\Sigma) < \infty$, then it is the smallest positive integer m such that $\Sigma(x_i x_j)^m = \Sigma$; while, if $m_{ij}(\Sigma) = \infty$ then there is no integer m with this property. Since $\mathrm{Chamb}_{\Sigma_{\{i,j\}}}(\Sigma)$ is isomorphic to $\mathrm{Chamb}(L_{ij}(\Sigma))$, to prove these assertions it suffices to consider the case of a polygon with 2m sides. But these assertions are obvious in this case. (See Figure 1 for the picture when m = 3.)

LEMMA 1.21. Suppose that (L_1, q_1) and (L_2, q_2) are simplicial complexes over Δ^n satisfying (1.10) and that $f : L_1 \dashrightarrow L_2$ is a simplicial map over Δ^n. Let $\tilde{f} : \mathrm{Chamb}(L_1) \dashrightarrow \mathrm{Chamb}(L_2)$ be the induced map. Then $\tilde{f}$ is G(X)-equivariant.

Proof. Let $\Sigma \in \mathrm{Chamb}(L_1)$. It suffices to prove that for each x_i in X, we have $f(\Sigma x_i) = (f(\Sigma))x_i$. The chambers $f(\Sigma)$ and $f(\Sigma x_i)$ are adjacent and their intersection has type $\{i\}$ (since f preserves adjacency and type). Therefore, $f(\Sigma x_i) = (f(\Sigma))x_i$.

Conversely, we have the following result.

THEOREM 1.22. Suppose that (L_1,q_1) and (L_2,q_2) are simplicial complexes over Δ^n and that ϕ : Chamb(L_1) --> Chamb(L_2) is a map of G(X)-sets. Then there is a unique simplicial map over Δ^n denoted $f_\phi : L_1 \dashrightarrow L_2$, such that $\bar{f}_\phi = \phi$.

In other words, there is a natural bijection between the set of simplicial maps over Δ^n from (L_1,q_1) to (L_2,q_2) and the set of G(X)-equivariant maps from Chamb(L_1) to Chamb(L_2).

Proof. We first remark that since ϕ is G(X)-equivariant, $\phi(\Sigma x_i) = (\phi(\Sigma))x_i$. Thus, the chambers $\phi(\Sigma)$ and $\phi(\Sigma x_i)$ are adjacent and their intersection is an (n-1)-simplex of type $\{i\}$. Let σ be a k-simplex in L_1. Put I = Type(σ). Choose a chamber Σ such that $\sigma < \Sigma$. Let

$$\alpha_I = \phi(\Sigma) \cap \bigcap_{i \in I} \phi(\Sigma x_i)$$

It follows from the above remarks that α is k-simplex of type I. Furthermore, by (1.19.4),

$$\alpha = \bigcap_{g \in G_+(X_I)} (\phi(\Sigma))g.$$

From this last equation we see that the definition of α is independent of the choice of $\Sigma \in \text{Chamb}_\sigma(L_1)$. Define $f_\phi : L_1 \dashrightarrow L_2$ by $f_\phi(\sigma) = \alpha$. It is clear that f_ϕ is a simplicial map over Δ^n and that $\bar{f}_\phi = \phi$. Uniqueness follows from Lemma (1.16).

Now suppose that K is an n-dimensional convex cell complex. We shall apply paragraphs (1.12) through (1.22) to the special case $L = K'$.

1.23. If $F \in K$, then $\{F\}$ is a vertex of K'. If $i \in \{0,1,\cdots,n\}$, then $\{i\}$ is a vertex of Δ^n. The map of vertex sets Vert(K') --> Vert(Δ^n) defined by $\{F\}$ --> $\{\dim F\}$ yields a simplicial projection $p' : K' \dashrightarrow \Delta^n$. Thus, <u>$K'$ has a canonical structure of a simplicial complex over</u> Δ^n.

1.24. Let K_1, K_2 be n-dimensional convex cell complexes. If $f : K_1 \dashrightarrow K_2$ is a combinatorial equivalence, then $f' : K_1' \dashrightarrow K_2'$ is a simplicial isomorphism over Δ^n. (In particular, f' preserves type; consequently, f preserves the dimension of cells. Thus, a combinatorial equivalence preserves the dimension of cells.)

Conversely, suppose that $g : K_1' \to K_2'$ is a simplicial map over Δ^n. Let $f : K_1 \to K_2$ be the induced map of vertex sets. Since g is compatible with the projections to Δ^n, it follows that f is a map of posets and that $f' = g$. If g is an isomorphism, then so is f. Hence, <u>the set of combinatorial equivalences from K_1 to K_2 is naturally bijective with the set of simplicial isomorphisms over Δ^n from K_1' to K_2'</u>.

1.25. Applying this in the special case $K = K_1 = K_2$, we see that Aut(K), the group of combinatorial symmetries of K, is canonically isomorphic to Aut(K',p), the group of simplicial isomorphisms of K' over Δ^n.

<u>From now on, we suppose that K satisfies (1.10)</u>. First we note the following.

1.26. K' also satisfies (1.10).

Using (1.25), corollary (1.17) can be translated as follows.

LEMMA 1.27. If K satisfies (1.10), then the group Aut(K) acts freely on Chamb(K').

1.28. Suppose that $\Sigma = \{F_0 < \cdots < F_n\}$ is a chamber in K'. For $1 \le i \le n$, let $L_i(\Sigma)$ be the 1-dimensional convex cell complex defined as follows:

$$L_1(\Sigma) = \partial F_2$$

$$L_i(\Sigma) = \mathrm{Link}(F_{i-2}, \partial F_{i+1}), \quad 2 \le i \le n-1,$$

$$L_n(\Sigma) = \mathrm{Link}(F_{n-2}, K), \quad n \ge 2.$$

For $1 \le i \le n-1$, $L_i(\Sigma)$ is a polygon; let $m_i(\Sigma)$ denote the number of its sides. The complex $L_n(\Sigma)$ is either a polygon, in which case $m_n(\Sigma)$ denotes the number of its sides, or the tessellation of $\mathbb{R}$ by intervals, in which case put $m_n(\Sigma) = \infty$.

Recall that for $0 \le i, j \le n$, $i \ne j$, in (1.18.1) we define complexes $L_{ij}(\Sigma) = \mathrm{Link}(\Sigma_{\{i,j\}}, K')$, where $\Sigma_{\{i,j\}}$ denotes the (n-2)-simplex $\Sigma - \{F_i, F_j\}$. For $\{i,j\} = \{k-1,k\}$, $L_{ij}(\Sigma)$ can be identified with the derived complex of $L_k(\Sigma)$. Thus,

$$m_{ij}(\Sigma) = m_k(\Sigma), \text{ whenever } \{i,j\} = \{k-1,k\} .$$

In the notation of (1.19) we have that, for $0 \le i \le n$, $\Sigma x_i = (\Sigma - \{F_i\}) \cup \hat{F}_i$ for some i-cell F_i in K. Suppose $|i-j| \ge 2$. Put $\Sigma(i,j) = (\Sigma - \{F_i, F_j\}) \cup \{\hat{F}_i, \hat{F}_j\}$. It follows from the hypothesis $|i-j| \ge 2$, that $\Sigma(i,j)$ is a chamber of K'. Moreover, $\Sigma(i,j)$ is the adjacent chamber to Σx_i (resp. to Σx_j) across the face of type $\{j\}$ (resp. type $\{i\}$). Thus, $\Sigma x_i x_j = \Sigma(i,j) = \Sigma x_j x_i$, from which it follows that $\Sigma(x_i x_j)^2 = \Sigma$.

In other words, $m_{ij}(\Sigma) = 2$ and $L_{ij}(\Sigma)$ is a quadrilateral, whenever $|i-j| \geq 2$. In summary,

$$m_{ij}(\Sigma) = \begin{cases} 1 & \text{, if } i = j \\ 2 & \text{, if } |i-j| \geq 2 \\ m_k(\Sigma) & \text{, if } \{i,j\} = \{k-1,k\} . \end{cases}$$

2. REGULAR SIMPLICIAL COMPLEXES OVER Δ^n

2.1. A Coxeter matrix of degree n + 1 is a symmetric (n+1) by (n+1) matrix $M = (m_{ij})_{0 \leq i,j \leq n}$, with each diagonal entry equal to 1 and with each off-diagonal entry either an integer ≥ 2 or ∞.

2.2. To a Coxeter matrix $M = (m_{ij})_{0 \leq i,j \leq n}$, one can associate a graph as follows. The graph has one vertex z_i for each integer i, $0 \leq i \leq n$. Distinct vertices z_i and z_j are connected by an edge if and only if $m_{ij} \neq 2$. The edge corresponding to $\{z_i, z_j\}$ is labelled m_{ij} if $m_{ij} > 3$ and it is left unlabelled if $m_{ij} = 3$. The resulting graph with edge labels is called the Coxeter diagram associated to M. The matrix M is clearly determined by its diagram, up to permutations of the indices.

2.3. Suppose that $X = \{x_0, \cdots, x_n\}$ is a set of n + 1 symbols and that $M = (m_{ij})_{0 \leq i,j \leq n}$ is a Coxeter matrix. The Coxeter group W associated to M is the quotient of the free group on X (denoted G(X)) by the normal subgroup generated by $\{(x_i x_j)^{m_{ij}}\}$, where $0 \leq i, j \leq n$, and $m_{ij} \neq \infty$.

2.4. Associated to M there is another matrix $C_M = (c_{ij})_{0 \leq i,j \leq n}$, called its cosine matrix, defined by the formula

$$c_{ij} = -\cos(\pi/m_{ij}).$$

2.5. Regard $\mathbb{R}^{n+1}$ as the set of all functions from $\{0,1,\cdots,n\}$ to $\mathbb{R}$ and let $\{e_0, e_1, \cdots, e_n\}$ be the standard basis. Let M, C_M, X, W be as above. Put $B_M(e_i, e_j) = c_{ij}$. This extends to a bilinear form

$$B_M : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \dashrightarrow \mathbb{R}.$$

For $0 \leq i \leq n$, define a linear reflection s_i on $\mathbb{R}^{n+1}$ by $s_i y = y - 2B_M(y,e_i)e_i$. (This is not a standard orthogonal reflection with respect to a hyperplane, rather it is orthogonal with respect to the symmetric bilinear form $B_M(\ ,\ )$. This bilinear form could be degenerate and/or indefinite.) It can be checked that the order of $(s_i s_j)$ in $GL(n+1, \mathbb{R})$ is m_{ij}. Hence, the map $X \dashrightarrow GL(n+1, \mathbb{R})$ defined by $x_i \dashrightarrow s_i$ extends to a representation $\phi : W \dashrightarrow GL(n+1, \mathbb{R})$ called the canonical representation of W. (For more details on the canonical representation, see Ch. V of [1].)

It follows from the existence of the representation ϕ that the natural map $X \dashrightarrow W$ is injective and that it takes each element of X to a non-trivial element of W. Henceforth, we shall identify X with its image in W. It also follows from the existence of ϕ that the element $(x_i x_j)$ has order m_{ij} in W (rather than just dividing m_{ij}). The pair (W,X) is called the Coxeter system associated to M. Any pair isomorphic to (W,X) is also called a Coxeter system.

2.6. Suppose that (L,q) is a simplicial complex over Δ^n (cf. (1.14)) and that L satisfies (1.10). From (1.18), we get a function $M : \text{Chamb}(L) \dashrightarrow$ {Coxeter matrices} defined by $M(\Sigma) = (m_{ij}(\Sigma))_{0 \leq i,j \leq n}$. We say that (L,q) is regular if M is a constant function. We denote its value by $M(L) = (m_{ij}(L))_{0 \leq i,j \leq n}$. We say that L is of type M(L).

2.7. Again, suppose that (L,q) is a simplicial complex over Δ^n and that L satisfies (1.10). Then (L,q) is symmetrically regular if its automorphism group Aut(L,q) acts transitively on Chamb(L).

Suppose (L,q) is symmetrically regular. Obviously, for any $a \in \text{Aut}(L,q)$ and $\Sigma \in \text{Chamb}(L)$, we have $M(a\Sigma) = M(\Sigma)$. Hence, symmetric regularity implies regularity.

We suppose for the remainder of this section that (L,q) is a regular simplicial complex over Δ^n.

2.8. Let (W,X) be the Coxeter system associated to the Coxeter matrix M(L). In (1.19) we defined a transitive nonsimplicial action of G(X) on Chamb(L). It follows from formula (1.20.1) that this action factors through W. Hence, W acts transitively (from the right) on Chamb(L). For any $\Sigma \in \text{Chamb}(L)$, let $\pi(L,\Sigma)$ denote the isotropy subgroup at Σ. We call $\pi(L,\Sigma)$ the group of L at Σ. As a right W-set, Chamb(L) is isomorphic to $\pi(L,\Sigma)\backslash W$. Hence, for each $w \in W$, $\pi(L,\Sigma w) = w^{-1}\pi(L,\Sigma)w$. In other words, as Σ ranges over Chamb(L), $\pi(L,\Sigma)$ ranges over the conjugacy class of a subgroup in W.

Next we state a few easy consequences of theorem 1.22.

PROPOSITION 2.9. For i = 1, 2, suppose that (L_i,q_i) is a regular simplical complex over Δ^n of type M and that $\Sigma_i \in \text{Chamb}(L_i)$. Let (W,X) be the Coxeter system associated to M. Then there is a simplicial map $f : L_1 \to L_2$ over Δ^n such that $f(\Sigma_1) = \Sigma_2$ if and only if $\pi(L_1,\Sigma_1)$ is a subgroup of $\pi(L_2,\Sigma_2)$. Moreover, f is an isomorphism if and only if $\pi(L_1,\Sigma_1) = \pi(L_2,\Sigma_2)$.

COROLLARY 2.10. For i = 1, 2, suppose that (L_i,q_i) is a regular simplicial complex over Δ^n and that $\Sigma_i \in \text{Chamb}(L_i)$. Then (L_1,q_1) and (L_2,q_2) are isomorphic over Δ^n if and only if $M(L_1) = M(L_2)$ and $\pi(L_1,\Sigma_1)$ and $\pi(L_2,\Sigma_2)$ are conjugate subgroups of the associated Coxeter group W.

THEOREM 2.11. Suppose that (L,q) is a regular simplicial complex over Δ^n and that $\Sigma \in \text{Chamb}(L)$.

(i) The automorphism group, Aut(L,q), is isomorphic to $N(\pi)/\pi$, where $\pi = \pi(L,\Sigma)$, and $N(\pi)$ denotes the normalizer of π in W.

(ii) (L,q) is symmetrically regular if and only if π is normal in W (and hence, $\text{Aut}(L,q) \cong W/\pi$).

2.12. We shall say that (L,q) is universal of type M if it has the following property: given any other regular simplicial complex (L_1,q_1) over Δ^n of type M and chambers $\Sigma \in \text{Chamb}(L)$ and $\Sigma_1 \in \text{Chamb}(L_1)$, there exists a unique $f : L \to L_1$ which is a simplicial map over Δ^n and which takes Σ to Σ_1. Clearly, a universal regular simplicial complex over Δ^n of type M is unique up to isomorphism over Δ^n.

Consider the following two conditions on a regular simplicial complex (L,q) of type M.

(i) $\pi(L,\Sigma)$ is the trivial group.

(ii) $\text{Aut}(L,q) \cong W$.

From (2.9) and (2.11) we see, just as in the theory of covering spaces, that these two conditions are equivalent and that either implies that (L,q) is universal.

2.13. We turn now to the question of existence of such universal complexes. Let $M = (m_{ij})_{0\le i,j\le n}$ be a Coxeter matrix and (W,X) the associated Coxeter system. It turns out that the universal complex of type M coincides with the well-known "Coxeter complex" of (W,X). We shall now recall its construction. For any proper subset I of $\{0,1,\cdots,n\}$, let W_I denote the subgroup of W generated by X_I, where $X_I = \{x_i \in X \mid i \in I\}$. Consider the simplicial complex $W \times \Delta^n$. Define an equivalence relation $\sim$ on $W \times \Delta^n$ by $(w,\sigma) \sim (w',\sigma')$ <=> the simplices σ and σ' are equal and $w^{-1}w' \in W_I$, $I = \mathrm{Type}(\sigma)$. (Type($\sigma$) is defined in (1.13).) The quotient of $W \times \Delta^n$ by $\sim$ is a simplicial complex which we shall denote by U_M. (U_M is the Coxeter complex.) (The geometric realization of U_M is the space formed by pasting together n-simplices, one for each element of W, in the obvious manner.) For the moment, write $[w,\sigma]$ for the simplex in U_M which is the equivalence class of (w,σ). We can identify Δ^n with the subcomplex in U_M consisting of all simplices of the form $[1,\sigma]$, $\sigma \in \Delta^n$. The natural projection $W \times \Delta^n \dashrightarrow \Delta^n$ induces a projection $p : U_M \dashrightarrow \Delta^n$. Thus, U_M is a simplicial complex over Δ^n. Define an action (from the left) of W on U_M by $w[w',\sigma] = [ww',\sigma]$. It is clear that W acts as a group of automorphisms over Δ^n and that W acts transitively on Chamb(U_M) (= $\{w\Delta^n\}_{w\in W}$); hence, $W = \mathrm{Aut}(U_M,p)$ and U_M is symmetrically regular. Thus, we have proved the following result.

2.13.1. The Coxeter complex U_M equipped with its canonical projection $p : U_M \dashrightarrow \Delta^n$ is the universal regular simplicial complex over Δ^n of type M.

Here is another property of U_M.

2.13.2. If W is finite, then U_M is homeomorphic to S^n. If W is infinite, then U_M is contractible. (For a proof, see [11], p. 108, or [5].)

2.14. If π is any subgroup of W, then U_M/π is naturally a simplicial complex over Δ^n; however, the quotient may fail to satisfy condition (1.10). (For example, if $\pi = W$, then $U_M/\pi \cong \Delta^n$, which does not satisfy (1.10).) Even if the quotient does satisfy (1.10), it might still fail to be regular of type M. We shall now determine the conditions on the subgroup π so that the quotient is regular of type M.

For $0 \leq i, j \leq n$, $i \neq j$, let W_{ij} denote the subgroup of W generated by $\{x_i,x_j\}$. Then W_{ij} is a dihedral group of order $2m_{ij}$.

2.14.1. If (L,q) is regular of type M, then for each Σ in Chamb(L), the intersection of $\pi(L,\Sigma)$ and W_{ij} is the trivial group. (To see this, note that the subgroup W_{ij} acts freely and transitively on Chamb($L_{ij}(\Sigma)$), where $L_{ij}(\Sigma)$ is defined in (1.18.1); hence, the intersection of W_{ij} with the isotropy subgroup $\pi(L,\Sigma)$ must be trivial.)

Consider the following condition on a subgroup π of W.

2.14.2. For $0 \leq i, j \leq n$, $i \neq j$, and for each $w \in W$, the intersection of $w^{-1}\pi w$ and W_{ij} is trivial.

It is now easy to see that we have the following result.

LEMMA 2.14.3. Let M be a Coxeter matrix and π a subgroup of W. Then U_M/π is a regular simplicial complex over Δ^n of type M if and only if π satisfies (2.14.2).

COROLLARY 2.15. Let M be a Coxeter matrix and (W,X) the associated Coxeter system. Then the set of isomorphism classes of regular simplicial complexes over Δ^n of type M is naturally bijective with the set of conjugacy classes of subgroups π of W such that π satisfies (2.14.2).

PROPOSITION 2.16. Let M be a Coxeter matrix. Then there exists a finite regular simplicial complex over Δ^n of type M if and only if each entry of M is $< \infty$.

Proof. If (L,q) is of type M and L is a finite complex, then obviously each m_{ij} is $< \infty$. Conversely, suppose each m_{ij} is $< \infty$. Since the associated Coxeter group W has a faithful representation in $GL(n+1,\mathbb{R})$, it is virtually torsion-free (cf. [10]). Hence, there is a torsion-free subgroup π of finite index in W. Since W_{ij} is finite and π is torsion-free the subgroup π satisfies (2.14.2). The proposition follows.

2.17. Next let us consider the action of the automorphism group W on U_M. For any proper subset I of $\{0,1,\cdots,n\}$ let (W_I, X_i) be as in (2.13). The maximal proper subsets of $\{0,1,\cdots,n\}$ are of the form $I(i) = \{0,1,\cdots,n\} - \{i\}$ for some i, with $0 \leq i \leq n$. To simplify notation, put $X_{(i)} = X_{I(i)}$ and $W_{(i)} = W_{I(i)}$. The isotropy subgroup at a simplex σ in Δ^n ($\Delta^n \subset U_M$) is $W_{Type(\sigma)}$. Similarly, the isotropy subgroup at $w\sigma$, $w \in W$, is $wW_{Type(\sigma)}w^{-1}$. It is easy to see that W acts properly on the geometric realization of U_M if and only if each isotropy subgroup is finite. In other words,

2.17.1. W acts properly on the geometric realization of U_M if and only if for each proper subset I of $\{0,1,\cdots,n\}$, the group W_I is finite.

Since each maximal subgroup of the form W_I is of the form $W_{(i)}$, we can rephrase (2.17.1) as follows: W acts properly on the geometric realization of U_M if and only if it satisfies the finiteness condition (FC) below.

(FC) For $0 \leq i \leq n$, the group $W_{(i)}$ is finite.

Next, suppose that π is any subgroup of W. The isotropy subgroup of π at a simplex $w\sigma$ in U_M is $\pi \cap wW_{\mathrm{Type}(\sigma)}w^{-1}$. Hence, the π-action on U_M is free and if and only if each of these intersections is the trivial group. Moreover, if this condition holds then it is also easy to see that the π-action is proper. Thus, π acts freely and properly on U_M if and only if it satisfies the following condition.

2.17.2. For $0 \leq i \leq n$ and for each $w \in W$, $w^{-1}\pi w \cap W_{(i)}$ is the trivial group.

LEMMA 2.18. Suppose that M is a Coxeter matrix, W is the associated Coxeter group, and that π is a subgroup of W.

(i) The natural map $q : U_M \to U_M/\pi$ is a topological covering if and only if π satisfies (2.17.2).

(ii) If π satisfies (2.17.2), then the fundamental group of U_M/π is isomorphic to π, (provided $n \geq 2$).

Proof. Statement (i) follows from the remarks in (2.17). Statement (ii) follows from the fact that U_M is simply connected (cf. (2.13.2)), if $n \geq 2$.

2.19. Let M be a Coxeter matrix and C_M (= $(c_{ij})_{0\leq i,j\leq n}$) its associated cosine matrix (cf. (2.2)). Let $C_{(i)}$ be the n by n matrix obtained from C_M by deleting the i^{th} row and i^{th} column. We suppose that, for $0 \leq i \leq n$, $C_{(i)}$ is positive definite. Then C must be one of the following three types: positive definite (type (+1)), positive semi-definite with 1-dimensional null-space (type (0)), or nondegenerate and indefinite of signature $(n,1)$ (type (-1)). We shall say that M is of <u>type ε</u>, $\varepsilon \in \{+1,0,-1\}$, if C_M is of type ε.

2.20. The complete simply connected Riemannian-manifolds, $n \geq 2$, of constant sectional curvature are the n-sphere, Euclidean n-space, and hyperbolic n-space (denoted H^n). Let Y^n_ε stand for S^n, $\mathbb{R}^n$, or H^n as $\varepsilon = +1$, 0, or -1.

Let $(y_0, \cdots, y_n)$ be linear coordinates on $\mathbb{R}^{n+1}$. For $\varepsilon \in \{+1,0,-1\}$, let q_ε be the quadratic form defined by $q_\varepsilon(y) = (y_0)^2 + \cdots + (y_{n-1})^2 + \varepsilon(y_n)^2$, and let $B_\varepsilon(\ ,\ )$ be the symmetric bilinear form associated to q_ε. For $\varepsilon = \pm 1$, we can identify Y^n with the hypersurface $q^n_\varepsilon(y) = \varepsilon$ in $\mathbb{R}^{n+1}$. (When $\varepsilon = -1$, we also require $y_n > 0$.) We identify Y^n_0 with the affine hyperplane $y_n = 1$. For $y \in Y^n_\varepsilon$, the tangent space of Y^n_ε can be identified with a linear hyperplane in $\mathbb{R}^{n+1}$; for $\varepsilon = \pm 1$, this is the hyperplane which is

B_ε-orthogonal to y. The Riemannian inner product on the tangent space, $T_y(Y^n_\varepsilon)$, is obtained by restricting B to the corresponding hyperplane. It follows that we can identify the group of isometries of Y^n_ε, denoted $\mathrm{Isom}(Y^n_\varepsilon)$, with the subgroup of $GL(n+1,\mathbb{R})$ which preserves q_ε.

By a <u>hyperplane</u> in Y^n_ε we shall mean the intersection of the linear hyperplane in $\mathbb{R}^{n+1}$ with Y^n_ε. (Such a hyperplane in Y^n_ε is a complete, totally geodesic submanifold of codimension one.) It follows that an isometric reflection on Y^n_ε across a hyperplane is the restriction to Y^n_ε of a linear reflection on $\mathbb{R}^{n+1}$.

2.21. Suppose that (W,X) is a Coxeter system of rank $n+1$. We are interested in finding a representation $\theta : W \to \mathrm{Isom}(Y^n_\varepsilon)$ satisfying the following condition.

2.21.1. The group $\theta(W)$ acts as a (discrete) reflection group on Y^n_ε (cf. [5]). Moreover, there is a convex n-simplex Σ in Y^n_ε with codimension-one faces denoted by $\Sigma_{(i)}$, $0 \le i \le n$, such that Σ is a fundamental chamber for $\theta(W)$ on Y^n_ε and such that for each $x_i \in X$, $\theta(x_i)$ is the reflection across the hyperplane supported by $\Sigma_{(i)}$.

We shall say that $\Sigma_{(i)}$ has type $\{i\}$. More generally, for any proper subset I of $\{0,1,\cdots,n\}$, the simplex $\Sigma_I = \cap_{i\in I}\Sigma_{(i)}$ is said to be of type I. This gives a isomorphism from Σ to the standard simplex Δ^n.

If W acts on Y^n_ε as in (2.21.1), then the translates of Σ by W give a triangulation of Y^n_ε by convex simplices. It is well-known that the underlying simplicial complex of this triangulation can be identified with the Coxeter complex U_M, where M is the Coxeter matrix associated to (W,X). (See, for example, Prop. 15.1, p. 318 in [5].)

The following result is classical (e.g., see [1] or [13]).

THEOREM 2.22. Let (W,X) be a Coxeter system of rank $n+1$ and C its associated cosine matrix. A necessary and sufficient condition for there to be a representation $\theta : W \to \mathrm{Isom}\,(Y^n_\varepsilon)$, $\varepsilon \in \{+1,0,-1\}$, as in (2.21.1) is that C be of type ε. Moreover, if θ exists and we regard it as a representation into $GL(n+1,\mathbb{R})$, then it is equivalent to the dual of the canonical representation (cf. (2.5)).

Proof. First we show necessity. Let Σ be as in (2.21.1). For $0 \le i \le n$, let u_i be the unit normal to $\Sigma_{(i)}$ which points inward. Since $\Sigma_{(i)}$ and $\Sigma_{(j)}$ must make a dihedral angle of π/m_{ij}, we have that

$$B_\varepsilon(u_i,u_j) = \langle u_i,u_j\rangle = -\cos(\pi/m_{ij}) = c_{ij}.$$

In other words, the cosine matrix C can be identified with the matrix of inner products $(B_\varepsilon(u_i,u_j))_{0\le i,j\le n}$. Let v_i denote the vertex of Σ of type $\{0,1,\cdots,n\} - \{i\}$. The set $\{u_j\}_{j\ne i}$ is a basis for $T_{v_i}(Y^n_\varepsilon)$. Hence, the matrix of dot products of this basis is positive definite; i.e., $C_{(i)}$ is positive definite. If $\varepsilon = \pm 1$, then $\{u_0,\cdots,u_n\}$ is a basis for $\mathbb{R}^{n+1}$ Since C is the matrix representation for B_ε with respect to this basis, it must be of Type(ε). The case $\varepsilon = 0$ also follows easily.

To prove the converse, suppose C is of Type(ε). To simplify the discussion, suppose $\varepsilon \ne 0$. Then we can find a basis $\{u_0,\cdots,u_n\}$ for $\mathbb{R}^{n+1}$ such that $B_\varepsilon(u_i,u_j) = c_{ij}$. The intersection of the half-spaces $B_\varepsilon(u_i,y) \ge 0$ with Y^n_ε is a simplex Σ. For each $x_i \in X$ let $\theta(x_i) \in \mathrm{Isom}(Y^n_\varepsilon)$ be the reflection across the hyperplane $B_\varepsilon(u_i,y) = 0$ given by $z \dashrightarrow z - 2B_\varepsilon(u_i,z)$. The map $\theta : X \to \mathrm{Isom}(Y^n_\varepsilon)$ extends to a representation $\theta : W \to \mathrm{Isom}(Y^n_\varepsilon)$ as in (2.21.1). Moreover, θ is obviously equivalent to the canonical representation (which is self-dual since B is nondegenerate). The argument must be modified somewhat when $\varepsilon = 0$. (See Ch. V, 4.9 of [1].)

COROLLARY 2.23. Let (W,X) be a Coxeter system and C its cosine matrix.

(i) The group W is finite if and only if C is positive define.

(ii) (W,X) satisfies condition (FC) (cf. (2.17)) if and only if C is of type ε for some $\varepsilon \in \{+1,0,-1\}$.

2.24. Suppose that M is a Coxeter matrix of degree (n+1) and of type ε. We shall say that the corresponding Coxeter complex U_M is a classical regular triangulation of Y^n_ε over Δ^n. More generally, if π is a subgroup of W such that each conjugate of π intersects $W_{(i)}$ trivially (i.e., if π satisfies (2.17.2)), then U_M/π is called a classical regular triangulation of Y^n_ε/π over Δ^n.

LEMMA 2.25. Suppose that (L,q) is a regular simplicial complex over Δ^n of type M, $n \ge 2$, and that (W,X) is the associated Coxeter system. Also, suppose that L is a finite complex. The following statements are equivalent.

(i) L is simply connected and satisfies (1.11.1).

(ii) L is PL-homeomorphic to S^n.

(iii) W is finite and the natural projection $p : U_M \to L$ is an isomorphism.

The implication (ii) $\Rightarrow$ (iii) means that any regular triangulation of S^n over Δ^n is combinatorially equivalent to a classical one.

Proof. It is fairly clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). (The implication (iii) $\Rightarrow$ (ii) follows from theorem (2.22) and from (2.13.1).)

Suppose, by induction, that the implication (i) $\Rightarrow$ (iii) holds in all dimensions ≥ 2 and $< n$. Let $\Sigma \in \text{Chamb}(L)$ and put $\pi = \pi(L,\Sigma)$. For $0 \leq i \leq n$, let v_i be the vertex of Σ which projects to $\{i\}$ in Δ^n. The complex $\text{Link}(v_i,L)$ is a regular simplicial complex over Δ^{n-1} ($\Delta^{n-1} \cong \text{Link}(\{i\},\Delta^n)$); its associated Coxeter system is $(W_{(i)},X_{(i)})$. Since L satisfies (1.11.1) so does $\text{Link}(v_i,L)$. If $n > 2$, it follows from the inductive hypothesis that $W_{(i)} \cap \pi = \{1\}$. If $n = 2$, the same conclusion follows from (2.14.3). Thus, π satisfies (2.17.2). By Lemma (2.18) (i), $p : U_M \to L$ is a covering projection. Since L is simply connected, p must be an isomorphism (i.e., π is trivial). Since L (and hence, U_M) is a finite complex and since W is bijective with $\text{Chamb}(U_M)$, the group W is finite. Thus, (iii) holds.

THEOREM 2.26. Suppose that (L,q) is a regular simplicial complex over Δ^n of type M and that (W,X) is the associated Coxeter system. The following statements are equivalent.

(i) L satisfies (1.11.1).

(ii) L is a PL-manifold.

(iii) (W,X) satisfies condition (FC) and the natural projection $p : U_M \to L$ is a topological covering.

(iv) M is of type ε, for some $\varepsilon \in \{+1,0,-1\}$, and L is equivalent to a classical regular triangulation of Y^n_ε/π (where $\pi \cong \pi(L,\Sigma)$).

In other words, if L is a PL-manifold, then it is isomorphic to a classical triangulation of a complete manifold of constant sectional curvature.

Proof. It follows from theorem 2.22 that (iii) $\Leftrightarrow$ (iv). Obviously, (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i). We shall show that (i) $\Rightarrow$ (iii). Suppose that L satisfies (1.11.1). Let $\Sigma \in \text{Chamb}(L)$, $\pi = \pi(L,\Sigma)$, and for $0 \leq i \leq n$, let v_i be the vertex of Σ which projects to $\{i\} \in \Delta^n$. Since $\text{Link}(v_i,L)$ is simply connected, it follows from the previous lemma, that $W_{(i)}$ is finite and that $W_{(i)} \cap \pi = \{1\}$. In other words, (W,X) satisfies (FC) and π satisfies 2.17.2. By lemma (2.18), $p : U_M \to L$ is a covering. Thus, (iii) holds.

3. REGULAR CONVEX CELL COMPLEXES

3.1. Suppose that K is an n-dimensional convex cell complex satisfying (1.10). Then K' is a simplicial complex over Δ^n (cf. (1.23)) and K' satisfies (1.10)(cf. (1.26)). We shall say that K is a regular convex cell complex if (K',p) is a regular simplicial complex over Δ^n.

3.2. Here is an equivalent definition. In (1.27), we associated to each Σ in Chamb(K') an n-tuple $(m_1(\Sigma), \cdots, m_n(\Sigma))$, where $m_i(\Sigma)$ is an integer ≥ 3 for $1 \leq i \leq n-1$, and where $m_n(\Sigma)$ is either an integer ≥ 3 or ∞. The Coxeter matrix $(m_{ij}(\Sigma))_{0 \leq i,j \leq n}$ is given by (1.28.1). It follows that K is regular if and only if the function from Chamb(K') to n-tuples is constant; it is then denoted by $(m_1(K), \cdots, m_n(K))$ (or simply by $(m_1, \cdots, m_n)$) and called the Schläfli symbol of K.

3.3. An n-dimensional convex cell complex K satisfying (1.10) is symmetrically regular if Aut(K) acts transitively on Chamb(K'). Since Aut(K) = Aut(K',p) (cf. (1.25)), we see that K is symmetrically regular if and only if (K',p) is symmetrically regular over Δ^n. It follows from (2.7) that the symmetric regularity of K implies its regularity.

From now on we suppose that K is a regular convex cell complex of dimension n.

3.4. The Coxeter matrix of K is the Coxeter matrix $M = (m_{ij})_{0 \leq i,j \leq n}$ associated to (K',p). It is related to the Schläfli symbol by formula (1.28.1) which we restate here:

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } |i - j| \geq 2 \\ m_k, & \text{if } \{i,j\} = \{k-1,k\} \end{cases} \qquad (3.4.1)$$

It follows that the Coxeter diagram of M is a connected line segment. Similarly, the Coxeter system of K is the Coxeter system (W,X) associated to M.

Using (1.23), (2.10) and (2.11) can be translated as follows.

PROPOSITION 3.5. For i = 1,2, suppose that K_i is a regular convex cell complex and that $\Sigma_i \in \text{Chamb}(K_i')$. Then K_1 and K_2 are combinatorially equivalent if and only if their Schläfli symbols are equal and $\pi(K_1',\Sigma_1)$ and $\pi(K_2',\Sigma_2)$ are conjugate subgroups of the associated Coxeter group.

PROPOSITION 3.6. (i) Aut(K) is isomorphic to $N(\pi)/\pi$, where $\pi = \pi(K',\Sigma)$ and $N(\pi)$ is the normalizer of π in W.

(ii) K is symmetrically regular if and only if π is normal in W (and hence, $\mathrm{Aut}(K) \cong W/\pi$).

3.7. A convex (n+1)-cell E is combinatorially regular if its boundary complex, denoted by ∂E, is regular in the sense of (3.1). A convex (n+1)-cell E in $\mathbb{R}^{n+1}$ is a classical regular convex polyhedron, if its group of isometric symmetries, denoted by Isom(E), acts transitively on Chamb$((\partial E)')$. If E is a classical regular polyhedron, then radial projection from the center gives a tessellation of S^n by convex cells such that Isom(E) acts transitively on the chambers in its barycentric subdivision. We call such a tessellation a classical regular tessellation of S^n, by convex cells. Conversely, given a classical regular tessellation of S^n, by taking the convex hull of its vertex set, we obtain a classical regular polyhedron in $\mathbb{R}^{n+1}$.

3.8. Suppose that K is a classical regular tessellation of S^n by convex cells. Let (W,X) be the associated Coxeter system, then $W \cong \mathrm{Aut}(K)$ and we can identify W with a finite reflection group in $\mathrm{Isom}(S^n)$ $(= O(n+1))$. The Coxeter diagram of (W,X) is a line segment. Conversely, there is the following classical result (cf. [3]).

THEOREM 3.9. Let (W,X) be a Coxeter system with Coxeter matrix $M = (m_{ij})_{0 \le i,j \le n}$. Suppose that W is finite and that the Coxeter diagram of M is a line segment. For $1 \le k \le n$, put $m_k = m_{ij}$, where $\{i,j\} = \{k-1,k\}$. Then there is a classical regular tessellation K of S^n by convex cells with Schläfli symbol $(m_1, \cdots, m_n)$. Moreover, K is unique up to isometries of S^n.

Proof. By theorem (2.22), there is an orthogonal action of W on S^n such that the corresponding triangulation can be identified with U_M. We want to show that the spherical simplices in S^n can be assembled into convex cells in S^n giving a convex cell complex K, with $K' = U_M$. Since the diagram of W is a line segment, the diagram of $W_{(i)}$ is either a line segment (if i = 0 or n) or two line segments (if $1 \le i \le n-1$). It follows that $W_{(i)} = G_i \times H_i$ where G_i is the subgroup generated by $x_0, \cdots, x_{i-1}$ and H_i is the subgroup generated by $x_{i+1}, \cdots, x_n$. Let Σ be a chamber in S^n as in (2.21.1) (so that Σ is a spherical n-simplex) and for $0 \le i \le n$, let v_i be the vertex of Σ which projects to $\{i\} \in \Delta^n$. Let σ_i be the face of σ spanned by $\{v_0, \cdots, v_i\}$. Put

$$E_i = \bigcup_{\omega \in G_i} \omega\sigma_i .$$

We claim that, for $0 \leq i \leq n$,

(a) E_i is a convex i-cell in the i-sphere fixed by H_i.

(b) The triangulation of E_i by the i-simplices $\omega\sigma_i$, $\omega \in G_i$, is the barycentric subdivision.

Once we have established (a) and (b) it then follows easily that the translates of these cells under W give a classical regular tessellation K of S^n and that $K' \cong U_M$. Let S^i be the i-sphere fixed by H_i. Obviously, $\sigma_i \subset S^i$. Since the G_i- and H_i-actions commute, we also have $E_i \subset S^i$. The convexity of E_i is then clear, i.e., (a) holds. Suppose, by induction, that the triangulation of E_i by translates of σ_i is the barycentric subdivision of E_i for $i < n$. (The case, $i = 0$, is trivial.) In particular, this holds for E_{n-1} and all of its translates under G_n. But the union of these translates in ∂E_n. Also, v_n is the central point of E_n. Therefore, (b) holds. The uniqueness of K follows from the last sentence of theorem (2.22).

3.10. The theorem above shows that the classification of classical regular convex polyhedra follows from the classification of finite Coxeter groups. Given a finite Coxeter group with diagram a line segment, we obtain a Schläfli symbol $(m_1, \cdots, m_n)$ by the formula $m_i = m_{ij}$, where $j = i-1$. Since we get the same diagram if we reverse the order of the indices, this Schläfli symbol is only well-defined up to reversing the order of the m_i's. Thus, a given diagram corresponds to classical regular polyhedron and its dual. The regular polyhedron is self-dual if and only if $(m_1, \cdots, m_n) = (m_n, \cdots, m_1)$

The diagram $\circ\overset{p}{\text{——}}\circ$, $p \geq 3$, has Schläfli symbol (p) corresponding to the regular p-gon which is self-dual. The diagram $\circ\text{——}\circ \cdots \circ\text{——}\circ$ (A_n) has Schläfli symbol $(3,3,\cdots,3)$ corresponding to the regular n-simplex (self-dual). The diagram $\circ\overset{4}{\text{——}}\circ \cdots \circ\text{——}\circ$ (B_n) has Schläfli symbol $(4,3,\cdots,3)$ or $(3,3,\cdots,4)$ corresponding to the regular n-cube or regular n-octahedron, respectively. The diagram $\circ\overset{5}{\text{——}}\circ\text{——}\circ$ (H_3) has symbol $(5,3)$ or $(3,5)$ corresponding to the dodecahedron or icosahedron. The diagram $\circ\overset{5}{\text{——}}\circ\text{——}\circ\text{——}\circ$ (H_4) has symbols $(5,3,3)$ or $(3,3,5)$; the corresponding regular 4-dimensional polyhedra are called, by Coxeter, the "120-cell" and the "600-cell" respectively. The diagram $\circ\text{——}\circ\overset{4}{\text{——}}\circ\text{——}\circ$ (F_4) has symbol $(3,4,3)$ and corresponds to the self-dual regular 4-dimensional polyhedron called the "24-cell" by Coxeter. Since these are the only finite Coxeter groups with diagram a line segment (see Table 1), we have listed above, all of the classical regular polyhedra.

Combining theorem (3.9) with lemma (2.25) we get the following result.

PROPOSITION 3.11. Suppose that K is a regular convex cell complex and that K is PL-homeomorphic to S^n. Then K is equivalent to a classical regular tessellation of S^n by convex cells.

COROLLARY 3.12. Suppose that E is a combinatorially regular convex cell. Then E is combinatorially equivalent to a classical regular convex polyhedron.

This corollary was proved by McMullen [9], under the slightly stronger hypothesis of combinatorial symmetric regularity. (See Chapter V of [8] for a somewhat different argument.)

3.13. Let $(m_1, \cdots, m_n)$ be an n-tuple of numbers. The <u>initial part</u> (resp. <u>final part</u>) of this n-tuple is the (n-1)-tuple $(m_1, \cdots, m_{n-1})$ (resp. $(m_2, \cdots, m_n)$). The initial part (resp. final part) is <u>spherical</u> if it is the Schläfli symbol of a classical regular convex polyhedron of dimension n. We shall say that $(m_1, \cdots, m_n)$ is an <u>admissible Schläfli symbol</u> if its initial part is spherical.

PROPOSITION 3.14. Let K be a regular convex cell complex with Schläfli symbol $(m_1, \cdots, m_n)$.

(i) Each k-cell in K, $0 \leq k \leq n$, is combinatorially regular and has Schläfli symbol $(m_1, \cdots, m_{k-1})$.

(i)' The link of each k-cell in K, $0 \leq k \leq n-2$, is a regular convex cell complex with Schläfli symbol $(m_{k+2}, \cdots, m_n)$.

(ii) Any two k-cells in K are combinatorially equivalent.

(ii)' Put $\pi = \pi(K', \Sigma)$. If each conjugate of π has trivial intersection with H_k (defined in the proof of Theorem (3.9)), $0 \leq k \leq n-2$, then the links of any two k-cells in K are combinatorially equivalent.

(iii) The Schläfli symbol of K is admissible.

Proof. Statements (i) and (i)' are immediate from the definitions. It follows from corollary (3.12) that any k-cell in K is classically regular with automorphism group G_k (defined in the proof of (3.9)). It follows that the group of such a k-cell is trivial; thus, statement (ii) follows from (i) and proposition (3.5). Similarly, (ii)' follows from (i)' and proposition (3.5). Statement (iii) follows from (i) (when k = n) and Corollary (3.12).

LEMMA 3.15. Suppose that (W, X) is the Coxeter system of K. Let $\Sigma \in \mathrm{Chamb}(K')$ and put $\pi = \pi(K', \Sigma)$. (π is a subgroup of W.) Then π satisfies the following two conditions:

3.15.1. Each conjugate of π has trivial intersection with $W_{(n)}$. (Recall that $W_{(n)}$ is the subgroup of W generated by $\{x_0, \cdots, x_{n-1}\}$.)

3.15.2. Each conjugate of π has trivial intersection with $W_{n-1,n}$, the subgroup of W generated by $\{x_{n-1}, x_n\}$.

Proof. The group $W_{(n)}$ is the Coxeter group associated to an n-cell in K. Since such an n-cell is classically regular (cf. (3.12)) we have condition (3.15.1). Condition (3.15.2) is a special case of (2.14.1) which holds for (K', p).

3.16. Let $(m_1, \cdots, m_n)$ be an admissible Schläfli symbol, M the associated Coxeter matrix, (W, X) the associated Coxeter system, and U_M the Coxeter complex. Let v_n be a vertex in U_M which projects to $\{n\} \in \Delta^n$. Then $\mathrm{Link}(v_n, U_M)$ is the Coxeter complex of the finite Coxeter group $W_{(n)}$. It follows that $\mathrm{Star}(v_n, U_M)$ can be identified with the barycentric subdivision of a regular n-cell, the Schläfli symbol of which is $(m_1, \cdots, m_{n-1})$. By assemblying the chambers of U_M which meet at each vertex of type $\{0, \cdots, n-1\}$ in this fashion, we obtain a convex cell complex $U(m_1, \cdots, m_n)$, the derived complex of which is U_M.

THEOREM 3.17. (i) Let K be a regular convex cell complex with Schläfli symbol $(m_1, \cdots, m_n)$ and let $\pi (= \pi(L, \Sigma))$ be its group. Then K is combinatorially equivalent to $U(m_1, \cdots, n_n)/\pi$.

(ii) Let $(m_1, \cdots, m_n)$ be an admissible Schläfli symbol, (W, X) the associated Coxeter system, and π a subgroup of W satisfying (3.15.1) and (3.15.2). Then $U(m_1, \cdots, m_n)/\pi$ is a regular convex cell complex with symbol $(m_1, \cdots, m_n)$.

Proof. Statement (i) follows from (2.13.1). Consider (ii). The fact that π satisfies (3.15.1) means that each cell in $U(m_1, \cdots, m_n)$ has trivial stabilizer in π. It follows that $U(m_1, \cdots, m_n)/\pi$ is a convex cell complex and that for each chamber Σ in its derived complex $m_i(\Sigma) = m_i$, $0 \leq i \leq n-1$. Condition (3.15.2) implies that $m_n(\Sigma) = m_n$ for each Σ. This proves (ii).

COROLLARY 3.18 (compare (2.15)). Let $(m_1, \cdots, m_n)$ be an admissible Schläfli symbol and (W,X) the associated Coxeter system. The set of combinatorial equivalence classes of regular convex cell complexes with symbol $(m_1, \cdots, m_n)$ is naturally bijective with the set of conjugacy classes of subgroups π of W satisfying (3.15.1) and (3.15.2).

Proposition (2.16) can be translated as follows:

PROPOSITION 3.19. Let $(m_1, \cdots, m_n)$ be an admissible Schläfli symbol and suppose that $m_n \neq \infty$. Then there exists a finite symmetrically regular convex cell complex with the given Schläfli symbol.

Proof. If W is finite, then choose π to be the trivial group. If W is infinite, then let π be any torsion-free normal subgroup of finite index in W (such π exist). Then $U(m_1, \cdots, m_n)/\pi$ is the desired finite convex cell complex.

3.20. Let $(m_1, \cdots, m_n)$ be an admissible Schläfli symbol, M the associated Coxeter matrix and (W,X) the associated Coxeter system. Suppose that the final part of $(m_1, \cdots, m_n)$ is also spherical (cf. (3.13)). This means that the subgroups $W_{(n)}$ and $W_{(0)}$ of W are finite. Since the diagram of W is a line segment, this implies that $W_{(i)}$ is finite for $0 \leq i \leq n$. In other words, (W,X) satisfies condition (FC) of (2.17). Therefore, M is of type ε for some $\varepsilon \in \{+1,0,-1\}$, (cf. (2.19)). We shall say that $(m_1, \cdots, m_n)$ is of type ε.

3.21. Suppose that $(m_1, \cdots, m_n)$ is of type ε. In theorem (2.22) we showed that U_M can be identified with a classical regular triangulation of Y^n_ε (cf. (2.24)). The proof of Theorem (3.9) shows that the simplices in Y^n_ε can be assembled into convex cells in Y^n_ε corresponding to the cells of $U(m_1, \cdots, m_n)$. In this way we identify W with a subgroup of $\mathrm{Isom}(Y^n_\varepsilon)$ (unique up to conjugation). We shall say that $U(m_1, \cdots, m_n)$ is a <u>classical regular tessellation of Y^n_ε by convex cells</u>. The Schläfli symbols of type ε are listed in Table 4. (We note from Table 4 that if $\varepsilon = -1$, then $n = 2$, 3, or 4. Thus, hyperbolic n-space admits a regular tessellation by convex cells only when $n = 2$, 3, or 4.) A subgroup π of W satisfies (2.17.2) if and only if it satisfies condition (3.15.1) and the following condition.

3.21.1. Each conjugate of π intersects $W_{(0)}$ trivially.

A subgroup π satisfying (3.15.1) and (3.21.1) acts freely on Y^n_ε. Conversely, any subgroup of W which acts freely on Y^n_ε satisfies (3.15.1) and (3.21.1). We note that if $\varepsilon = 0$ or -1, this means that a subgroup π satisfies (3.15.1) and (3.21.1) if and only if it is torsion-free.

If π satisfies (3.15.1) and (3.21.1), then we shall say that $U(m_1, \cdots, m_n)/\pi$ is a <u>classical regular tessellation of</u> Y^n_ε/π <u>by convex cells</u>.

From theorem (2.26) we immediately get the following result.

THEOREM 3.22. Suppose that K is a regular convex cell complex with Schläfli symbol $(m_1, \cdots, m_n)$ and associated Coxeter system (W,X). The following statements are equivalent.

(i) K satisfies (1.11.1).

(ii) K is a PL-manifold.

(iii) The final part of $(m_1, \cdots, m_n)$ is spherical and K is equivalent to $U(m_1, \cdots, m_n)/\pi$, where $\pi = \pi(K', \Sigma)$.

(iv) The symbol $(m_1, \cdots, m_n)$ is of type ε for some $\varepsilon \in \{+1, 0, -1\}$ and K is equivalent to a classical regular tessellation of Y^n_ε/π by convex cells.

REMARK. This result implies the theorem in the Introduction. It was first proved by Kato [7] by a different method.

EXAMPLES 3.23. (i) Suppose that K is a classical tessellation of S^n by convex cells and that W contains the element -1. ($W \cong \mathrm{Isom}(K) \subset O(n+1)$.) (This always happens except in the case where K is the boundary of a regular (n+1)-simplex or in the case where n = 1 and K is a polygon with an odd number of sides, cf. [3].) Then $K/\{\pm 1\}$ is a symmetrically regular tessellation of $\mathbb{R}P^n$.

(ii) A (p,q,2)-triangle group W gives a classical tesselation of Y^2_ε with Schläfli symbol (p,q), where $\varepsilon = +1$, 0, or -1 as $(p^{-1} + q^{-1} + 2^{-1})$ is greater than, equal to, or less than 1, respectively. For $\varepsilon = 0$ or -1 and π any torsion-free subgroup of finite index in W, the complex K/π is a tessellation of the closed surface Y^n_ε/π. (See [6].)

(iii) Suppose that K is a classical tessellation of $\mathbb{R}^n$ and that π ($\cong \mathbb{Z}^n$) is the subgroup of translations in W. Then K/π is a symmetrically regular tessellation of a torus.

(iv) Let K be the classical tesselation of S^3 with Schläfli symbol (5,3,3) and let K^* be the dual tessellation with symbol (3,3,5). The automorphism group W contains a subgroup π which is isomorphic to the binary icosahedral group and which acts freely on S^3. The quotient S^3/π is Poincaré's homology 3-sphere. It follows that K/π and K^*/π are regular convex cell complexes (at least, in the sense of (1.9)). The complex K/π has only one 3-cell, a dodecahedron; the complex K^*/π is tessellated by 5 icosahedra. These complexes are not symmetrically regular (by theorem (4.7) in the next section).

If K is not required to be a PL-manifold or to satisfy (1.11.1), then nothing prevents one of the above examples from occuring as the link of a cell.

4. SYMMETRIC REGULARITY

The conclusions of theorems (2.26) and (3.22) can be substantially improved if we add the hypothesis of symmetric regularity. (see theorem (4.7) below.)

4.1. A <u>pre-Coxeter system</u> (G,S) of rank n+1 consists of a group G and a set $S = \{s_0, \cdots, s_n\}$ of involutions in G such that S generates G. Associated to a pre-Coxeter system (G,S) there is a Coxeter matrix $M = (m_{ij})_{0 \leq i,j \leq n}$ given by the formula, $m_{ij} = \text{order}(s_i s_j)$. If (W,X) is the Coxeter system associated to M, then the map $X \to S$ given by $x_i \to s_i$ extends to an epimorphism $\Lambda : W \to G$. Let π denote the kernel of Λ. Let U(G,S) denote the quotient of the Coxeter complex U_M by π. The group G acts on U(G,S) and there is a natural projection $p : U(G,S) \to \Delta^n$. Thus, U(G,S) is a symmetrically regular simplicial complex over Δ^n (cf. (2.7)).

Let R denote the set of conjugates of S in G. Put U = U(G,S). For each r in R, the fixed point set of r on U is denoted by U_r and is called a <u>wall</u> of U.

LEMMA 4.1.1. Suppose that r and r' are elements of R such that $U_r = U_{r'}$. Then r = r'.

Proof. By definition r is conjugate to some element s in S. Choose g in G so that $g^{-1}rg = s$. Also, choose a point x in the relative interior of the (n-1)-simplex $\Delta^n \cap s\Delta^n$. The isotropy subgroup at x is the cyclic group of order two generated by s. Hence, the isotropy subgroup at gx is the cyclic group of order two generated by r. But r' fixes gx (since $gx \in U_{r'}$) and this forces r' = r.

The next result gives a characterization of Coxeter systems among pre-Coxeter systems.

LEMMA 4.2. Suppose that (G,S) is a pre-Coxeter system. The following statements are equivalent.

(i) For each s in S, $U - U_s$ is not connected.

(ii) (G,S) is a Coxeter system.

Moreover, if one of these conditions holds, then $U - U_r$ has exactly two components, for each $r \in R$.

Proof. (i) $\Rightarrow$ (ii). On p. 18 of [1] we find the following result.

LEMMA 4.2.1 (Bourbaki). Let (G,S) be a pre-Coxeter system and let $(P_s)_{s \in S}$ be a family of subsets of G satisfying the following conditions.

(A) For each $s \in S$, $1 \in P_s$.

(B) For each $s \in S$, $P_s \cap sP_s = \phi$.

(C) For elements s,s' in S and g in G, if $g \in P_s$ and $gs' \notin P_s$, then $sg = gs'$.

Then (G,S) is a Coxeter system. Moreover, $P_s = \{g \in G | \ell(sg) > \ell(g)\}$, where $\ell(g)$ denotes the word length of g with respect to the generating set S.

Supposing that (i) holds, we apply this lemma as follows. For each $s \in S$, let P_s denote the set of g in G such that the open chambers $\mathring{\Delta}^n$ and $g\mathring{\Delta}^n$ belong to the same component of $U - U_s$. Condition (A) holds trivially. Since U is connected, s must permute the components of $U - U_s$; hence, $g\mathring{\Delta}$ and $sg\mathring{\Delta}$ are contained in different components of $U - U_s$. This implies (B). To verify (C), suppose $g \in P_s$ and $gs' \notin P_s$. Then $g\mathring{\Delta}^n$ and $gs'\mathring{\Delta}^n$ lie in different components of $U - U_s$. Hence, $\mathring{\Delta}^n$ lie in different components of $U - g^{-1}U_s$. Putting $r = g^{-1}sg$, we have $g^{-1}U_s = U_r$; hence, U_r separates the adjacent chambers Δ^n and $s'\Delta^n$. But $U_{s'}$ is the unique wall with this property. Therefore, $U_r = U_{s'}$. Using Lemma (4.1.1), this implies that $r = s'$, i.e., $g^{-1}sg = s'$. This verifies (C). Consequently, Lemma (4.2.1) shows that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). We suppose that (G,S) is a Coxeter system. Here are some basic facts about Coxeter systems, the proofs of which can be found in [1].

4.2.2. For each $g \in G$ and $r \in R$, the parity of the number of times a gallery in U from Δ^n to $g\Delta^n$ crosses U_r depends only on g and r (and not on the gallery). This gives a mapping $n : G \times R \to \{\pm 1\}$ defined by $n(g,r) = -1$ (resp. +1) if a gallery from Δ^n to $g\Delta^n$ crosses U_r an odd (resp. even) number of times.

4.2.3. If $n(g,r) = -1$ (resp. $+1$), then a minimal gallery from Δ^n to $g\Delta^n$ crosses U_r exactly once (resp. does not cross U_r).

4.2.4. We have $n(g,r) = -1$ if and only if $\ell(rg) < \ell(g)$. From (4.2.2) and (4.2.3), we see that $\mathring{\Delta}^n$ and $g\mathring{\Delta}^n$ (resp. $sg\mathring{\Delta}^n$) belong to the same component of $U - U_s$ if and only if $n(g,s) = +1$ (resp. $n(g,s) = -1$). It follows that $U - U_s$ has exactly two components.

This shows that (ii) $\Rightarrow$ (i) and it also proves the last sentence of lemma (4.2).

4.3. Suppose that (L,q) is a symmetrically regular simplicial complex over Δ^n, that M is its Coxeter matrix, and that (W,X) is its Coxeter system. Put $G = \mathrm{Aut}(L,q)$. Then $G \cong W/\pi$, where π is the normal subgroup $\pi(L,\Sigma)$. Let S denote the image of X in G. Then (G,S) is a pre-Coxeter system, and (W,X) is its associated Coxeter system. Moreover, L and $U(G,S)$ are isomorphic over Δ^n (cf. (2.13.1)).

PROPOSITION 4.4. Let (L,q) be a symmetrically regular simplicial complex over Δ^n of type M. Suppose that L is a PL-manifold and that one of the following two conditions holds:

(a) $\dim L = 1$, or

(b) $H_1(L; \mathbb{Z}/2) = 0$.

Then (G,S) is a Coxeter system and $L \cong U_M$ (and consequently, L is a classical triangulation of Y^n_ε (cf. (2.24)).

Proof. If (G,S) is a Coxeter system, then $W \cong G$ and consequently, π is trivial and $L \cong U_M$. By lemma (4.2), it suffices to prove that for each $s \in S$, $L - L_s$ is not connected. (L_s denotes the fixed point set of s.) If (a) holds, this is obvious. In general, if follows from Smith theory that L_s is a $\mathbb{Z}/2$-homology manifold. Let $\hat{L}_s$ be the component of L_s containing Δ^n_s, the $(n-1)$-face of Δ^n corresponding to s. Then $\hat{L}_s$ is a $\mathbb{Z}/2$-homology manifold of codimension one in L. If $L - \hat{L}_s$ is connected, then the fundamental class in $H^n_c(L; \mathbb{Z}/2)$ must be in the image of $H^{n-1}_c(L_s; \mathbb{Z}/2)$, and hence, the Poincaré dual of $\hat{L}_s$ must represent a nonzero element of $H_1(L; \mathbb{Z}/2)$. It follows that condition (b) implies that $L - \hat{L}_s$ is not connected. This completes the proof.

4.5. The following condition on a n-dimensional simplicial complex L is a weak version of (1.11.1).

4.5.1. For each k-simplex σ in L, with $k \leq n-3$, $H_1(\mathrm{Link}(\sigma,L); \mathbb{Z}/2) = 0$.

LEMMA 4.6 (Compare lemma (2.25)). Suppose that (L,q) is a symmetrically regular simplicial complex over Δ^n, $n \geq 2$, that L is a finite complex, that L satisfies (4.5.1), and that $H_1(L; \mathbb{Z}/2) = 0$. Then L is isomorphic to a classical triangulation of S^n. (In particular, L is PL-homeomorphic to S^n.)

Proof. It is easy to see that since L is symmetrically regular, then so is the link of each simplex in L. Suppose, by induction, that the lemma holds in dimensions < n. Since Link(σ,L) satisfies the inductive hypothesis for each k-simplex σ, $k \leq n-3$, Link(σ,L) = S^{n-k-1}. Thus, L is a PL-manifold. Since $H_1(L; \mathbb{Z}/2) = 0$, it follows from the previous proposition, that L is isomorphic to the Coxeter complex U_M. Since L is finite, the associated Coxeter group W (= Aut(L,q)) must also be finite. Thus, U_M is a classical triangulation of S^n (cf. (2.22)).

THEOREM 4.7. Suppose that (L,q) is a symmetrically regular simplicial complex over Δ^n, that L is locally finite and that L satisfies (4.5.1). Then L is a PL-manifold. Consequently, L is equivalent to a classical triangulation of Y^n_ε/π for some $\varepsilon \in \{+1,0,-1\}$.

Proof. This follows from the previous lemma and (2.26).

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APPENDIX

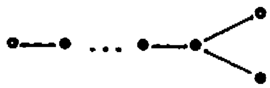
TABLE 1. Elliptic Coxeter Systems ($\varepsilon = +1$)

The Irreducible Diagrams

A_n

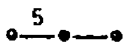
B_n

D_n



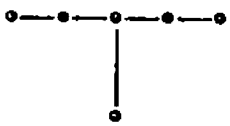
$I_2(p)$

H_3

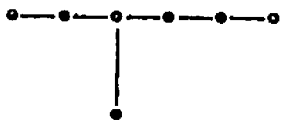


F_4

E_6



E_7



E_8

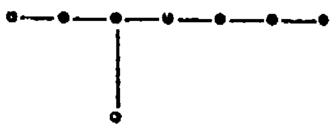


TABLE 2. Flat Coxeter Systems ($\varepsilon = 0$) with Fundamental Chamber an n-simplex

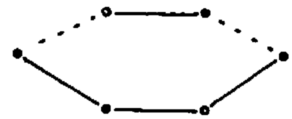
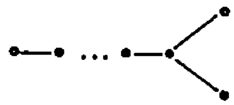
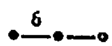
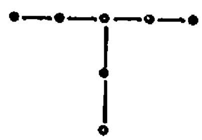
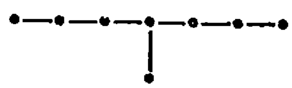
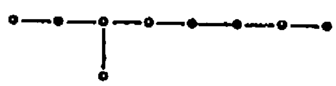
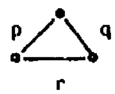
$\tilde{A}_n (n \geq 2)$	
$\tilde{B}_n (n \geq 3)$	
$\tilde{C}_n (n \geq 3)$	
$\tilde{D}_n (n \geq 4)$	
$\tilde{A}_1$	
$\tilde{B}_2$	
$\tilde{G}_2$	
$\tilde{F}_4$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	

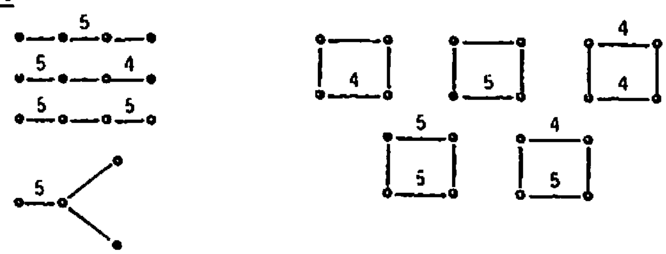
TABLE 3. Hyperbolic Coxeter Systems ($\varepsilon = -1$) with Fundamental Chamber an n-simplex

n = 2



with $(p^{-1} + q^{-1} + r^{-1}) < 1$

n = 3



n = 4

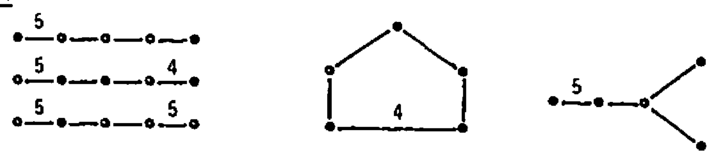


TABLE 4. The Classical Schläfli Symbols (only one is listed for each pair of dual tessellations).

n	ε = +1	ε = 0	ε = -1
1	(p)	(∞)	
2	(3,3) (4,3) (5,3)	(4,4) (6,3)	(p,q) , with $(p^{-1}+q^{-1}+2^{-1}) < 1$
3	(3,3,3) (4,3,3) (5,3,3) (3,4,3)	(4,3,4)	(3,5,3) (5,3,4) (5,3,5)
4	(3,3,3,3) (4,3,3,3)	(4,3,3,4) (3,4,3,3)	(5,3,3,3) (5,3,3,4) (5,3,3,5)
$n \geq 5$	(3,3,...,3) (4,3,...,3)	(4,3,...,3,4)	

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My lecture at the conference was on a different topic, namely, on the material in the two preprints, "The homology of a space on which a reflection group acts" and "Some aspherical manifolds".

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