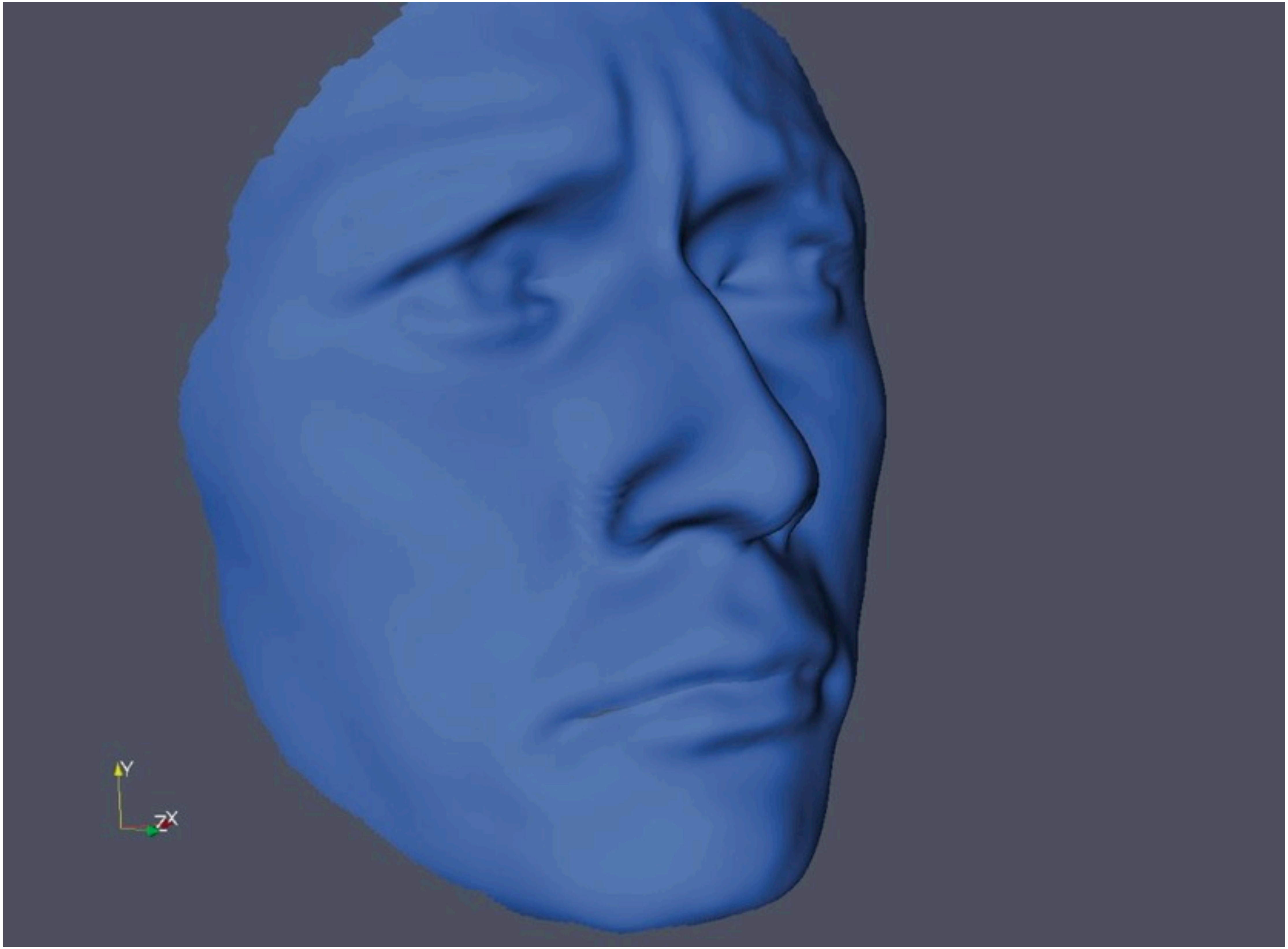
Quick review of other methods for shape matching

- Shape distributions
- Shape contexts
- Hamza-Krim
- Boutin-Kemper

Assignment: write a 1/2 page summary of each of those approaches. Papers are posted on course webpage. Due Monday Feb. 24th.


Y
Z
X

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Shape Distributions [Osada-et-al]

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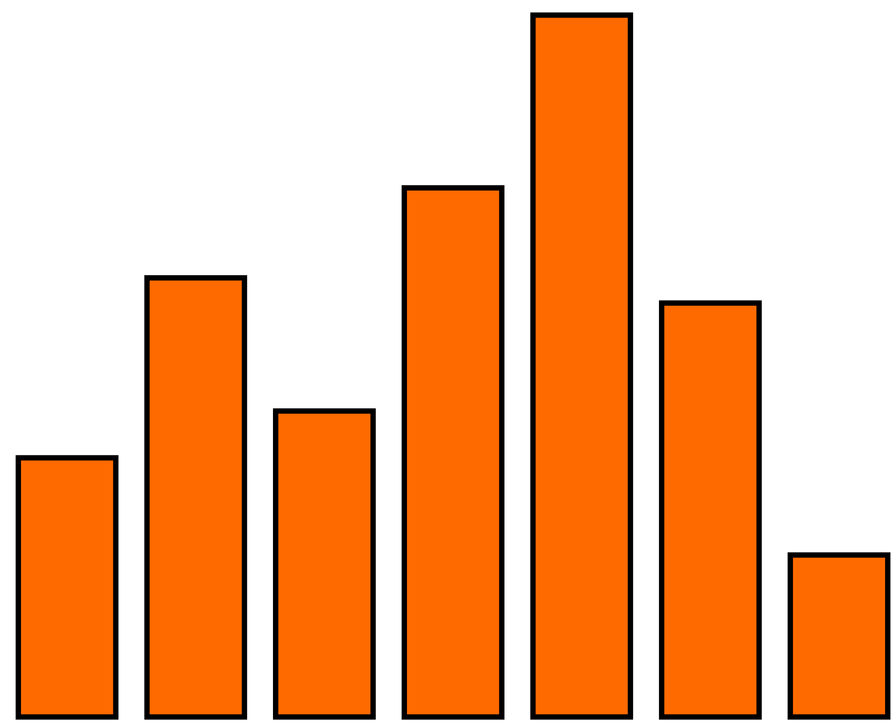

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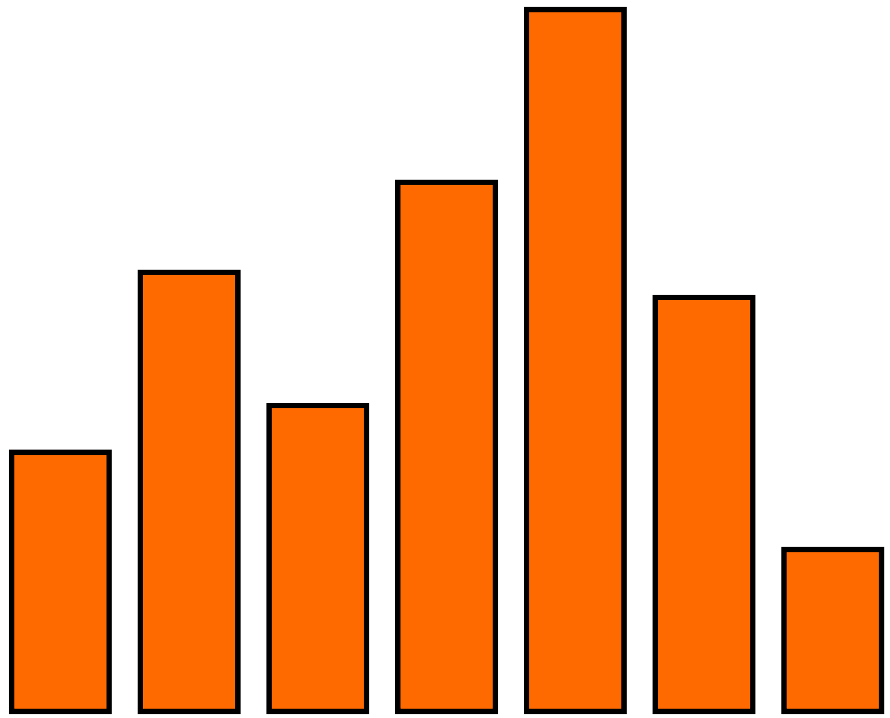

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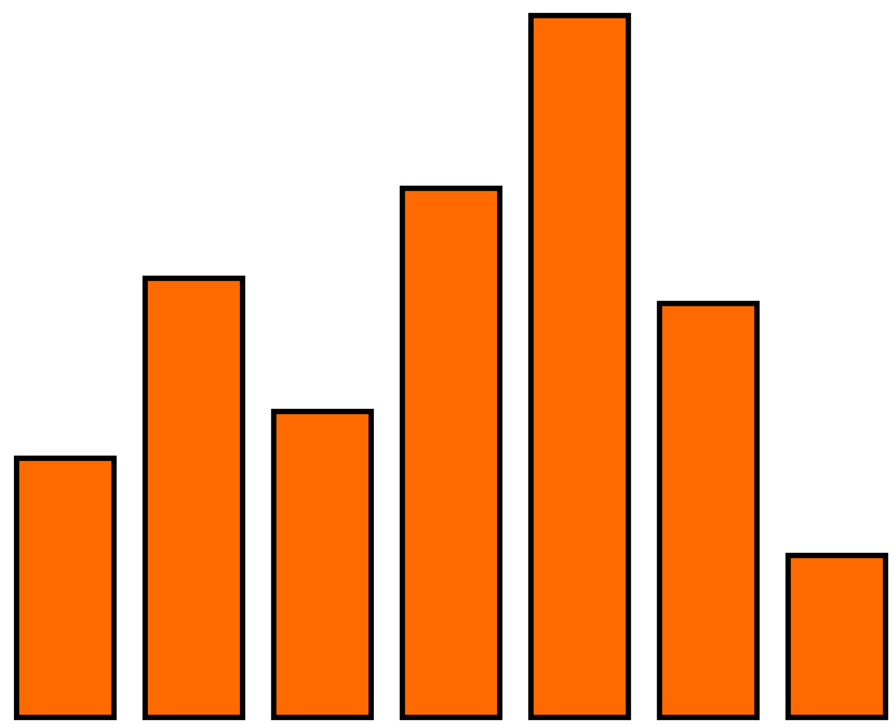
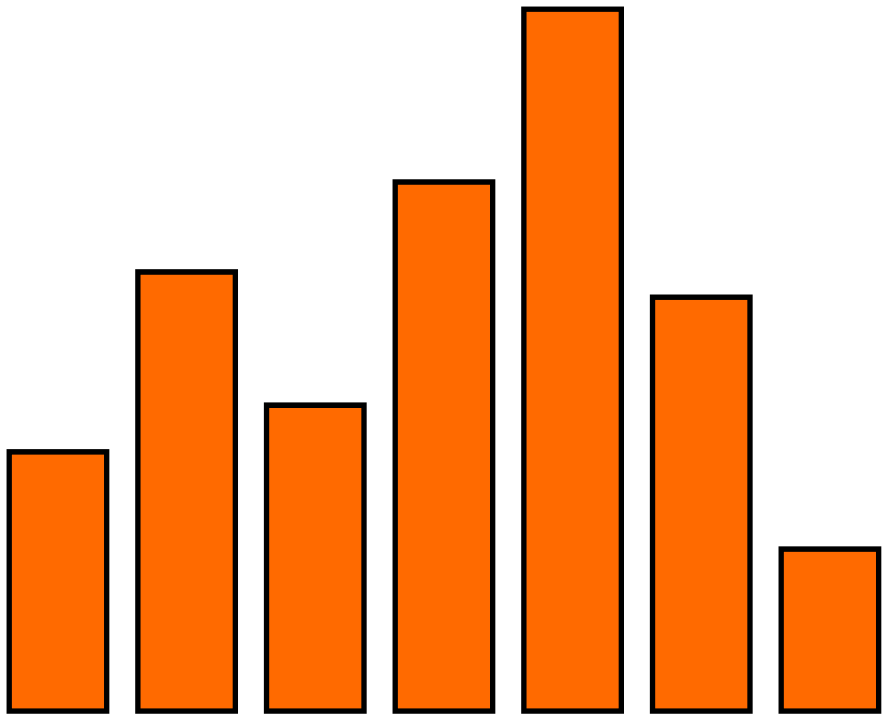
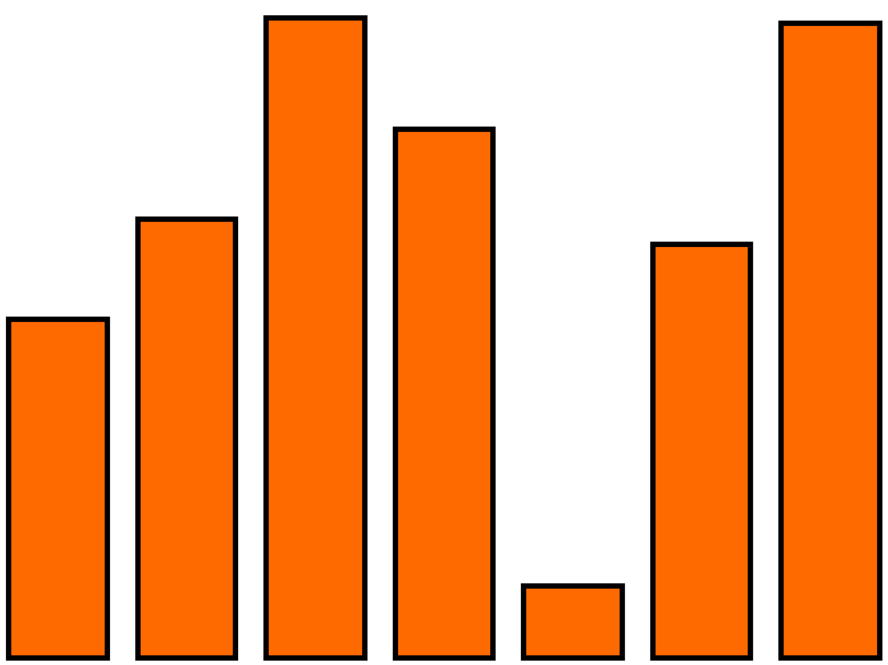
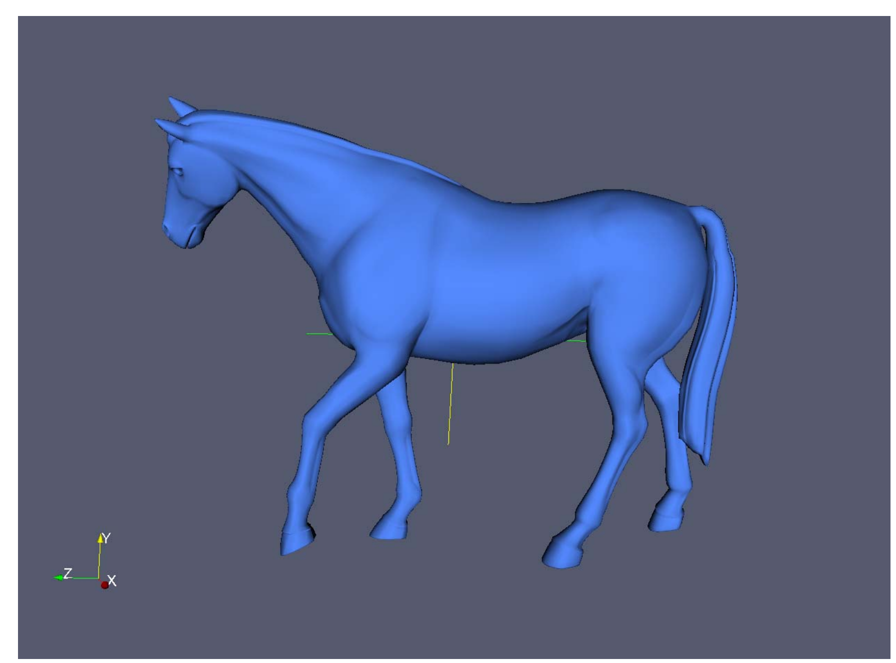
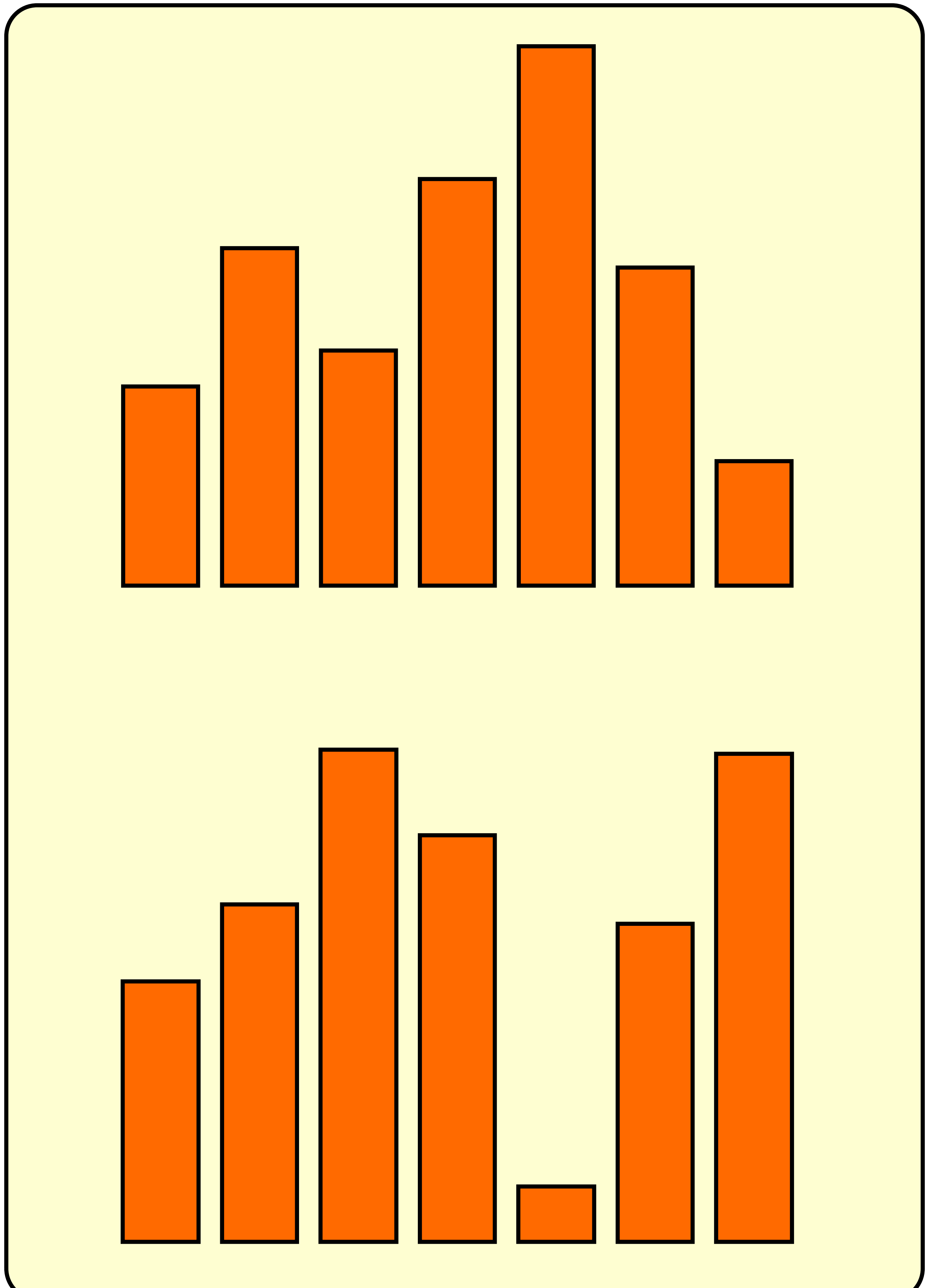
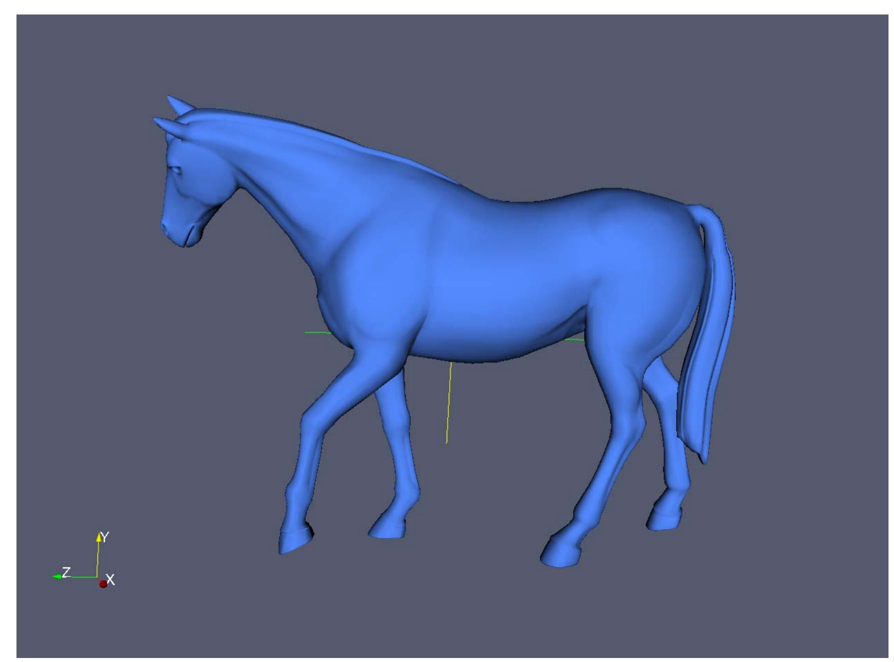






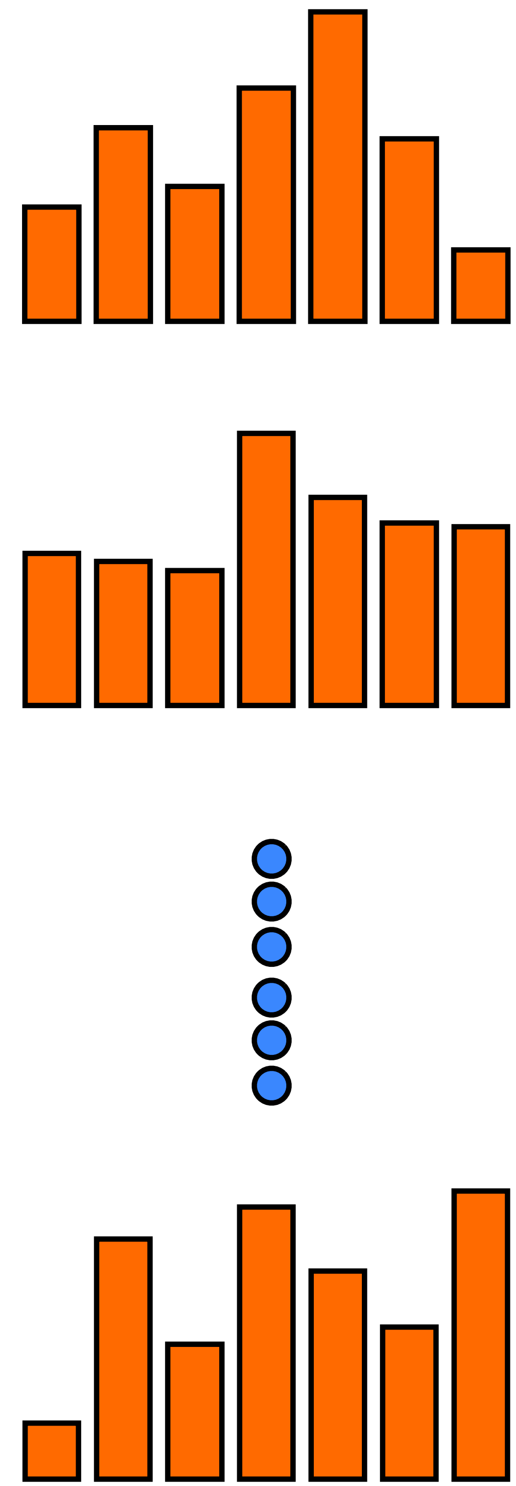






Shape Contexts

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



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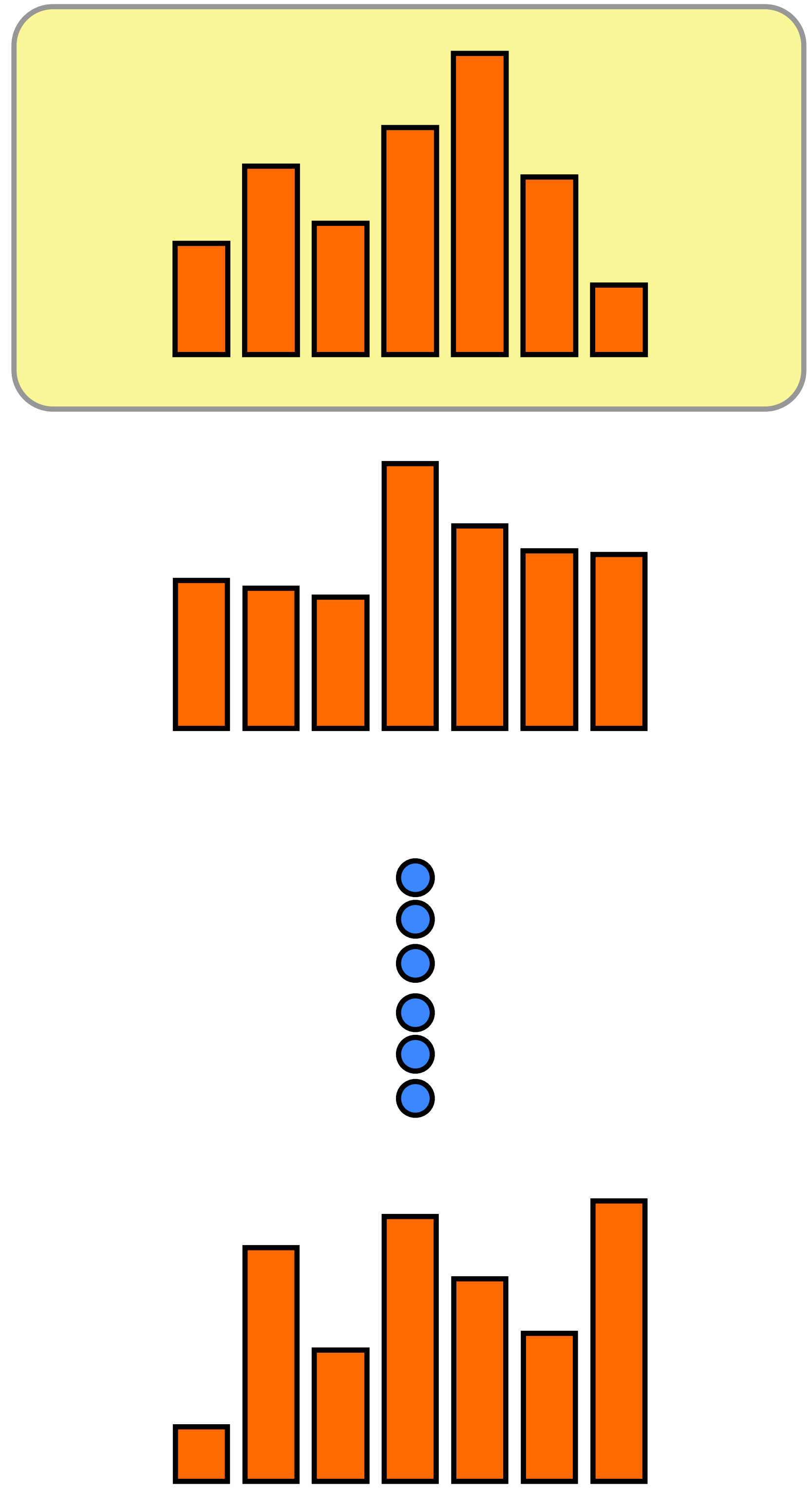

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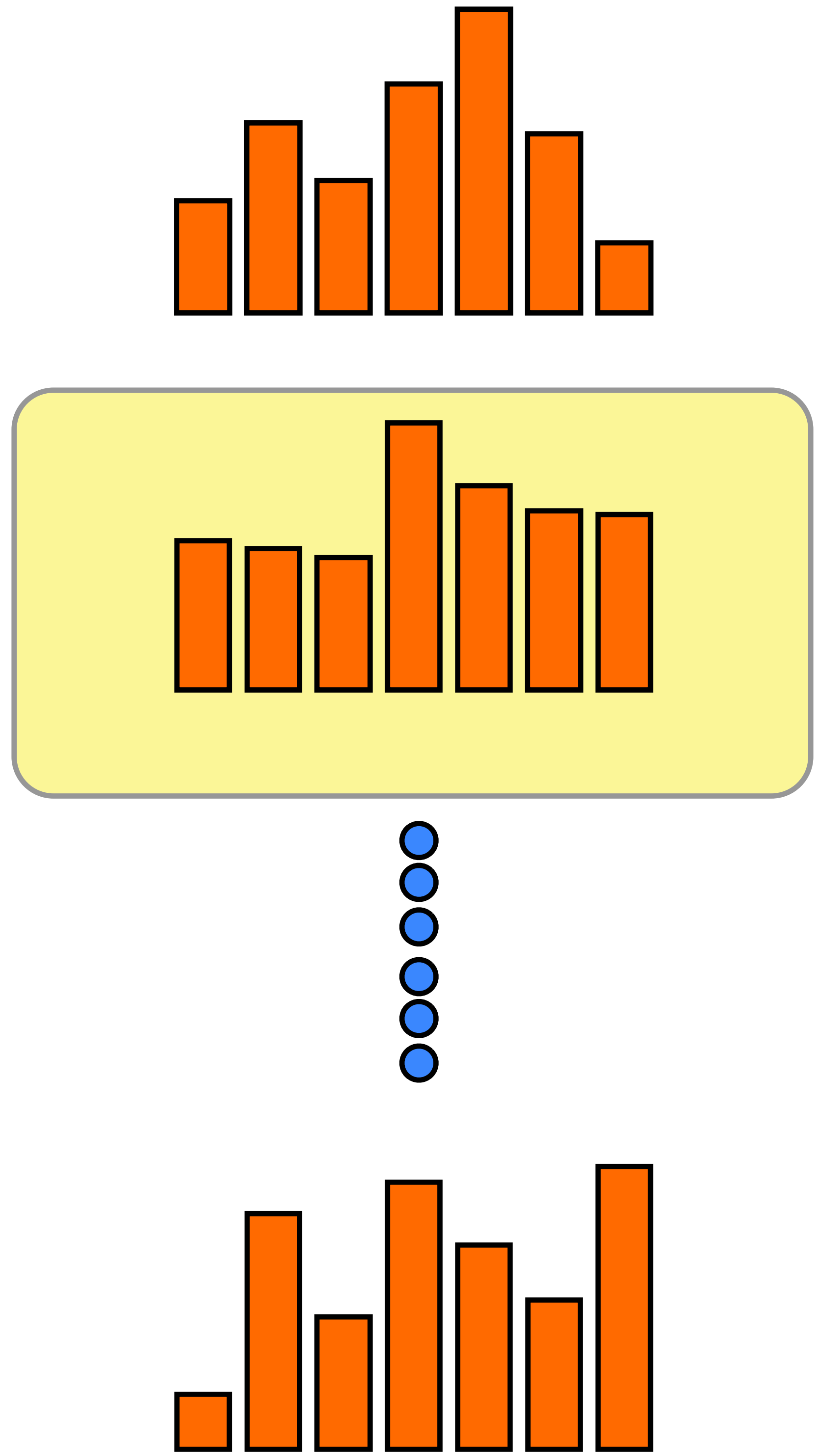

Eccentricities (Hamza-Krim)

$$\frac{\sum_j d_{1,j}}{N}$$

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$$\vdots$$

$$\frac{\sum_j d_{N,j}}{N}$$

Let's go back to formulation: from hard objects to soft objects

Theoretical consequences: from sets to probability measures.

Practical consequences: from combinatorial optimization to continuous optimization

- **Hausdorff distance**. For (compact) subsets A, B of a (compact) metric space (Z, d), the *Hausdorff distance* between them, $d_{\mathcal{H}}^{Z}(A, B)$, is defined to be the infimal $\varepsilon > 0$ s.t.

$$A \subset B^{\varepsilon} \text{ and } B \subset A^{\varepsilon}$$

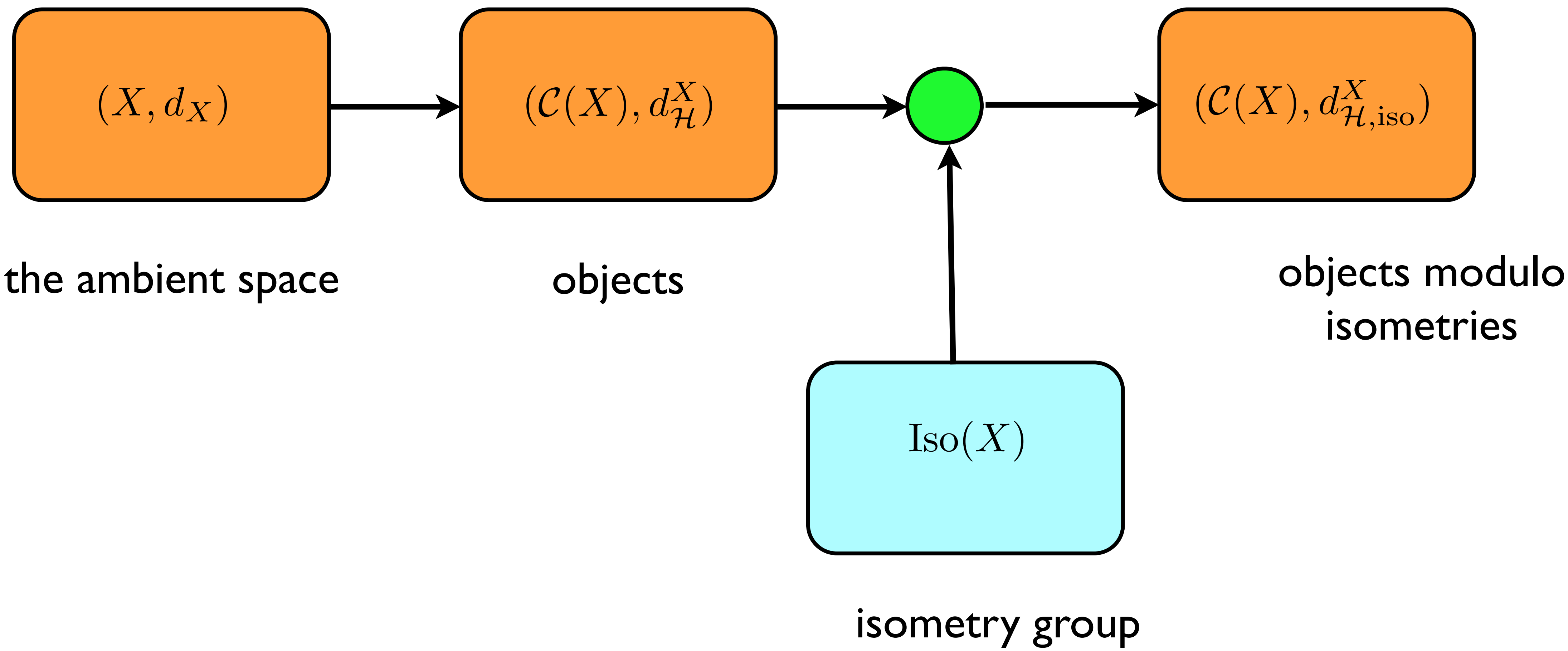
Equivalently,

$$d_{\mathcal{H}}^{Z}(A, B) = \max(\max_{b \in B} \min_{a \in A} d(a, b), \max_{a \in A} \min_{b \in B} d(a, b)).$$

For a subset A of a metric space (X, d) we will use the notation $d(x, A) := \inf_{a \in A} d(x, a)$.

Theorem ([**BBI**], Proposition 7.7.3)**.** *The Hausdorff distance is a* metric *on the set of all objects (i.e.* compact *subsets) of* X, $\mathcal{C}(X)$.

The usual setup for **Extrinsic** shape matching



Theorem ([**BBI**], Blaschke's theorem)**.** *If* (X, d_X) *is compact, then* $(\mathcal{C}(X), d_{\mathcal{H}}^X)$ *is also compact.*

correspondences

Definition [Correspondences]

For sets A and B, a subset $R \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\ a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\ b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B.

Remark. *Note that* $\mathcal{R}(A, B) \neq \emptyset$. *Indeed,* $A \times B$ *is always in* $\mathcal{R}(A, B)$.

correspondences

Note that when A and B are finite, $R \in \mathcal{R}(A, B)$ can be represented by a matrix $((r_{a,b})) \in \{0,1\}^{n_A \times n_B}$ s.t.

$$\sum_{a \in A} r_{ab} \geq 1 \;\; \forall b \in B$$

$$\sum_{b \in B} r_{ab} \geq 1 \;\; \forall a \in A$$

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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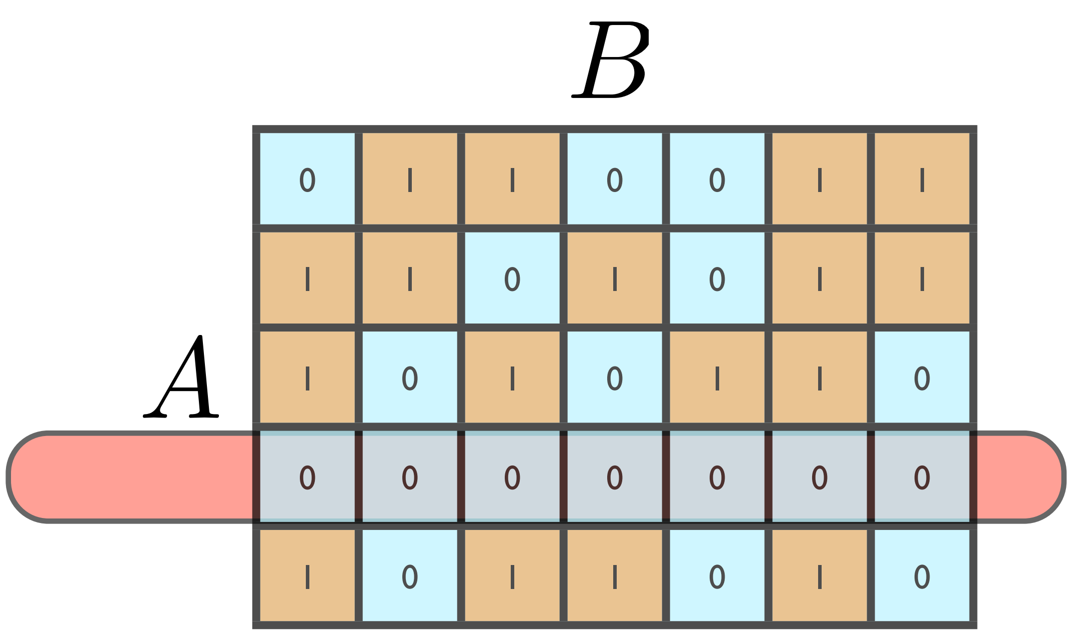

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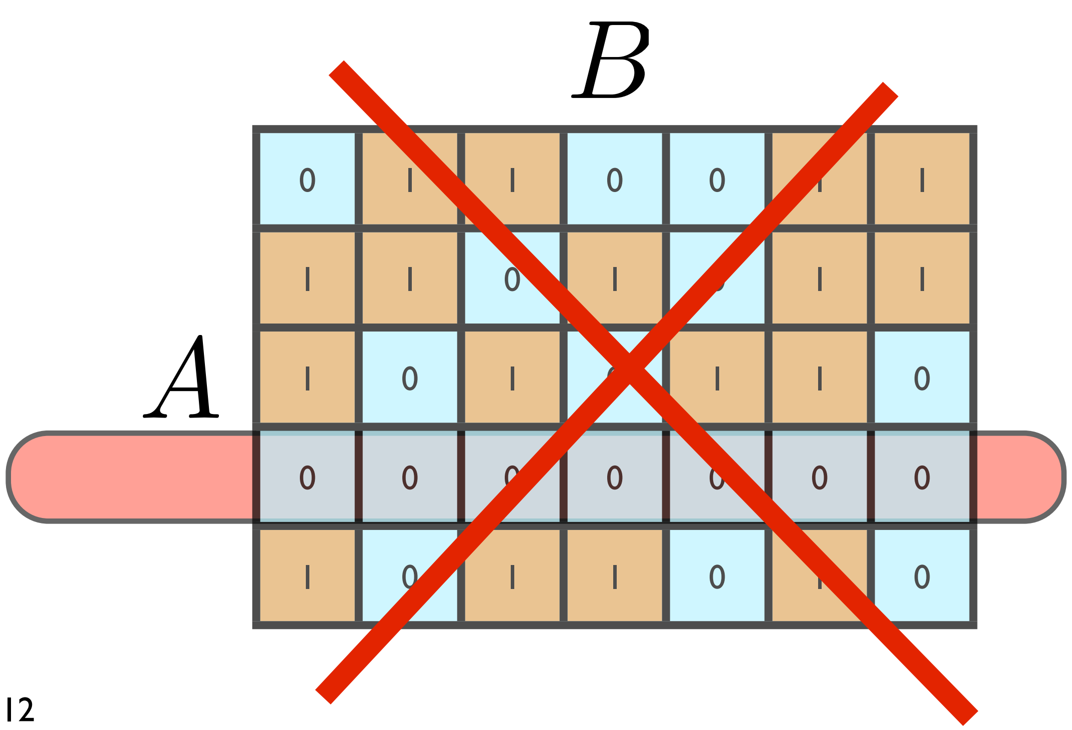

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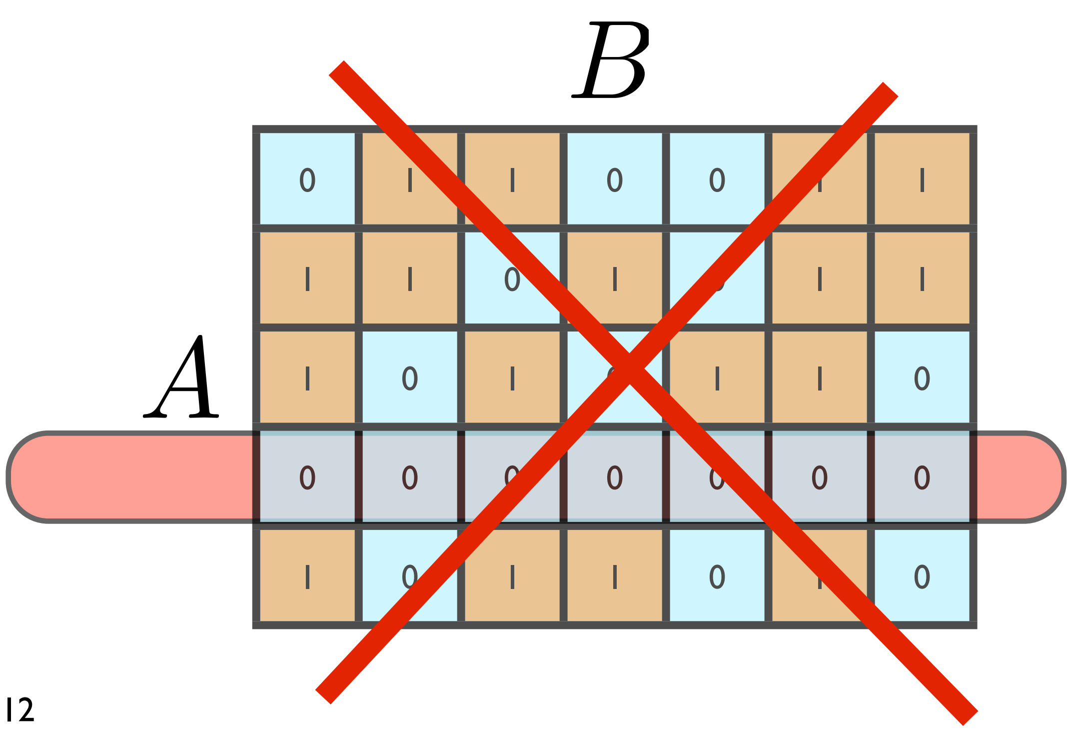

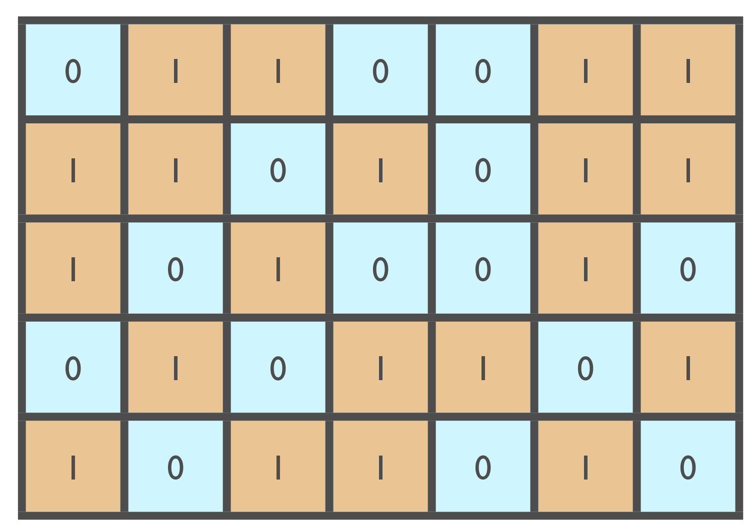

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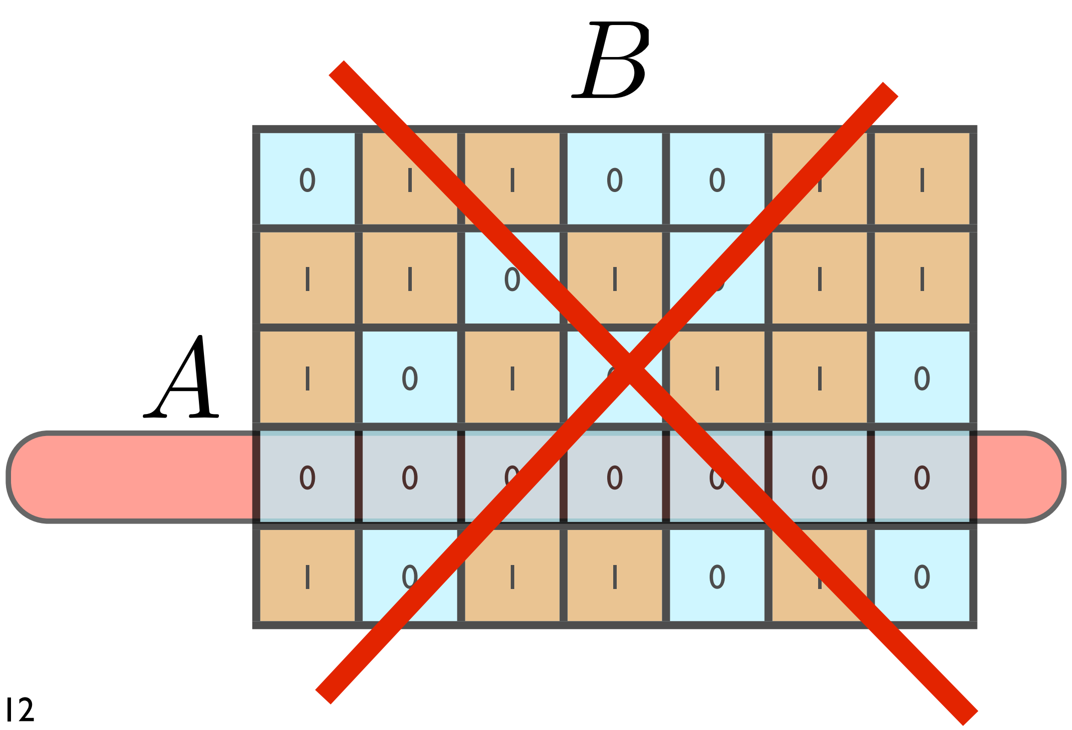
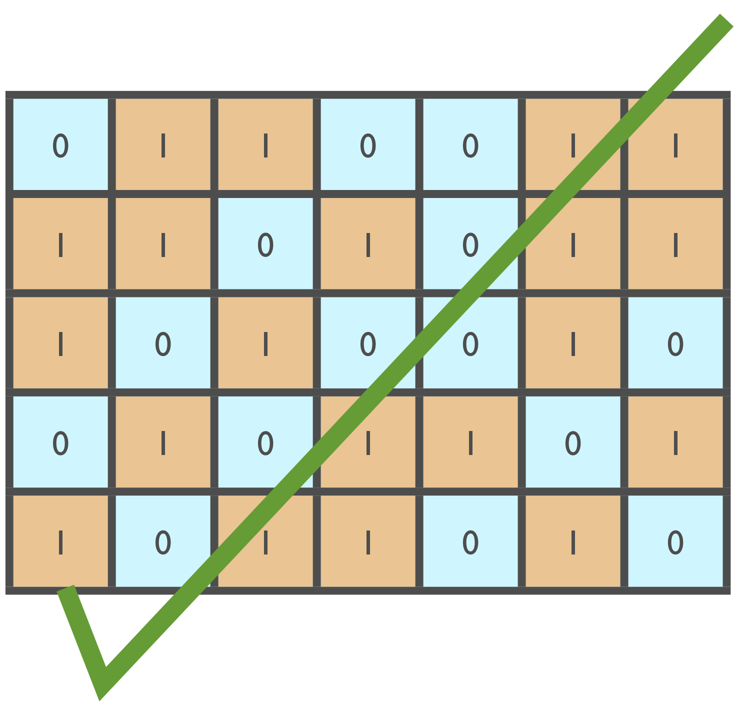




Examples and remarks

The proof of the results below is an exercise.

- If $A = \{a_1, \ldots, a_n\}$ and $B = \{p\}$, then, $\mathcal{R}(A, B) = \{R\}$, where $R = \{(x_i, p),\ 1 \leq i \leq n\}$..

- If $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, then for all $\pi \in \Pi_n$ (permutations of $\{1, \ldots, n\}$), $\{(a_i, b_{\pi_i}),\ 1 \leq i \leq n\} \in \mathcal{R}(A, B)$. Hence, correspondences include bijections (when these exist).

- **Composition of correspondences.** If A, B, C are sets and $R \in \mathcal{R}(A, B)$ and $S \in \mathcal{R}(B, C)$, then

$$T := \{(a, c) |\, \exists b \in B \,\text{s.t.}\, (a, b) \in R \,\text{and}\, (b, c) \in S\}$$

  belongs to $\mathcal{R}(A, C)$.

- Let $f : A \to B$ and $g : B \to A$ be given. Then,

$$\{(a, f(a)),\ a \in A\} \cup \{\ (g(b), b),\ b \in B\} \in \mathcal{R}(A, B).$$

Theorem (An important observation, **[M07]**). *Let (X, d) be a compact metric space. Then, for all compact $A, B \subset X$,*

$$d_{\mathcal{H}}^{X}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \sup_{(a,b) \in R} d(a, b).$$

Proof. Exercise. □

Remark. *We will use the following notation: for a function $f : Z \to \mathbb{R}$ and $C \subset Z$, we let*

$$\|f\|_{L^\infty(C)} := \sup_{z \in C} |f(c)|.$$

Remark. *Then, we can write in a somewhat abbreviated way that will be used for reasoning about potential candidates for* **dist**,

$$d_{\mathcal{H}}^{X}(A, B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}.$$

Exercise. *Using the expression for the H-distance above and the remark on composition of correspondences prove the triangle inequality for the H-distance.*

Soft objects

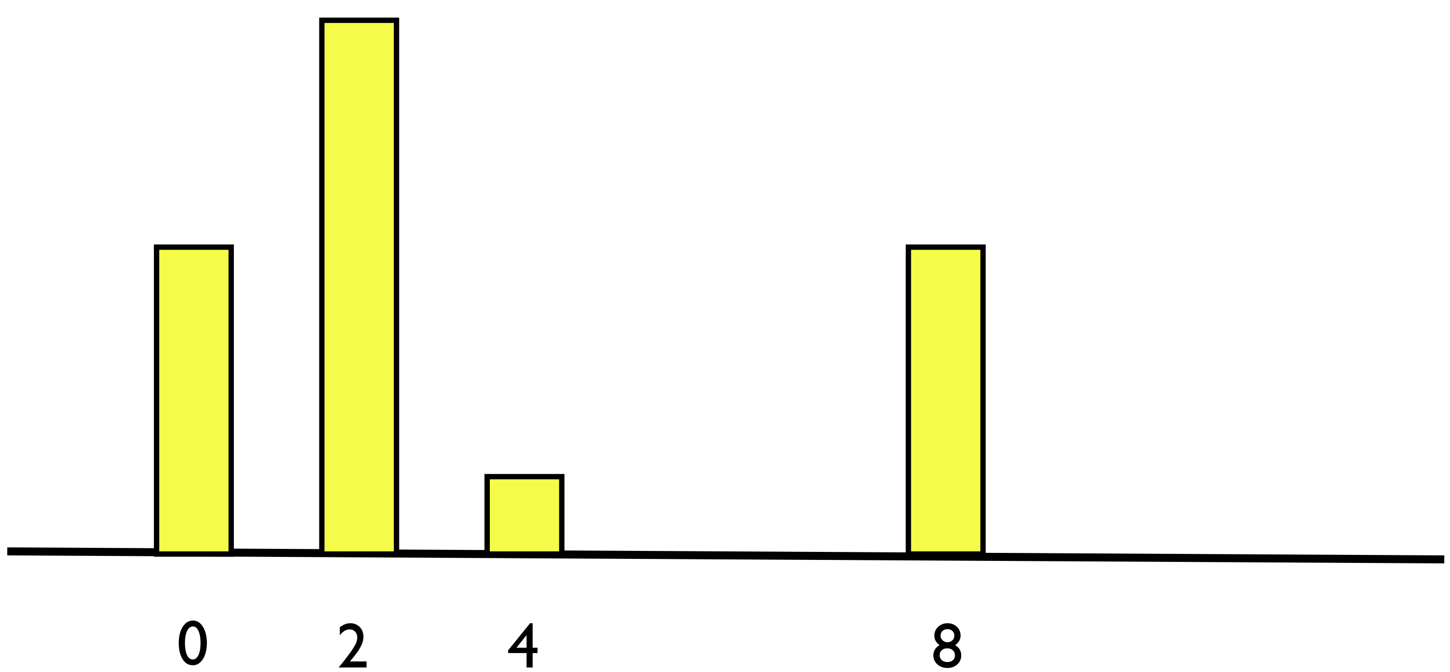
• **Probability Measures.** Consider a finite set $A = \{a_1, \ldots, a_n\}$. A set of *weights*, $W = \{w_1, \ldots, w_n\}$ on A is called a *probability measure* on A if $w_i \geq 0$ and $\sum_i w_1 = 1$.

Probability measures can be interpreted as a way of assigning (relative) importance to different points.

There is a more general definition that we do not need (today). But you should become familiar with it for general culture, see [**BBI, Def. 1.7.1**].

• **Support of a measure.** Given a metric space (X, d) and a probability measure ν on X, the *support* of ν consists of the points of X with non-zero *mass*. We use the notation $\mathrm{supp}(\nu)$ for the support of a probability measure ν on X.

Example. *Consider for example the case of $X = \mathbb{R}$ with the usual metric. Let ν be the probability measure on the real line that assigns mass 1/4, 5/12, 1/12 and 1/4 to points 0, 2, 4 and 8, respectively. Then, there is no mass anywhere else and $supp(\nu) = \{0, 2, 4, 8\}$.*



correspondences and measure couplings

Let A and B be compact subsets of the compact metric space (X, d) and μ_A and μ_B be **probability measures** supported in A and B respectively.

Definition [Measure coupling] Is a probability measure μ on $A \times B$ s.t. (in the finite case this means $((\mu_{a,b})) \in [0,1]^{n_A \times n_B}$, i.e. μ is a *matrix*.)

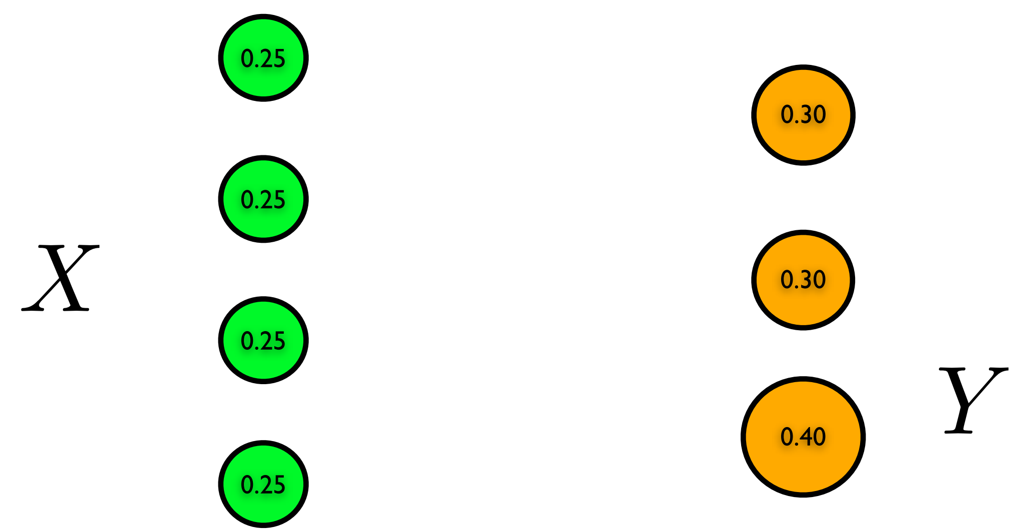
- $\sum_{a \in A} \mu_{ab} = \mu_B(b) \ \ \forall b \in B$
- $\sum_{b \in B} \mu_{ab} = \mu_A(a) \ \ \forall a \in A$

Let $\mathcal{M}(\mu_A, \mu_B)$ be the set of all couplings of μ_A and μ_B.
Notice that in the finite case, $((\mu_{a,b}))$ must satisfy $n_A + n_B$ *linear* constraints.

Y

X

	0.30	0.30	0.40
0.25	0.25	0	0
0.25	0	0.15	0.10
0.25	0	0.1	0.15
0.25	0.05	0.05	0.15



Example. *In this example,*

$$supp(\mu) = \{(x_1, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_2), (x_3, y_3), (x_4, y_1), (x_4, y_2), (x_4, y_3)\}.$$

Example. *Assume* $X = \{x_1, \dots, x_n\}$ *and* $Y = \{p\}$, *together with an arbitrary* μ_X *supported on* X *and* μ_Y *s.t.* $\mu_Y(p) = 1$ *(all the mass is in* p*). Prove that*

$$\mathcal{M}(\mu_X, \mu_Y) = \{\mu_X\}.$$

(compare with exercise for correspondences)

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X

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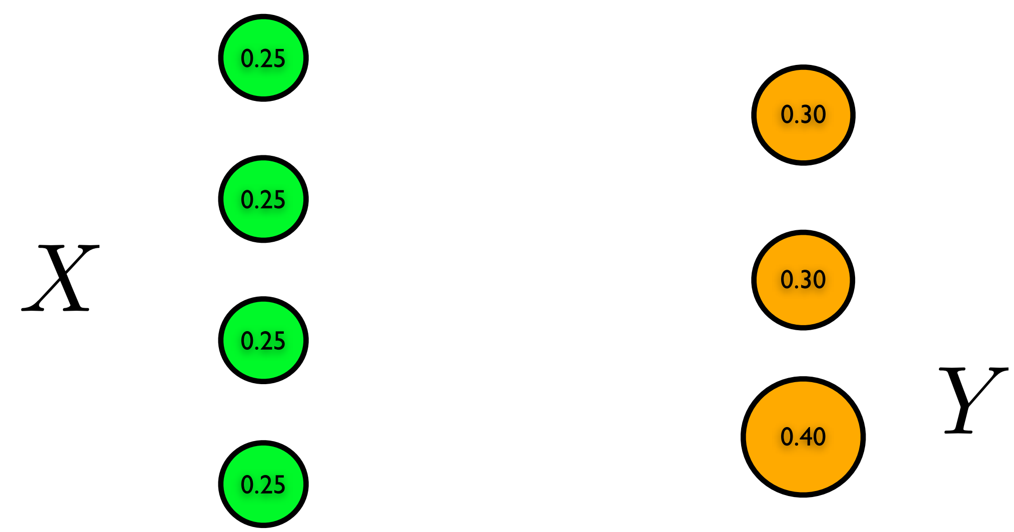

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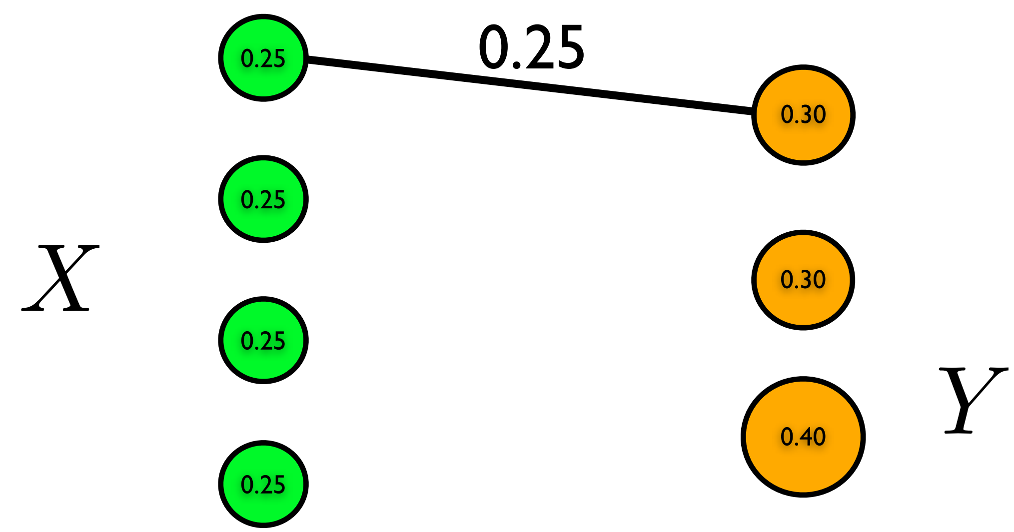

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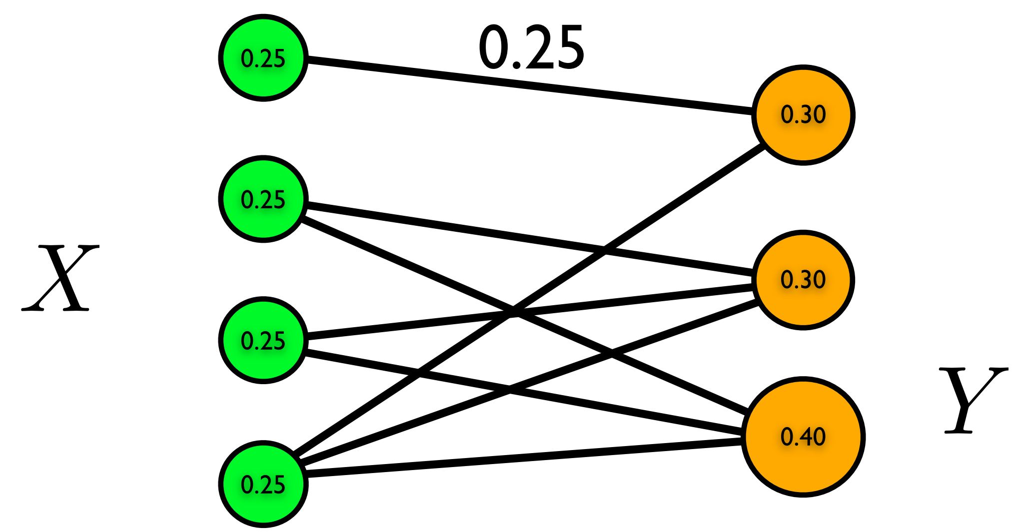

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$$\mathcal{M}(\mu_X, \mu_Y) = \{\mu_X\}.$$

(compare with exercise for correspondences)

• **Composition of measure couplings**. Can you guess what is the construction in this setting? Cf. with "composition of correspondences".

Remark. *You should gain some intuition about the duality between correspondences and measure couplings. Think about this on your own.*

• **Product measure**. Assume A and B are finite sets and μ_A and μ_B are probability measures on A and B, respectively. We define a probability measure on $A \times B$, called the *product measure* and denoted $\mu_A \otimes \mu_B$ s.t.

$$\mu_A \otimes \mu_B(a, b) = \mu_A(a) \times \mu_B(b).$$

Remark. *It is then clear that* $\mathcal{M}(\mu_A, \mu_B) \neq \emptyset$ *as (exercise!!)* $\mu_A \otimes \mu_B \in \mathcal{M}(\mu_X, \mu_Y)$.

Proposition ($\mu \leftrightarrow R$)**.** *Let A, B be sets.*

- *Given (A, μ_A) and (B, μ_B), and $\mu \in \mathcal{M}(\mu_A, \mu_B)$, then*

$$R(\mu) := supp(\mu) \in \mathcal{R}(A, B).$$

- *König's Lemma. [gives conditions for $R \to \mu$] [We don't need precise statement.]*

Proof. Omitted! □

Remark. *Let $f : X \to$ be a function and ν a probability measure on X. Then, for $p \geq 1$ the L^p norm of f w.r.t. to ν is (in the case of X finite)*

$$\|f\|_{L^p(\nu)} := \left(\int_X |f(x)|^p \nu(x)\right)^{1/p} = \left(\sum_{x \in X} \nu(x)|f(x)|^p\right)^{1/p}$$

Remark. • *Correspondences and measure couplings provide two different ways of putting objects in correspondence. This is necessary whenever one tries to compare two objects.*

- *correspondence are* combinatorial gadgets. *They pairings they encode are* hard *as opposed to the* soft *or relaxed notion provided by measure couplings.*

- *Measure couplings are* continous *gadgets. As a general, imprecise rule, using them instead will lead to continuous optimization problems instead of combinatorial optimization problems. "CnOPs are easier to deal with than CbOPs ".*

- Now, given (X, d_X) we looked at $\mathcal{C}(X)$: all the objects (closed subsets) in X and endowed that with the Hausdorff distance. These are called 'hard' objects.

- We can instead look at $\mathcal{C}^w(X)$: all the probability measures on X and try to put a metric there (spoiler: it will be called Wasserstein distance)

- Notice that probability measures have more information than sets:

$$\mathcal{C}^w(X) \to \mathcal{C}(X)$$

  but given $A \in \mathcal{C}(X)$ there may be many elements in $\mathcal{C}^w(X)$ compatible with A!

Wasserstein distance

$$d_{\mathcal{H}}(A,B) = \inf_{R \in \mathcal{R}(A,B)} \|d\|_{L^\infty(R)}$$

$$\Downarrow \; (R \leftrightarrow \mu)$$

$$d_{\mathcal{W},\infty}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^\infty(R(\mu))}$$

$$\Downarrow \; (L^\infty \leftrightarrow L^p)$$

$$d_{\mathcal{W},p}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^p(A\times B,\mu)}$$

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$$\boxed{d_{\mathcal{W},p}(A,B) = \inf_{\mu \in \mathcal{M}(\mu_A,\mu_B)} \|d\|_{L^p(A \times B,\mu)}}$$

Remark. *When A and B are finite, a more explicit expression for the W-distance is*

$$d^X_{\mathcal{W},p}(A,B) := \min_{\mu} \left(\sum_{a,b} d(a,b)^p \, \mu(a,b) \right)^{1/p}$$

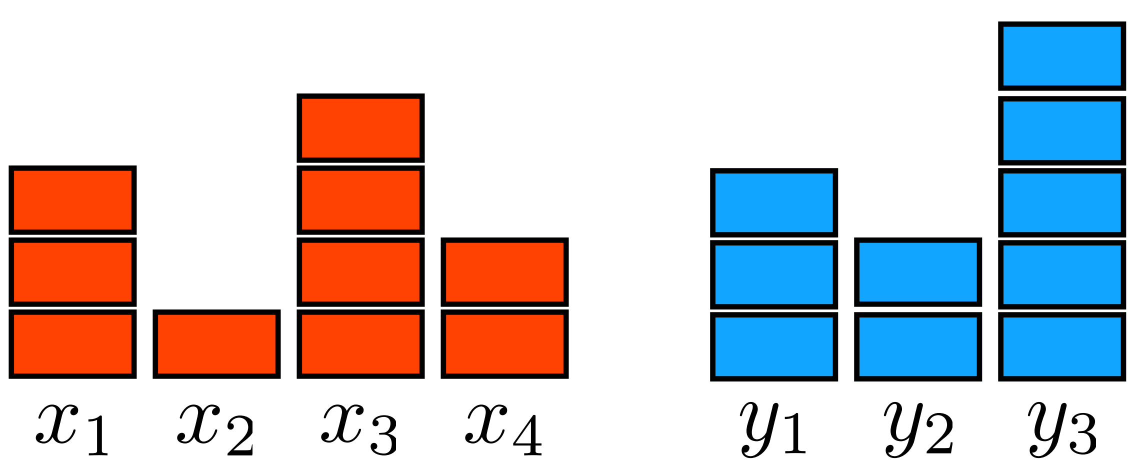
where $\mu \in \mathcal{M}(\mu_A, \mu_B)$.

Remark. *Notice that computing the W-distance leads to solving an* ***LOP*** *with linear and bound constraints.*

Remark. *We will see that $d^X_{\mathcal{H}} \leq d^X_{\mathcal{W},\infty}$. Can you prove this?*

Remark.

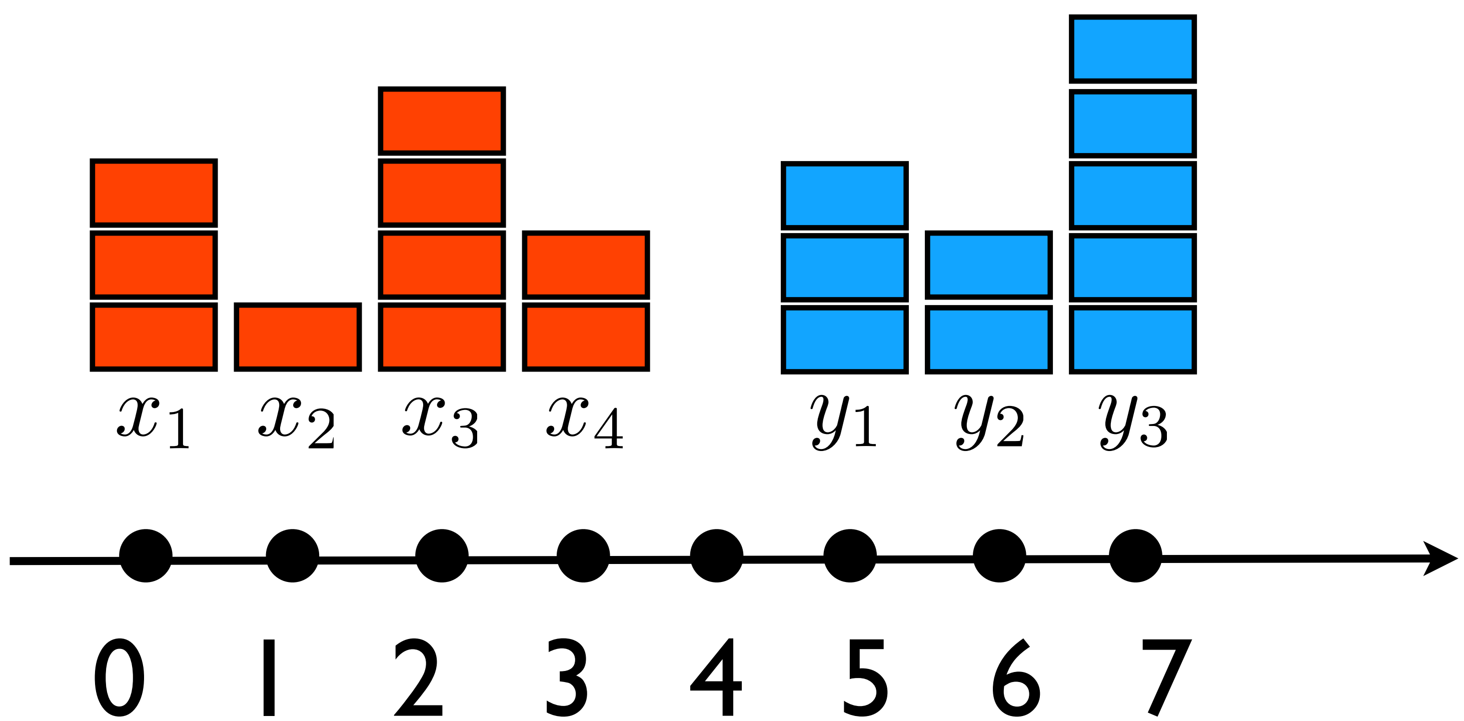
- *The Wasserstein distance is a.k.a. EMD (Eart Mover's Distance) a.k.a. Kantorovich-Rubinstein.*
- *there is a very nice physical interpretation: μ_A represent a certain* source *profile of n_A bricks that must be moved from a certain location to another. The* target *profile at the destination, represented by μ_B, is such that the total number of bricks used is* equal *to n_A.*
- *The* cost *of moving a brick from location x to location y is $d(x, y)$, the horizontal distance between x and y.*
- *A measure coupling, in a first approximation, is a integer valued matrix that tells you how to distribute bricks in a source pile to the destination.*



	3	4	5
3			
1			
4			
2			

	3	2	5
3	**3**	0	0
1	0	0	**1**
4	0	0	**4**
2	0	**2**	0

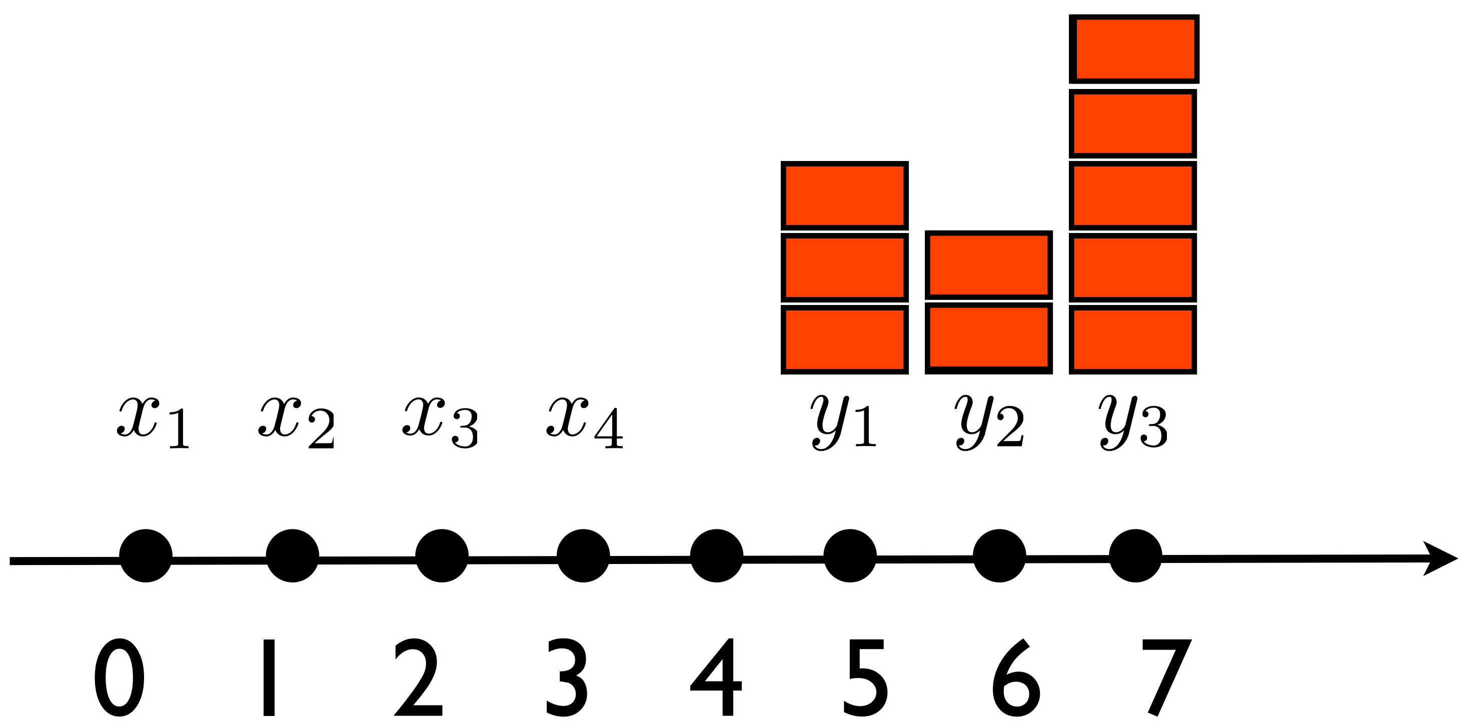
$$\mathbf{cost}(\mu) = \sum_{x,y} d(x,y)\mu_{x,y}$$



$$\mathbf{cost}(\mu) = (3 \cdot 5 + 1 \cdot 6 + 4 \cdot 5 + 2 \cdot 3) = 47$$

	3	2	5
3	**3**	0	0
1	0	0	**1**
4	0	0	**4**
2	0	**2**	0

$$\mathbf{cost}(\mu) = \sum_{x,y} d(x,y)\mu_{x,y}$$



$$\mathbf{cost}(\mu) = (3 \cdot 5 + 1 \cdot 6 + 4 \cdot 5 + 2 \cdot 3) = 47$$

For a compact metric space (X,d) let

$$\mathcal{C}^w(X) := \{(A,\mu_A),\ A \in \mathcal{C}(X) \text{ and} \operatorname{supp}(\mu_A) = A\}.$$

This is the collection of all *weighted objects* in X.

Theorem ([**Villani**], see [**M07**])**.** *Let (X,d_X) be a compact metric space. The Wasserstein distance is a metric on $\mathcal{C}^w(X)$.*

Theorem ([**Villani**], Prokhorov's theorem)**.** *Let (X,d_X) be a compact metric space. Then, $\mathcal{C}^w(X)$ with the Wasserstein distance is also a compact metric space.*

This is an analogue to Blacshke's theorem (meditate about this!)

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Transportation theory (mathematics)

From Wikipedia, the free encyclopedia

In mathematics and economics, **transportation theory** is a name given to the study of optimal transportation and allocation of resources.

The problem was formalized by the French mathematician Gaspard Monge in 1781.[1]

In the 1920s A.N. Tolstoi was one of the first to study the transportation problem mathematically. In 1930, in the collection *Transportation Planning Volume I* for the National Commissariat of Transportation of the Soviet Union, he published a paper "Methods of Finding the Minimal Kilometrage in Cargo-transportation in space".[2][3]

Major advances were made in the field during World War II by the Soviet/Russian mathematician and economist Leonid Kantorovich.[4] Consequently, the problem as it is stated is sometimes known as the **Monge–Kantorovich transportation problem**.

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Mines and factories [edit]

Suppose that we have a collection of n mines mining iron ore, and a collection of n factories which consume the iron ore that the mines produce. Suppose for the sake of argument that these mines and factories form two disjoint subsets M and F of the Euclidean plane $\mathbf{R}^2$. Suppose also that we have a *cost function* $c : \mathbf{R}^2 \times \mathbf{R}^2 \to [0, \infty)$, so that $c(x, y)$ is the cost of transporting one shipment of iron from x to y. For simplicity, we ignore the time taken to do the transporting. We are also assume that each mine can supply only one factory (no splitting of shipments) and that each factory requires precisely one shipment to be in operation (factories cannot work at half- or double-capacity). Having made the above assumptions, a *transport plan* is a bijection $T : M \to F$ – i.e. an arrangement whereby each mine $m \in M$ supplies precisely one factory $T(m) \in F$. We wish to find the *optimal transport plan*, the plan T whose *total cost*

$$c(T) := \sum_{m \in M} c(m, T(m))$$

is the least of all possible transport plans from M to F. This motivating special case of the transportation problem is in fact an instance of the assignment problem.

Gaspard Monge

Born	May 9, 1746 Beaune, Côte-d'Or
Died	July 28, 1818 Paris
Nationality	French

Leonid Kantorovich

Leonid Kantorovich in 1975

Born	19 January 1912 Saint Petersburg, Russian Empire
Died	7 April 1986 (aged 74) Moscow, Russia, USSR
Nationality	Soviet

Leonid Kantorovich

From Wikipedia, the free encyclopedia

Leonid Vitaliyevich Kantorovich (Russian: Леони́д Вита́льевич Канторо́вич, IPA: [lʲɪɐˈnʲit vʲɪˈtalʲɪvʲɪt͡ɕ kɐntɐˈrovʲɪt͡ɕ] (listen)) (19 January 1912 – 7 April 1986) was a Soviet mathematician and economist, known for his theory and development of techniques for the optimal allocation of resources. He was the winner of the Nobel Prize in Economics in 1975 and the only winner of this prize from the USSR.

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Biography [edit]

Kantorovich was born on 19 January 1912, to a Russian Jewish family.[1] His father was a doctor practicing in Saint Petersburg.[2] In 1926, at the age of fourteen, he began his studies at the Leningrad University. He graduated from the Faculty of Mathematics in 1930, and began his graduate studies. In 1934, at the age of 22 years, he became a full professor.

Later, Kantorovich worked for the Soviet government. He was given the task of optimizing production in a plywood industry. He came up (1939) with the mathematical technique now known as linear programming, some

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Earth mover's distance

From Wikipedia, the free encyclopedia

This article **needs attention from an expert in Mathematics**. Please add a *reason* or a *talk* parameter to this template to explain the issue with the article. WikiProject Mathematics (or its Portal) may be able to help recruit an expert. *(October 2008)*

In computer science, the **earth mover's distance (EMD)** is a measure of the distance between two probability distributions over a region *D*. In mathematics, this is known as the Wasserstein metric. Informally, if the distributions are interpreted as two different ways of piling up a certain amount of dirt over the region *D*, the EMD is the minimum cost of turning one pile into the other; where the cost is assumed to be amount of dirt moved times the distance by which it is moved.[1]

The above definition is valid only if the two distributions have the same integral (informally, if the two piles have the same amount of dirt), as in normalized histograms or probability density functions. In that case, the EMD is equivalent to the 1st Mallows distance or 1st Wasserstein distance between the two distributions.[2][3]

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See papers under Week of Feb 24-28