# On well-quasi-ordering lower sets of finite trees, a new proof

Yared Nigussie *
Department of Mathematics
East Tennessee State University
nigussie@etsu.edu

September 14, 2008

**Abstract**

In 1964, [3], Nash-Williams proved that the iterated lower sets (ideals) of finite trees are well-quasi-ordered by the subset relation. However, acknowledges that his proof is very complicated. Robertson in 1997, [8], conjectured a "Lifting lemma" which claims that every ideal has a finite description tree in such way that an embedding relation between two description trees of two ideals lifts back to the subset relation between the corresponding two ideals. We prove the lifting conjecture in the affirmative. As a corollary, a simple proof of the result of [3] follows.

## 1 Introduction

We start with an easy analogous example. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $F\mathbb{N}$ be the set of finite subsets of $\mathbb{N}$ which have the property that if $k \in S \in F\mathbb{N}$ and $j \leq k$ then $j \in S$. Then clearly, $S = \{0, 1, 2, \ldots, m\} = [m]$ for some $m \in \mathbb{N}$. If $S = [m]$ and $S' = [m']$, then $S \subseteq S'$ if and only if $m \leq m'$. What is nice here is that by replacing the $\subseteq$ relation in $F\mathbb{N}$ by the *basic* relation $\leq$ of $\mathbb{N}$, and by representing each element of $F\mathbb{N}$ by its maximal element, we lose no information about the order in $F\mathbb{N}$. Define $F^n\mathbb{N} = F(F^{(n-1)}\mathbb{N})$, and we observe that inductively $\mathbb{N}$ is totally ordered implies so is $F^n\mathbb{N}$, for any $n$. This example is trivial of course, since we have sets that are finite and totally ordered. However, it shows the key idea of our proof. Although finite, we can have arbitrarily long anti-chain of trees. The classes of trees we study are usually infinite and so have no maximal element. Nevertheless, we ask if there exists a similar representation of a class of trees by a finite tree in a such a way that the *basic* relation between the representative trees governs the subset relation between the iterated sets? The conjecture of Robertson is positive to this question. We start with basic definitions and state the conjecture formally.

A *quasi-order* $\leq$ on a set $Q$ is a reflexive and transitive relation. Let $Q$ be a set quasi-ordered by $\leq$. A *lower set* of a set $Q$ (or an *ideal*, for short) is a subset $I$ of

*Partially supported by DIMATIA and by a Grant 1M002160808, also partially supported by the National Science Foundation under NSF Grant DMS 9701317.

$Q$ which is closed down under $\leq$. That is, for all $y \in I$ and all $x \in Q$, if $x \leq y$ then $x \in I$. We say $Q$ is *well-quasi-ordered*, (*wqo*), if every infinite sequence in $Q$ is "good", (A sequence $q_1, q_2, \ldots$ of $Q$ is *good* if there exist indexes $i, j$ such that $i < j$ and $q_i \leq q_j$).

Trees in this paper are finite, rooted and directed away from the root. We assume the edge set $E(T)$ of a tree $T$ is labeled from a finite ordinal, say $[m] = \{0, 1, \ldots, m\}$, and the vertex set $V(T)$ is labeled from a set $Q$, that is quasi-ordered by $\leq_Q$. The root of $T$, denoted $root(T)$, is a special vertex that has a *root-edge label* from $[m]$, in addition to its vertex label from $Q$. Hence, we assume every vertex $v \in V(T)$ has a pair of labels, $l(v) = (p, q)$, where $p \in [m], q \in Q$, and refer to $p$ as $l_1(v)$ and to $q$ as $l_2(v)$. If $v$ is not a root, then we can treat $l_1(v)$ as labeling the unique edge $e = uv$ and will sometimes refer to it as the *edge-label of e*. Where no misunderstanding can occur we will write $l(e)$ instead of $l_1(v)$ for emphasis and clarity. Given trees $T, T'$, we say that $T$ is a *topological minor of* $T'$ (and write $T \leq_t T'$) if a subdivision of $T$ is isomorphic (without the labels) to a subtree $T''$ of $T'$, such that $l_2(v) \leq_Q l_2(v'')$ for each $v \in V(T)$, where $v''$ is the isomorphism corresponding vertex in $V(T'')$. Note that the relation $T \leq_t T'$ is an injective mapping $f : V(T) \to V(T')$ and also that it ignores the edge label $l_1(v)$.

However, if $T \leq_t T'$, then $f$ maps every edge $e$ of $T$ to a unique path $f(e)$ in $T'$. The map $f$ is said to satisfy the *Kruskal-Friedman gap condition* (written $T \leq_{KF} T'$) if in addition $l(e) \leq l(e')$ for every edge $e'$ in $f(e)$. A stronger version of $\leq_{KF}$ has one more condition that $l_1(root(T)) \leq l_1(v')$ for every $v' \in V(T')$ that is on the path from $root(T')$ to $f(root(T))$. This stronger version is what we mean by the relation $\leq_{KF}$ throughout this paper. Also, unless specified otherwise, by ideal we mean a set of trees closed under $\leq_{KF}$. The following is Robertson's Lifting Conjecture.

**Conjecture 1 (Lifting Lemma)** *For every ideal $\mathcal{I}$, there exists a finite structure-tree $\psi(\mathcal{I})$ that is labeled from a wqo set such that for any two ideals $\mathcal{I}, \mathcal{I}'$ if $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$ then $\mathcal{I} \subseteq \mathcal{I}'$.*

Let $\mathcal{F}(Q)$ denote the set of all finite trees with vertices labeled from $Q$, and let $\mathcal{F}(m, Q)$ denote the set of all finite trees we get from $\mathcal{F}(Q)$ by labeling the edges from $[m], m \geq 0$. Note that when $m = 0$, the relation $\leq_{KF}$ reduces to $\leq_t$ and so is a natural generalization of Kruskal's relation $\leq_t$. Kruskal [2] proved in 1960 that $\mathcal{F}(Q)$ is wqo by $\leq_t$.

**Theorem 2** *(**J. Kruskal [2]**) If $Q$ is wqo, then $\mathcal{F}(Q)$ is wqo by $\leq_t$.*

Friedman [1] strengthened Kruskal's Theorem by the following:

**Theorem 3** *(**H. Friedman, [1]**) If $Q$ wqo, then $\mathcal{F}(m, Q)$ is wqo by $\leq_{KF}$.*

The class $LQ$ of ideals of $Q$ is quasi-ordered by the subset relation " $\subseteq$ ". Inductively, $L^n Q = L(L^{n-1} Q)$, is defined to be $n$-th iterated ideal of ideals. Note that a set $Q$ is wqo by $\leq$ relation does not imply $LQ$ is wqo by the subset relation $\subseteq$. In fact, Kruskal has shown by an unpublished example that for each $n$ there exists a set $Q(n)$ such that $L^n Q(n)$ is wqo and $L^{(n+1)} Q(n)$ is not. Then, Nash-Williams in [3] proved this does not occur for finite trees:

**Theorem 4** *(**Nash-Williams**) For all $n \in \mathbb{N}$, if $L^n Q$ is wqo, then $L^n \mathcal{F}(Q)$ is wqo by $\leq_t$.*

We assume the following result from [5]: (see next section for the definition of $\psi$)

**Theorem 5** *For every ideal $\mathcal{I} \subseteq \mathcal{F}(m, Q)$, $m \geq 0$ an integer and $Q$ wqo, there exists a wqo set $Q'$ and a finite tree $\psi(\mathcal{I})$ in $\mathcal{F}(2m+4, Q')$ that constructs precisely the elements of $\mathcal{I}$.*

By applying Theorem5 and the Lifting Lemma, we have:

**Theorem 6** *For all $n \in \mathbb{N}$, if $L^n Q$ is wqo, then $L^n \mathcal{F}(m, Q)$ is wqo.*

## 2 Proof of the Lifting Lemma and Theorem 6

First, we define a basic operation of constructing a new tree by "tree-summing" a finite number of trees in $\mathcal{F}(m, Q)$. The *null-tree*, $\Gamma$, is defined by $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, $\Gamma$ is not a rooted tree. Let $T_1, T_2, \ldots, T_n, n \geq 0$, be pairwise vertex disjoint trees or $\Gamma$. For $p \in [m]$ and $q \in Q$ the *tree-sum* of $T_1, T_2, \ldots, T_n$ is given by $T = Tree(T_1, T_2, \ldots, T_n)_{(p,q)}$ where for all $i$ the labels of each *summand* $T_i$ is preserved, $t_0$ is a new vertex (the root of $T$) labeled by $(p, q)$, $V(T) = V(T_1) \cup V(T_2) \cup ... \cup V(T_n) \cup \{t_0\}$, and its set of edges is $E(T_1) \cup E(T_2) \cup ... \cup E(T_n) \cup \{e_i = (t_0, root(T_i)) : 1 \leq i \leq n, T_i \neq \Gamma\}$. (note that $Tree(T_1, T_2, \Gamma)_{(p,q)}$ is isomorphic to $Tree(T_1, T_2)_{(p,q)}$ but allowing the null-tree to appear in a tree-sum will be convenient later). Note also that for each new edge $e_i$ we have $l(e_i) = l_1(root(T_i))$. As a notational convenience, if $T$ has height zero, then we write $T = (p, q)$, instead of $Tree()_{(p,q)}$.

Next we define a tool called "rst-cell", that is used to construct trees by using tree-sum operation:

$$H = (k; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{I}_\infty)_{\mathcal{I}_0} \tag{1}$$

is an *rst-cell* where the parameters in $H$ are defined as follows: $k \in \mathbb{N} \cup \{0, \infty\}$ is the *width* of $H$ and is denoted by $k(H)$; we allow $k(H) = \infty$ only if $n = 0$ and $\mathcal{I}_\infty = \emptyset$; $\mathcal{I}_i$ for $i = \infty$ or $i = 0, 1, \ldots, n$ is an ideal of $\mathcal{F}(m, Q)$; If $1 \leq i \leq n$, then $\mathcal{I}_i \neq \emptyset$. Every tree $T$ in $\mathcal{I}_0$ has height zero. Hence, $\mathcal{I}_0 = ([p], \mathcal{Q}), p \in [m], \mathcal{Q} \in LQ$. If $0 < i < \infty$, we say $\mathcal{I}_i$ is a *middle-component* of $H$. We say $\mathcal{I}_\infty$ is the *right-component* of $H$, and $\mathcal{I}_0$ is the *root-component* of $H$. Two rst-cells $H$ and $H'$ are assumed to be *equal* if they differ only by a permutation of their middle-components. Let $\mathcal{H}$ be a set of rst-cells and $H \in \mathcal{H}$ be as in Equation (1). An ideal $\mathcal{I}$ is said to satisfy condition

(*) if $Tree(T_1, \ldots, T_k, T_{k+1}, \ldots, T_{k+n}, T_{k+n+1}, \ldots, T_{k+n+s})_{(p,q)}$ belongs to $\mathcal{I}$, where $s \geq 0$, $(p, q) \in \mathcal{I}_0$ is called the **root-part**; for $a = 1, \ldots, k$, $T_a \in \mathcal{I} \cup \{\Gamma\}$ and collectively the $k$ trees are called the **left-part**; for $a = k+1, \ldots, k+n$, $T_a \in \mathcal{I}_a \cup \{\Gamma\}$ and collectively called the **middle-part**; and for $a = k+n+1, \ldots, s$, $T_a \in \mathcal{I}_\infty$ and are collectively called the **right-part**. We define the ideal $I(\mathcal{H})$ to be the intersection of all ideals satisfying (*). Let $\alpha(\mathcal{I}) = \max\{l_1(root(T)) : T \in \mathcal{I}\}$. Let $T$ be a tree, $\mathcal{H}$ be a set of rst-cells and $H \in \mathcal{H}$ be as in Equation (1). Then, we say that $T$ *conforms to $H$ in $I(\mathcal{H})$* if either $l_1(root(T)) \leq \alpha(\mathcal{I}_0)$ and $T \in \mathcal{I}_i, 1 \leq i \leq \infty$ or $T$ is a tree-sum as in (*) where the left-part is from $I(\mathcal{H}) \cup \{\Gamma\}$. The following is obvious by definition and by the fact that $I(\mathcal{H})$ is an ideal:

**Theorem 7** *Let $\mathcal{H}$ be a set of rst-cells. Let $T$ be a tree. Then, $T \in I(\mathcal{H})$ if and only if $T$ conforms to $H$ in $I(\mathcal{H})$.*

Let $H$ be a width zero rst-cell. If $\mathcal{I}'$ is a component of $H$ and if $I(\{H\}) \subseteq \mathcal{I}'$, then we assume $H$ is of the form $(0; \mathcal{I}'; \emptyset)_{\mathcal{I}_0}$. Further, if $\alpha(\mathcal{I}_0) < \alpha(\mathcal{I}')$, then we call $H$ a *trimmer-cell* and denote it by a unique symbol $H^*$. Note that if $H^* = (0; \mathcal{I}'; \emptyset)_{\mathcal{I}_0}$ is a trimmer-cell then $\mathcal{I}' \not\subseteq I(\{H^*\}) \subset \mathcal{I}'$. A set of rst-cells $\mathcal{H}$ need not have a trimmer-cell. So if $\mathcal{H}$ has no trimmer-cell we assume $H^*$ is null. For convenience, if $H^*$ is null then we assume $\{H^*, H_1, H_2 \ldots, H_t\} = \{H_1, H_2 \ldots, H_t\}$.

**Definition 8 (Proper Span)** *Let $\mathcal{I}$ be an ideal such that $\alpha(\mathcal{I}) = m$. Then, a set of rst-cells $\mathcal{H} = \{H^*, H_1, H_2 \ldots, H_t\}$ with at most one trimmer-cell $H^*$ is a proper spanning set of $\mathcal{I}$ if the following three properties hold:*

**(P1)** *$I(\{H^*\}) \subset \mathcal{I}$ and if $\mathcal{I}'$ is a component of $H$ in $\mathcal{H}$-$\{H^*\}$, then $\mathcal{I}' \subset \mathcal{I}$. (induction axiom)*
**(P2)** *$I(\mathcal{H}) = \mathcal{I}$. (spanning axiom)*
**(P3)** *for any component $\mathcal{I}'$ that is not a root-component of $H^*$, $\alpha(\mathcal{I}') = m$. (chain axiom)*

**Remark.** Note that in Definition8, we assume $\alpha(\mathcal{I}) = m$. In general, we see that $\alpha(\mathcal{I}) < m$ if and only if $(\alpha(\mathcal{I}) + 1, q_0)$ is an obstruction of $\mathcal{I}$. The ideal $\mathcal{I}'$ which forbids all obstructions of $\mathcal{I}$ except $(\alpha(\mathcal{I}) + 1, q_0)$ has $\alpha(\mathcal{I}') = m$. Hence, For both $\mathcal{I}$ and $\mathcal{I}'$ we use the same proper span. However, in Definition 9 below, we assign them distinct structure-trees (See Equation 2).

Let $\mathcal{I}$ be an ideal and $\mathcal{H}$ be a proper span of $\mathcal{I}$. A middle or a right-component $\mathcal{I}'$ in $\mathcal{H}$ a *child of* $\mathcal{I}$ (alternately $\mathcal{I}$ is a *parent* of $\mathcal{I}'$). Call a sequence $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ a *lineage-path*, if for each $i \geq 1$, $\mathcal{I}_i$ is a child of $\mathcal{I}_{i-1}$. Let $X = (\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \ldots)$ be a lineage-path. For some $i \geq 1$, if $\mathcal{I}_{i+1}$ is a child in trimmer-cell and equals $\mathcal{I}_j$, for some $j < i$, then we call its parent $\mathcal{I}_i$ an *ancestor-bearer* or more specifically $\mathcal{I}_j$*-bearer*. If $\mathcal{I}_j \neq \mathcal{I}_{i+1}$ for all $j, 0 \leq j < i$, then we say $\mathcal{I}_i$ is *ancestor-free*. A recursive definition of "Structure-tree" follows: (Recall that $q_0 \in Q$ is assumed to be uniquely minimal and also let $\hat{q} \in Q$, such that $\hat{q} \neq q_0$.)

**Definition 9 (Structure-tree)** *Let $\mathcal{I}$ be an ideal in $\mathcal{F}(m, Q)$, where $m \geq 0$ is an integer and $Q$ is a wqo set. Let $\mathcal{H}(\mathcal{I}) = \{H^*, H_1, H_2, \ldots, H_t\}$ be a proper span of $\mathcal{I}$. If $\mathcal{I} = \emptyset$, then set $\psi(\mathcal{I}) = (0, q_0)$. Otherwise,*

$$\psi(\mathcal{I}) = Tree(R^*(\mathcal{I}), R_1(\mathcal{I}), R_2(\mathcal{I}), \ldots, R_t(\mathcal{I}))_{(\alpha(\mathcal{I}), \hat{q})}, \tag{2}$$

*where $R^*(\mathcal{I})$ corresponds to $H^* \in \mathcal{H}(\mathcal{I})$ and for $i = 1, 2, \ldots t$, $R_i(\mathcal{I})$ corresponds to $H_i = (k_i; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n_i}; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$, such that :*

*(i) If $H^*(\mathcal{I})$ is null, then $R^*(\mathcal{I}) = \Gamma$. Otherwise $R^*(\mathcal{I}) = Tree(\psi(I(\{H^*\}))_{(2m+4,0)}$.*

*(ii) For each $i, 1 \leq i \leq t$, construct the summand $R_i(\mathcal{I})$ on the path $P_{n_i+3}$ with vertices $v_r, v_\infty, v_0, v_1, v_2, \ldots, v_{n_i}$, rooted at $v_r$ as follows:*

*(a) Let $l(v_r(R_i)) = (2m + 3, (k_i, \mathcal{I}_0))$ and $l(v_\infty(R_i)) = (2m + 2, q_0)$.*
*For $0 \leq j \leq n_i$, $l(v_j(R_i)) = (2m + 1, q_0)$.*

*(b) Join $(0, q_0)$ to $v_0(R_i)$ by an edge. For each $j, j = \infty$ or $1 \leq j \leq n_i$, using a proper span $\mathcal{H}(\mathcal{I}_j)$ of $\mathcal{I}_j$, join the root of $\phi(\mathcal{I}_j)$ to $v_j$ by an edge,*

*where $\phi(\mathcal{I}_j) = \psi(\mathcal{I}_j)$, if $\mathcal{I}_j$ is ancestor-free. Otherwise, $\phi(\mathcal{I}_j)$ is obtained from Equation 2 of $\psi(\mathcal{I}_j)$, by adding $1 + \alpha(I(\{H^*(\mathcal{I}_j)\}))$ to $l_1(\psi(\mathcal{I}_j))$ and setting $H^*(\mathcal{I}_j)$ to null, and proceeding to step (i).*

**Proof.** (**of the Lifting Lemma1**) Let $\mathcal{I}$ and $\mathcal{I}' \in L\mathcal{F}(m, Q)$ with their structure-trees $\psi(\mathcal{I}) = Tree(R^*, R_1, \ldots, R_t)_{(\alpha(\mathcal{I}),\hat{q})}$ and $\psi(\mathcal{I}') = Tree(R'^*, R'_1, \ldots, R'_{t'})_{(\alpha(\mathcal{I}'),\hat{q})}$ be given. Their summands correspond to the rst-cells $\mathcal{H} = \{H^*, H_1, H_2, \ldots, H_t\}$ and $\mathcal{H}' = \{H'^*, H'_1, H'_2, \ldots, H'_{t'}\}$, proper spans of $\mathcal{I}$ and $\mathcal{I}'$, respectively. For a contradiction, assume $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$ by a gap embedding $f$ and that $\mathcal{I} \not\subseteq \mathcal{I}'$, where the ideal $\mathcal{I}$ is chosen to be as small as possible. For the chosen $\mathcal{I}$ we may also choose $\psi(\mathcal{I}')$ so that $f(root(\psi(\mathcal{I}))) = root(\psi(\mathcal{I}'))$. By the gap condition, we see that for any edge $e$ of $\psi(\mathcal{I})$, if $l(e) \geq 2m + 2$, then $f(e) = e'$ is an edge of $\psi(\mathcal{I}')$. If $R^* \neq \Gamma$, then its root has unique largest label $2m + 4$, and so we have $R^* \leq_{KF} R'^*$ and $f(root(R^*)) = root(R'^*)$. Hence, we have $\psi(I(\{H^*\})) \leq_{KF} \psi(I(\{H'^*\}))$ and so by induction $I(\{H^*\}) \subseteq I(\{H'^*\})$. Similarly and by rearranging if necessary, we have $R_i \leq_{KF} R'_i$ and $f(root(R_i)) = v_r(R_i) = v_r(R'_i) = root(R'_i)$, for $i = 1, 2, \ldots, t$. Let the corresponding rst-cells be $H_i = (k_i; \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{n_i}; \mathcal{I}_\infty)_{\mathcal{I}_0} \in \mathcal{H}$, and $H'_i = (k'_i; \mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_{n'_i}; \mathcal{I}'_\infty)_{\mathcal{I}'_0} \in \mathcal{H}'$.

We show that every tree $T \in I(\mathcal{H}) - I(\mathcal{H}')$ of minimal height $h, h \geq 1$, that conforms to $H_i$ in $I(\mathcal{H})$ also conforms to $H'_i$ in $I(\mathcal{H}')$, which is absurd. From $f(root(R_i)) = root(R'_i)$, we have $k_i \leq k'_i$ and $\mathcal{I}_0 \subseteq \mathcal{I}'_0$. Next, we show that for every component $\mathcal{I}_a$ of $H_i$ there is a unique component $\mathcal{I}'_{a'}$ of $H'_i$ such that $\mathcal{I}_a \subseteq \mathcal{I}'_{a'}$. Also, if $a = \infty$, then we shall show that $a' = \infty$. As a result, using induction on $h$, we obtain the desired contradiction. Let $P \subseteq \psi(\mathcal{I})$ be the path induced by $v_r(R_i), v_\infty(R_i), v_0(R_i), v_1(R_i), \ldots, v_{n_i}(R_i)$, and also $P' \subseteq \psi(\mathcal{I}')$ be the path induced by $v_r(R'_i), v_\infty(R'_i), v_0(R'_i), v_1(R'_i), \ldots, v_{n'_i}(R'_i)$. By the gap condition, we have $f(P) \subseteq P'$, since any other edge incident to $P'$ has label at most $2m$. In particular, $f(v_\infty(R_i)) = v_\infty(R'_i)$. We have $\phi(\mathcal{I}_a) \leq_{KF} \phi(\mathcal{I}'_{f(a)})$, for some $\mathcal{I}'_{f(a)}$, a descendant of $\mathcal{I}'$. Since $\hat{q} \not\leq q_0$, we may assume that $f(root(\phi(\mathcal{I}_a))) = root(\phi(\mathcal{I}'_{f(a)}))$. Then, either $\mathcal{I}'_{f(a)} = \mathcal{I}'_{a'}$ is in $H'_i$ or is a descendant of $\mathcal{I}'_{a'}$. Furthermore, since $l_1(v_\infty(R_i)) = 2m+2$, and since $l_1(v_0(R_i)) = 2m + 1$ and that it exists for any $n_i \geq 0$, we deduce that if $a = \infty$ then $a' = \infty$.

We claim that $\psi(\mathcal{I}_a) \leq_{KF} \psi(\mathcal{I}'_{f(a)})$. If $\mathcal{I}_a$ ancestor-free, then $\phi(\mathcal{I}_a) = \psi(\mathcal{I}_a)$ and since $\phi(\mathcal{I}'_{f(a)})$ is a subtree of $\psi(\mathcal{I}'_{f(a)})$, the claim holds. If it is $\mathcal{I}$-bearer, then $l_1(root(\phi(\mathcal{I}_a))) \geq m + 1$. So, by the gap condition, $\mathcal{I}'_{f(a)}$ and all of its ancestors (except $\mathcal{I}'$) are also ancestor-bearers. Let $\mathcal{I}' = \mathcal{I}'_{a_r}, \mathcal{I}'_{a_{(r-1)}}, \ldots, \mathcal{I}'_{a_1}, \mathcal{I}'_{a_0} = \mathcal{I}'_{f(a)}, r \geq 1$ be the subsequence from the lineage-path starting with $\mathcal{I}'$ and ending with $\mathcal{I}'_{f(a)}$, such that $\mathcal{I}'_{a_i}$ is $\mathcal{I}'_{a_{i+1}}$-bearer, $0 \leq i \leq r - 1$. By the gap condition, we have $l_1(root(\phi(\mathcal{I}'_{a_i}))) \geq l_1(root(\phi(\mathcal{I}_a)))$ for each such $i$. Let $\tau = \psi(I(\{H^*(\mathcal{I}_a)\}))$ and $\tau_i = \psi(I(\{H^*(\mathcal{I}'_{a_i})\}))$. Since we have $\phi(\mathcal{I}_a) \leq_{KF} \phi(\mathcal{I}'_{f(a)})$, to attain our claim, it suffices to show that $\tau \leq_{KF} \tau_0$, because we join $Tree(\tau)_{(2m+4,0)}$ to $root(\phi(\mathcal{I}_a))$ to get $\psi(\mathcal{I}_a)$ (See Def. 9 (i)). We construct $\psi(\mathcal{I}'_{f(a)})$ similarly using $\tau_0$. Note that, in $\tau_0$ a path $P^*$ of length $2r - 1$ exists, starting from $root(\tau_0)$, with edges labeled by alternating $l_1(root(\phi(\mathcal{I}'_{a_i}))) - (m + 1)$, and $2m + 4$, for $0 \leq i \leq r - 1$ (See Def. 9 (ii)). But then $l_1(root(\tau)) = l_1(root(\phi(\mathcal{I}_a))) - (m+1)$, and so we can map $root(\tau)$ to

$root(\tau_{a_{r-1}})$, because the gap condition is satisfied and $\tau$ is identical to $\psi(\mathcal{I})$ (except for its root label $\alpha$) and $\tau_{a_{r-1}}$ is identical to $\psi(\mathcal{I}')$ (except for its root label $\alpha'$) and we know $\alpha \leq \alpha'$ from the gap condition. The claim follows and hence by the choice of $\mathcal{I}$ we have $\mathcal{I}_a \subseteq \mathcal{I}'_{f(a)}$.

By (P3), for every ancestor $\mathcal{I}''$ of $\mathcal{I}'_{f(a)}$ we have $\alpha(\mathcal{I}'') = m$. Hence using (P1) recursively, we have $\mathcal{I}' \supset \mathcal{I}'_{a'} \supset \ldots \supset \mathcal{I}'' \supset \ldots \supset \mathcal{I}'_{f(a)}$, where $\mathcal{I}'_{a'}$ in $H'_i$. Hence, $\mathcal{I}_a \subseteq \mathcal{I}'_{a'}$ as needed. □

**Proof.** **(of Theorem 4)** Let $X = \mathcal{I}_1, \mathcal{I}_2, \ldots$ be any infinite sequence of $L^n\mathcal{F}(m, Q)$. The case $n = 1$ is obvious. Inductively, let $\bar{\mathcal{I}}_i = \{T \in \mathcal{F}(2^n(m+4) - 4, (\mathbb{N}, L^n Q)) : T \leq_{KF} \psi(\mathcal{I}), \mathcal{I} \in \mathcal{I}_i\}, i \geq 1$. Let $\psi(\bar{\mathcal{I}}_1), \psi(\bar{\mathcal{I}}_2), \cdots$ be the structure-tree sequence in $\mathcal{F}(2^{(n+1)}(m+4) - 4, (\mathbb{N}, L^{n+1}Q))$. By Friedman' Theorem, we have $i, j, i < j$ such that $\psi(\bar{\mathcal{I}}_i) \leq_{KF} \psi(\bar{\mathcal{I}}_j)$ and by the Lifting Lemma, $\bar{\mathcal{I}}_i \subseteq \bar{\mathcal{I}}_j$. Now let $\mathcal{I} \in \mathcal{I}_i$. Then $\psi(\mathcal{I})$ and $\psi(\mathcal{I}')$ for some $\mathcal{I}' \in \mathcal{I}_j$, are in $\bar{\mathcal{I}}_j$ such that $\psi(\mathcal{I}) \leq_{KF} \psi(\mathcal{I}')$. This lifts to $\mathcal{I} \subseteq \mathcal{I}'$. Since $\mathcal{I}_j$ is closed under $\subseteq$, we have $\mathcal{I} \in \mathcal{I}_j$. Hence $\mathcal{I}_i \subseteq \mathcal{I}_j$. So, $X$ is good.□

In our next paper [6], we will consider iterated ideals beyond the finite case and obtain a new proof of [4].

**Acknowledgment** A first talk on this result was given in 2006 at a conference in honor of the 60th birth day of J. Nešetřil at Charles University, Prague, and I am thankful to Charles University KAM/ITI for providing ample research environment.

## References

[1] H. Friedman, N. Robertson, P. Seymour, The metamathematics of graph minor theorem. In Logic and Combinatorics (Arata, CA, 1985) (65) Contemporary Math., 229-261, AMS, Providence, RI 1987.

[2] J. Kruskal, Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. Trans. American Mathematical Society, 95:210-223, 1960

[3] C.St.J.A. Nash-Williams, On well-quasi-ordering lower sets of finite trees. Proc. Cambridge Phil. Soc., 60:369-384, 1964.

[4] C.St.J.A. Nash-Williams, On better-quasi-ordering transfinite Sequences. Proc. Cambridge Phil. Soc., 64:273-2090, 1968.

[5] Y. Nigussie, Structural descriptions of Kruskal-Friedman ideals of finite trees, (Submitted)

[6] Y. Nigussie, On better-quasi-ordering transfinite sequences, a new proof, (in preparation)

[7] Y. Nigussie, N. Robertson, On Structural Description of lower Ideals of Tress, Journal of Graph Theory, 50(2005), 3 221-230.

[8] N.Robertson, (Graph Theory Lectures, 1997 OSU, Columbus)