# Vanishing results for the cohomology of complex toric hyperplane complements

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**Abstract**

Suppose $\mathcal{R}$ is the complement of an essential arrangement of toric hyperlanes in the complex torus $(\mathbb{C}^*)^n$ and $\pi = \pi_1(\mathcal{R})$. We show that $H^*(\mathcal{R}; A)$ vanishes except in the top degree $n$ when $A$ is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra $\mathcal{N}\pi$, or (c) the group ring $\mathbb{Z}\pi$. In case (a) the dimension of $H^n$ is $|e(\mathcal{R})|$ where $e(\mathcal{R})$ denotes the Euler characteristic, and in case (b) the $n^{\text{th}}$ $\ell^2$ Betti number is also $|e(\mathcal{R})|$.

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## 1 Introduction

A *complex toric arrangement* is a family of complex subtori of a complex torus $(\mathbb{C}^*)^n$. The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer [16, 17], Dimca-Lehrer [11], Douglass [12], Looijenga [18] and Macmeikan [20, 21].

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The foundation of the topic can be traced to the paper [10] by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of [10] is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see [9].)

The next step is the work of Moci, in particular his papers [22, 23, 24], developing the theory with a special focus on combinatorics. In [25] Moci and the second author study the homotopy type of the complement of a special class of toric arrangements which they call *thick*. In [3] D'Antonio and Delucchi generalize results in [25] to a wider class of toric arrangements which they call *complexified* because of structural affinity with the case of hyperplane arrangements. They also prove that complements of complexified toric arrangements are minimal (see [4]).

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement $M(\mathcal{A})$ of an arrangement $\mathcal{A}$ with coefficients in a complex local system $A$. Necessary conditions for $H^k(M(\mathcal{A}); A) = 0$ if $k \neq n$, i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [15], Esnault, Schechtman and Viehweg [13], Davis, Januszkiewicz and Leary [5], Schechtman, Terao and Varchenko [28] and Cohen and Orlik [2]. In particular, in [28] (see also [2]) it is proved that the cohomology of the complement $M(\mathcal{A})$ of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain *nonresonance* conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called *denses*.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary and Okun, [5, 6, 7, 8]. One considers an open cover of the complement $M$ by "small" open sets each homeomorphic to the complement of a central arrangement. In the cases of nonresonant rank one local coefficients or $\ell^2$ coefficients, the $E_1$ page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$ where $\Sigma$ is the union of all hyperplanes in the arrangement. (The simplicial complex $N(\mathcal{U})$ is the nerve of an open cover of $\mathbb{C}^n$ and $N(\mathcal{U}_{\mathrm{sing}})$ is a subcomplex.)

It follows that the $E_2$ page can be nonzero only in position $(l, 0)$. One also can prove that for an affine hyperplane arrangement of rank $l$ only the

$l^{\text{th}}$ $\ell^2$-Betti number of the complement $M$ can be nonzero and that it is equal to the rank of the reduced $(l-1)$-homology of $\Sigma$ (cf. [5]). Similarly, with coefficients in the group ring, $\mathbb{Z}\pi$, for $\pi = \pi_1(M)$, $H^*(M; \mathbb{Z}\pi)$ is nonzero only in degree $l$ (cf. [6]). We generalize all three of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [27], Papadima and Suciu generalize the result in [2] to arbitrary minimal CW-complex, i.e., a complex having as many $k$-cells as the $k$-th Betti number. It would be very interesting to decide if the complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by "small" convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system $A$ vanishes except in the top degree when $A$ is a nonresonant local system, the von Neumann algebra $\mathcal{N}\pi$ or the group ring $\mathbb{Z}\pi$.

# 2 Affine and toric hyperplane arrangements

**Affine hyperplanes arrangements** A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbb{C}^n$. A *subspace* of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its *rank*, $\mathrm{rk}(G)$, is the codimension of $G$ in $\mathbb{C}^n$. The minimal elements of $L(\mathcal{A})$ form a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement $\mathcal{A}$ is the rank of a minimal element in $L(\mathcal{A})$. $\mathcal{A}$ is *essential* if $\mathrm{rk}(\mathcal{A}) = n$.

The *singular set* $\Sigma(\mathcal{A})$ of $\mathcal{A}$ is the union of hyperplanes in $\mathcal{A}$. The complement of $\Sigma(\mathcal{A})$ in $\mathbb{C}^n$ is denoted $M(\mathcal{A})$.

**Toric arrangements** Let $T = (\mathbb{C}^*)^n$ be a complex torus and let $\Lambda = Hom(T, \mathbb{C}^*)$ denote the group of characters of $T$. Then $\Lambda \cong \mathbb{Z}^n$. A character is *primitive* if it is a primitive vector in $\Lambda$. Given a primitive character $\chi$ and an element $a \in \mathbb{C}^*$ put

$$H_{\chi,a} = \{t \in T \mid \chi(t) = a\}.$$

The subtorus $H_{\chi,a}$ is a *toric hyperplane*. A finite subset $X \subset \Lambda \times \mathbb{C}^*$ defines a *toric arrangement*,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a)\in X}$$

The projection of $X$ onto the first factor is denoted $p(X)$ and is called the *character set* of $\mathcal{T}_X$. (Thus, $p(X) := \{\chi \mid (\chi, a) \in X\}$.) The *singular set*, $\Sigma_X$, is the union of toric hyperplanes in the arrangement. Its complement, $T - \Sigma_X$, is denoted $\mathcal{R}_X$. The *intersection poset* $L_X$ is the set of nonempty intersections of toric hyperplanes and $\overline{L}_X = L_X \cup \{T\}$. $\overline{L}_X$ is partially ordered by inclusion. The *rank* of the arrangement is the dimension of the linear subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $p(X)$. The arrangement is *essential* if its rank is $n$.

Suppose $G \in L_X$. Choose a point $x \in G$. The *tangential arrangement along* $G$ is the arrangement $\mathcal{A}_G$ of linear hyperplanes which are tangent to the complex toric hyperplanes containing $G$ (i.e., all hyperplanes of the form $T_x(H_{\chi,a})$ where $T_x(G) \subset T_x(H_{\chi,a})$). It is a central hyperplane arrangement of rank equal to $n - \dim G$.

Given a toric arrangement $\mathcal{T}_X$ of rank $l$, let $K_X$ denote the identity component of the intersection of all kernels in $p(X)$, i.e., $K_X$ is the identity component of

$$\bigcap_{\chi\in p(X)} \operatorname{Ker}\chi = \{t \in T \mid \chi(t) = 1,\ \forall\chi \in p(X)\}.$$

Put $\overline{T}_X := T/K_X$. Thus, $K_X$ and $\overline{T}_X$ are tori of dimensions $n - l$ and $l$, respectively. ($K_X \cong (\mathbb{C}^*)^{n-l}$ and $\overline{T}_X \cong (\mathbb{C}^*)^l$.) Let $\overline{\Sigma}_X$ denote the image of $\Sigma_X$ in $\overline{T}_X$. Since $T \to T/K_X$ is a trivial $K_X$-bundle, we have a homeomorphism of pairs,

$$(T, \Sigma_X) \cong K_X \times (\overline{T}_X, \overline{\Sigma}_X). \tag{1}$$

In other words, the arrangement in $T$ is just the product of the arrangement in $\overline{T}_X$ with the torus $K_X$. We call $\overline{\mathcal{T}}_X$ the *essentialization* of $\mathcal{T}_X$. So, it is not restrictive to consider essential toric arrangements.

**Lemma 2.1.** (cf. [5, Prop. 2.1]). *Suppose $\mathcal{T}_X$ is an essential toric arrangement on $T$ and $\Sigma = \Sigma_X$. Then $H_*(T, \Sigma)$ is free abelian and concentrated in degree $n$.*

*Proof.* We follow the "deletion-restriction" argument in [5, Prop. 2.1]) using induction on $\operatorname{Card}(\mathcal{T}_X)$. Choose a toric hyperplane $H \in \mathcal{T}_X$. Let $\mathcal{T}' = \mathcal{T}_X -$

$\{H\}$ and let $\mathcal{T}''$ be the restriction of $\mathcal{T}_X$ to $H$, i.e., $\mathcal{T}'' = \{H \cap H' \mid H' \in \mathcal{T}_X\}$. Let $\Sigma'$ and $\Sigma''$ denote the singular sets of $\mathcal{T}'$ and $\mathcal{T}''$, respectively. Consider the exact sequence of the triple $(T, \Sigma, \Sigma')$,

$$\to H_*(T, \Sigma') \to H_*(T, \Sigma) \to H_{*-1}(\Sigma, \Sigma') \to \tag{2}$$

There is an excision, $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma'')$. The rank of $\mathcal{T}'$ is either $n$ or $n-1$, while the rank of $\mathcal{T}''$ is always $n-1$. The argument breaks into two cases depending on the rank of $\mathcal{T}'$.

*Case 1*: the rank of $\mathcal{T}'$ is $n$. By induction, $H_*(T, \Sigma')$ and $H_*(H, H \cap \Sigma)$ are free abelian and concentrated in degrees $n$ and $n-1$, respectively. So, (2) becomes,

$$0 \to H_n(T, \Sigma') \to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma') \to 0$$

and all other terms are 0. Therefore, $H_*(T, \Sigma)$ is concentrated in degree $n$ and $H_n(T, \Sigma)$ is free abelian.

*Case 2*: the rank of $\mathcal{T}'$ is $n-1$. Then the projection $T \to \overline{T}$ takes $H$ isomorphically onto $\overline{T}$ and the arrangement $\mathcal{T}''$ on $H$ maps isomorphically to the arrangement $\overline{\mathcal{T}}_X$ on $\overline{T}$. So, $(H, H \cap \Sigma') \cong (\overline{T}, \overline{\Sigma})$. By (1), $(T, \Sigma') \cong K_X \times (H, H \cap \Sigma') \cong \mathbb{C}^* \times (H, H \cap \Sigma')$. By the Künneth Formula, $H_*(T, \Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H \cap \Sigma')$. So,

$$\begin{aligned} H_{n-1}(T, \Sigma') &\cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \quad \text{and} \\ H_n(T, \Sigma') &\cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma'); \end{aligned}$$

moreover, the first isomorphism is induced by the inclusion $(H, H \cap \Sigma') \to (T, \Sigma')$. So, (2) becomes,

$$\begin{aligned} 0 \to H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') &\to H_n(T, \Sigma) \to H_{n-1}(H, H \cap \Sigma') \\ &\to H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \end{aligned}$$

where the last map is an isomorphism. It follows that $H_{n-1}(T, \Sigma) = 0$ and that $H_n(T, \Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$, which, by inductive hypothesis, is free abelian. This proves the lemma. $\square$

**Complexified toric arrangements** In [3] D'Antonio-Delucchi consider the case of "complexified toric arrangements." This means that for each $(\chi, a) \in X$, the complex number $a$ has modulus 1 (where $X \subset \Lambda \times \mathbb{C}^*$ is a set defining a toric arrangement $\mathcal{T}_X$). Let $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$ be the compact torus. Then for each $H \in \mathcal{T}_X$, $H \cap T^{\text{cpt}}$ is a compact subtorus of $T^{\text{cpt}}$. The set of subtori, $\mathcal{T}_X^{\text{cpt}} := \{H \cap \mathcal{S} \mid H \in \mathcal{T}\}$, is called the associated *compact arrangement*.

Let $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$. We note that $(T, \Sigma_X)$ deformation retracts onto $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$. Here are a few obervations.

(i) The universal cover of $T^{\text{cpt}}$ is $\mathbb{R}^n$ (actually the subspace $i\mathbb{R}^n \subset \mathbb{C}^n$). Let $\pi : \mathbb{R}^n \to T^{\text{cpt}}$ be the covering projection. Then for each $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$, each component of $\pi^{-1}(H^{\text{cpt}})$ is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in $\mathbb{R}^n$.

(ii) If $\mathcal{T}_X$ is essential, then $\Sigma^{\text{cpt}}$ cuts $T^{\text{cpt}}$ into a disjoint union of convex polytopes, called *chambers* (see [25]) . The inverse images of these polytopes under $\pi$ give a tiling of $\mathbb{R}^n$.

(iii) When $\mathcal{T}_X$ is essential, it follows from (ii) that for $n \geq 2$, $\Sigma^{\text{cpt}}$ is connected and that for $n \geq 3$, $\pi_1(\Sigma^{\text{cpt}}) = \pi_1(T^{\text{cpt}})$.

(iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}}) \cong H_*(\coprod(P_i, \partial P_i))$ where each chamber $P_i$ is an $n$-dimensional convex polytope. Hence, $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is concentrated in degree $n$ and is free abelian. Moreover, the rank of $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is the number of chambers.

(v) Let $\widetilde{\Sigma}^{\text{cpt}}$ denote the inverse image of $\Sigma^{\text{cpt}}$ in $\mathbb{R}^n$ and let $\widetilde{\Sigma}_X$ be the induced cover of $\Sigma_X$. Suppose $\mathcal{T}_X$ is essential. Then $\widetilde{\Sigma}^{\text{cpt}}$ cuts $\mathbb{R}^n$ into compact chambers. It follows that $\widetilde{\Sigma}^{\text{cpt}}$ (and hence, $\widetilde{\Sigma}$) is homotopy equivalent to a wedge of $(n-1)$-spheres.

# 3 Certain covers and their nerves

Equip the torus $T = (\mathbb{C}^*)^n$ with an invariant metric. This lifts to a Euclidean metric on $\mathbb{C}^n$ induced from an inner product. Hence, geodesics in $T$ lift to straight lines in $\mathbb{C}^n$ and each component of the inverse image of a subtorus of $T$ is an affine subspace of $\mathbb{C}^n$. A *convex subset* of $T$ means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of $T$ is a convex subset of $\mathbb{C}^n$.

The intersection of an open convex subset of $T$ with the toric hyperplanes in $\mathcal{T}_X$ is equivalent to an affine arrangement. An open convex subset $U \subset T$ is *small* (with respect to $\mathcal{T}_X$) if this affine arrangement is central. In other words, $U$ is *small* if the following two conditions hold (cf. [5, 6]):

(i) $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\mathrm{Min}(U)$.

(ii) A toric hyperplane $H \in \mathcal{T}_X$ has nonempty intersection with $U$ if and only if $\mathrm{Min}(U) \subset H$.

If (i) and (ii) hold, then the arrangement in $U$ is equivalent to the tangential arrangement along $\mathrm{Min}(U)$, which we denote by $\mathcal{A}_{\mathrm{Min}(U)}$. The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $T$ by small convex sets, put

$$\mathcal{U}_{\mathrm{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma_X \neq \emptyset\}.$$

Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. The *nerve* $N(\mathcal{U})$ of $\mathcal{U}$ is the simplicial complex defined as follows. Its vertex set is $I$ and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if $U_\sigma$ is nonempty. We have the following lemma.

**Lemma 3.1.** *Suppose $\mathcal{T}_X$ is essential. $N(\mathcal{U})$ is homotopy equivalent to $T$ and $N(\mathcal{U}_{\mathrm{sing}})$ is a subcomplex homotopy equivalent to $\Sigma_X$. Moreover, $H_*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$ is concentrated in degree $n$ and $H_n(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$ is free abelian.*

*Proof.* $\mathcal{U}_{\mathrm{sing}}$ is an open cover of a neighborhood of $\Sigma_X$ which deformation retracts onto $\Sigma_X$. For each simplex $\sigma$ of $N(\mathcal{U})$, $U_\sigma$ is contractible (in fact, it is a small convex open set). By a well-known result (see [14, Cor. 4G.3 and Ex. 4G(4)]) $N(\mathcal{U})$ is homotopy equivalent to $T$ and $N(\mathcal{U}_{\mathrm{sing}})$ is homotopy equivalent to $\Sigma_X$. The last sentence of the lemma follows from Lemma 2.1. □

**Definition 3.2.** $\beta(\mathcal{T}_X)$ is the rank of $H_n(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$.

Equivalently, $\beta(\mathcal{T}_X)$ is the rank of $H_n(T, \Sigma_X)$. It is not difficult to see that, for essential arrangements, $(-1)^n \beta(\mathcal{T}_X) = e(T, \Sigma_X) = -e(\Sigma_X) = e(\mathcal{R}_X)$, where $e(\ )$ denotes Euler characteristic.

# 4 Local coefficients

**Generic and nonresonant coefficients** Consider an affine arrangement $\mathcal{A}$ The fundamental group $\pi$ of its complement, $M(\mathcal{A})$, is generated by loops $a_H$ for $H \in \mathcal{A}$, where the loop $a_H$ goes once around the hyperplane $H$ in the "positive" direction. Let $\alpha_H$ denote the image of $a_H$ in $H_1(M(\mathcal{A}))$. Then $H_1(M(\mathcal{A}))$ is free abelian with basis $\{\alpha_H\}_{H\in\mathcal{A}}$. So, a homomorphism $H_1(M(\mathcal{A})) \to \mathbb{C}^*$ is determined by an $\mathcal{A}$-tuple $\Lambda \in (\mathbb{C}^*)^{\mathcal{A}}$, where $\Lambda = (\lambda_H)_{H\in\mathcal{A}}$ corresponds to the homomorphism sending $\alpha_H$ to $\lambda_H$. Let $\psi_\Lambda : \pi \to \mathbb{C}^*$ be the composition of this homomorphism with the abelianization map $\pi \to H_1(M(\mathcal{A}))$. The resulting rank one local coefficient system on $M(\mathcal{A})$ is denoted $A_\Lambda$.

Returning to the case where $\mathcal{T}_X$ is a toric arrangement, for each simplex $\sigma$ in $N(\widehat{\mathcal{U}})$, let $\mathcal{A}_\sigma := \mathcal{A}_{\mathrm{Min}(U_\sigma)}$ be the corresponding central arrangement (so that $\widehat{U}_\sigma \cong M(\mathcal{A}_\sigma)$). Given $\Lambda_\sigma \in (\mathbb{C}^*)^{\mathcal{A}_\sigma}$, put

$$\lambda_\sigma := \prod_{H\in\mathcal{A}_\sigma} \lambda_H.$$

Let $A_{\Lambda_T} \in \mathrm{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*)$ be a local coefficient system on $\mathcal{R}_X$. The localization of $A_{\Lambda_T}$ on the open set $\widehat{U}_\sigma$ has the form $A_{\Lambda_\sigma}$, where $\Lambda_\sigma$ is a $\mathcal{A}_\sigma$-tuple in $\mathbb{C}^*$. We call $\Lambda_T$ *generic* if $\lambda_\sigma \neq 1$ for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$. We call $\Lambda_T$ *nonresonant* if $\Lambda_\sigma$ is nonresonant in the sense of [2] for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$ i.e., if the Betti numbers of $M(\mathcal{A}_\sigma)$ with coefficients in $A_{\Lambda_\sigma}$ are minimal.

**$\ell^2$-cohomology and coeffiicients in a group von Neumann algebra** For a discrete group $\pi$, $\ell^2\pi$ denotes the Hilbert space of complex-valued, square integrable functions on $\pi$. There are unitary $\pi$-actions on $\ell^2\pi$ by either left or right multiplication; hence, $\mathbb{C}\pi$ acts either from the left or right as an algebra of operators. The *associated von Neumann algebra* $\mathcal{N}\pi$ is the commutant of $\mathbb{C}\pi$ (acting from, say, the right on $\ell^2\pi$).

Given a finite CW complex $Y$ with fundamental group $\pi$, the space of $\ell^2$-cochains on the universal cover $\widetilde{Y}$ is equal to $C^*(Y; \ell^2\pi)$, the cochains with local coefficients in $\ell^2\pi$. The image of the coboundary map need not be closed; hence, $H^*(Y; \ell^2\pi)$ need not be a Hilbert space. To remedy this, one defines the *reduced* $\ell^2$-cohomology $H^*_{\mathrm{red}}(Y; \ell^2\pi)$ to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a "dimension,"

$\dim_{\mathcal{N}\pi} V$, to any closed, $\pi$-stable subspace $V$ of a finite direct sum of copies of $\ell^2\pi$ (it is the trace of orthogonal projection onto $V$). The nonnegative real number $\dim_{\mathcal{N}\pi}(H^p_{\mathrm{red}}(Y;\ell^2\pi))$ is the $p^{\mathrm{th}}$ *$\ell^2$-Betti number* of $Y$.

A technical advance of Lück [19, Ch. 6] is the use local coefficients in $\mathcal{N}\pi$ in place of the previous version of $\ell^2$-cohomology. He shows there is a well-defined dimension function on $\mathcal{N}\pi$-modules, $A \to \dim_{\mathcal{N}\pi} A$, which gives the same answer for $\ell^2$-Betti numbers, i.e., for each $p$ one has that $\dim_{\mathcal{N}\pi} H^p(Y;\mathcal{N}\pi) = \dim_{\mathcal{N}\pi} H^p_{\mathrm{red}}(Y;\ell^2\pi)$.

**Group ring coeffiicients** Let $Y$ be a connected CW complex, $\pi = \pi_1(Y)$ and $r : \widetilde{Y} \to Y$ the universal cover. There is a well-defined action of $\pi$ on $\widetilde{Y}$ and hence, on the cellular chain complex of $\widetilde{Y}$. Given the left $\pi$-module $\mathbb{Z}\pi$, define the cochain complex with *group ring coefficients*

$$C^*(Y;\mathbb{Z}\pi) := \mathrm{Hom}_\pi(C_*(\widetilde{Y}), \mathbb{Z}\pi).$$

Taking cohomology gives $H^*(Y;\mathbb{Z}\pi)$.

# 5 The Mayer-Vietoris spectral sequence

**Statements of the main theorems** Suppose $\mathcal{T}_X$ is an essential toric arrangement in $T$ and $\pi = \pi_1(\mathcal{R}_X)$.

**Theorem 5.1.** *Let $\Lambda_T$ be a generic $X$-tuple with entries in $k^*$. Then $H^*(\mathcal{R}_X; A_{\Lambda_T})$ is concentrated in degree $n$ and*

$$\dim_k H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X).$$

**Theorem 5.2.** (cf. [7]). *The $\ell^2$-Betti numbers of $\mathcal{R}_X$ are $0$ except in degree $n$ and $\ell^2 b_n(\mathcal{R}_X) = \beta(\mathcal{T}_X)$.*

**Theorem 5.3.** (cf. [6, 8]). *$H^*(\mathcal{R}_X;\mathbb{Z}\pi)$ vanishes except in degree $n$ and $H^n(\mathcal{R}_X;\mathbb{Z}\pi)$ is free abelian.*

*Remark* 5.4. Suppose $W$ is a Euclidean reflection group acting on $\mathbb{R}^n$ and that $\mathbb{Z}^n \subset W$ is the subgroup of translations. The quotient $W' := W/\mathbb{Z}^n$ is a finite Coxeter group. The reflection group $W$ acts on the complexification $\mathbb{C}^n$ and $W'$ acts on the torus $T = \mathbb{C}^n/\mathbb{Z}^n$. The image of the affine reflection arrangement in $\mathbb{C}^n$ gives a toric arrangement $\mathcal{T}_X$ in $T$. The fundamental group of $\mathcal{R}_X$ is the Artin group $A$ associated to $W$ and $\mathcal{R}_X$ is the Salvetti complex associated to $A$. The quotient of the compact torus by $W'$ can be identified with the fundamental simplex $\Delta$ of $W$ on $\mathbb{R}^n$. (If $W$ is irreducible, then $\Delta$ is a simplex.) It follows that $\beta(\mathcal{T}_x)$ is the order of $W'$ (i.e., the index of $\mathbb{Z}^n$ in $W$). So, in this case Theorem 5.2 is a special case of the main result of [7] and Theorem 5.3 is a special case of a result of [8, Thm. 4.1].

**Lemma 5.5.** *Suppose $\mathcal{A}$ is a finite, central arrangement of affine hyperplanes. Let $\pi' = \pi_1(M(\mathcal{A}))$. Then*

(i) (cf. [28, 2, 5]). *For any generic system of local coefficients $A$, $H^*(M(\mathcal{A}); A)$ vanishes in all degrees.*

(ii) (cf. [5]). *$H^*(M(\mathcal{A}); \mathcal{N}\pi')$ vanishes in all degrees. Hence, all $\ell^2$-Betti numbers are $0$.*

(iii) (cf. [6]). *If the rank of $\mathcal{A}$ is $l$, then $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$ vanishes except in the top degree, $l$.*

**Proofs using the Mayer-Vietoris spectral sequence** The proofs of these three theorems closely follow the argument in [7], [5] and particularly, in [6]. For $\pi = \pi_1(\mathcal{R}_X)$, let $A$ denote one of the left $\pi$-modules in Section 4.

Let $\mathcal{U} = \{U_i\}$ be an open cover of $T$ by small convex sets. We may suppose that $\mathcal{U}$ is finite and that it is closed under taking intersections. For each $G \in \overline{L}_X$, put

$$\begin{aligned}\mathcal{U}_G &:= \{U \in \mathcal{U} \mid \mathrm{Min}(U) \leq G\},\\ \mathcal{U}_G^{\mathrm{sing}} &:= \{U \in \mathcal{U} \mid \mathrm{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.\end{aligned}$$

The open cover $\mathcal{U}$ restricts to an open cover $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U\in\mathcal{U}}$ of $\mathcal{R}_X$. Any element $\widehat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_{\mathrm{Min}(U)})$.

Suppose $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ and $N(\mathcal{U}_G)$ is the subcomplex defined by $\mathcal{U}_G$. Since $N(\mathcal{U}_G)$ and $N(\mathcal{U}_G^{\mathrm{sing}})$ are nerves of covers of $G$ and $\Sigma_X \cap G$, respectively, by contractible open subsets, we have that for each $G \in \overline{L}(\mathcal{A})$,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}})) = H^*(G, \Sigma(\mathcal{T}_X \cap G)). \tag{3}$$

For each $k$-simplex $\sigma = \{i_0, \dots, i_k\}$ in $N(\mathcal{U})$, let

$$U_\sigma := U_{i_0} \cap \cdots \cap U_{i_k}$$

denote the corresponding intersection.

Let $r : \widetilde{\mathcal{R}}_X \to \mathcal{R}_X$ be the universal cover. The induced open cover $\{r^{-1}(\widehat{U})\}$ of $\widetilde{\mathcal{R}}_X$ has the same nerve $N(\widehat{\mathcal{U}})$ $(= N(\mathcal{U}))$. We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of $i$-simplices in $N(\mathcal{U})$ (cf. [1, Ch. VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \mathrm{Hom}_\pi(C_{i,j}, A), \tag{4}$$

where $\pi = \pi_1(\mathcal{R}_X)$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\mathrm{Gr}\, H^m(\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

By first using the horizontal differential, there is a spectral sequence with $E_1$ page

$$E_1^{i,j} = C^i(N(\mathcal{U}); \mathcal{H}^j(A))$$

where $\mathcal{H}^j(A)$ is the coefficient system on $N(\mathcal{U})$ defined by

$$\sigma \mapsto H^j(\widehat{\mathcal{U}}_\sigma; A),$$

where $\widehat{\mathcal{U}}_\sigma \cong M(\mathcal{A}_{\mathrm{Min}(U_\sigma)})$. For $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$ these coefficients are 0 for $G \neq T$. For $A = \mathbb{Z}\pi$, they are 0 for $j \neq \dim(G)$. Hence, in all cases, for any coface $\sigma'$ of $\sigma$, if $G' := \mathrm{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j(M(\mathcal{A}_G); A) \to H^j(M(\mathcal{A}_{G'}); A)$ is the zero map. Moreover, the $E_1$ page of the spectral sequence decomposes as a direct sum (cf. [8, Lemma 2.2]). In fact, for a fixed $j$, by using Lemma 5.5, we see that the $E_1^{i,j}$ term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \overline{L}_X^{n-j}} C^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); A)),$$

where we have constant coefficients in each summand. Hence, at $E_2$ we have

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{G\in \overline{L}_X^{n-j}} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); A)) \\ &= \bigoplus_{G\in \overline{L}_X^{n-j}} H^i(G, \Sigma_X \cap G; H^j(M(\mathcal{A}_G); A)), \end{aligned} \tag{5}$$

where the second equation follows from (3).

When $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$, all summands vanish for $G \neq T$ and $j \neq 0$. So, we are left with $E_2^{n,0} = H^n(T, \Sigma_X; A)$, which is isomorphic to the tensor product free abelian group of rank $\beta(\mathcal{T}_X)$ with $A$. It follows that $H^*(\mathcal{R}_X; A)$ is concentrated in degree $n$ and that $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X) = \dim_{\mathcal{N}\pi} H^n(\mathcal{R}_X; \mathcal{N}\pi)$. This proves Theorems 5.1 and 5.2.

Consider formula (5) for $A = \mathbb{Z}\pi$. By Lemma 2.1, $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E_2^{i,j}$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at $E_2$, i.e., $E_2 = E_\infty$. This proves Theorem 5.3.

*Remark* 5.6. Let us remark that the statement of Theorem 5.1 holds even if the local system $\Lambda_T$ is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets , i.e. $\Lambda_\sigma$ verifies the nonresonance conditions in [28] for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$. Indeed under these conditions Lemma 5.5 holds.

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