Minimal Universal and Dense Minor Closed Classes

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Abstract

We study the homomorphism (coloring) order induced on minor closed classes. In [8], the minor closed class $\mathcal{P}$ of directed paths is shown to be universal and in [9], $\mathcal{P}$ is shown to contain a dense subset. In this note we prove $\mathcal{P}$ is a unique minimal class of oriented graphs which is both universal and dense. Moreover, we show a dichotomy result for any minor closed class $\mathcal{K}$ of directed trees: $\mathcal{K}$ is either universal or it is wqo. We also prove structure theorems about series-parallel graphs (SPG), in an attempt to determine the minimal universal and dense minor closed classes of undirected graphs. We show non-existence of a universal classes in certain subclasses of SPG. Also for basic graphs in the class of SPG, we show there is a linear time algorithm that decides if such a graph is core or not. We also give constructive description of arbitrary 2-connected graphs in SPG.

1 Introduction

In this paper we consider minor closed classes of directed trees and finite series-parallel graphs (SPG). (Recall that a graph G is a series-parallel graph if and only if K_4 is not a minor of G.) We consider minor closed subclasses of SPG and prove results from the point of view of universality and density of related partial order. We introduce some relevant notions first:
Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency. That is, $f(u)f(v) \in E(G')$ whenever $uv \in E(G)$. We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G < G'$ means $G \leq G' \not\leq G$ whereas $G \sim G'$ means $G \leq G' \leq G$. If $G \sim G'$, we say G and G' are *hom-equivalent*. The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G. This will be denoted by $H \subseteq G$. See [5]

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for graphs and their homomorphisms.

We say a graph G is a *minor* of G', written $G \preceq G'$, if G can be obtained from G' by deleting and contracting edges of G'. The class of graphs that is closed under the minor inequality is called *a minor closed class.* It is well known [11] that graphs are well-quasi-ordered (wqo) under the minor relation $\preceq$. However, this is not true for the homomorphism relation $\leq$. On the contrary, it is known that even simple classes such as the class of all directed paths, or for the undirected case, a proper subclass of SPG induces (under $\leq$) a 'universal' partial order (see [7]). Recall that class $\mathcal{K}$ of finite graphs is said to induce a *universal* partial order, [2], [3], if $\mathcal{K}$ together with the homomorphism order $\leq$ represents any countable partial order.

It is indeed surprising that such simple classes of graphs can be used to represent universality (or density). A related natural question arises: Does there exist even a simpler minor closed class of graphs which is universal (or dense)? That is, could we find a proper minor closed subclass or a minor closed class not containing the class given in [7], [8] and still preserve universality and density? In the next section we give a negative answer to these questions for directed trees.
In the consecutive sections we study SPG graphs (recall these are undirected K_4-minor-free graphs). In Section 3 we exhibit the class of graphs that forbids K_4 and the graph we call "3-pentagon" (see Figure 1) and prove that it induces a linear order. Particularly, we show the core of every graph G forbidding the two graphs as a minor is its smallest odd-cycle. In Section 4 we study a class of graphs we call "ear-faces" that are basic structural elements of graphs in SPG.
We prove ear-faces do not form an infinite anti-chain. We deduce that a universal minor closed class consisting of ear-faces does not exist. In Section 5 we show every 2-connected graph in SPG is a finite 'ear-face recursion'. In Section 6 we include some remarks and open problems.

2 Unique minimal universal minor closed class of directed trees

We study directed trees in this section. In follows from [8] that any minor closed class $\mathcal{K}$ of directed trees containing the set of finite directed paths $\mathcal{P}$ is universal. $\mathcal{P}$ is also known to be dense by the result of [9]. We prove here that the condition $\mathcal{P} \subseteq \mathcal{K}$ is also necessary: If $\mathcal{P} \not\subseteq \mathcal{K}$, then $\mathcal{K}$ is neither universal nor it contains a dense subset. Moreover, we show if $\mathcal{K}$ forbids any path, then $\mathcal{K}$ is wqo by the homomorphism order.
The main result of this section (Theorem **??**) is proved using Higman's Lemma [6], and additional two easy (and perhaps well known) lemmas below.

Lemma 1 *(Higman) Let $(Q, \leq_Q)$ be a well-quasi-order (wqo). Then $(Q^{<\omega}, \leq)$ is wqo, where $Q^{<\omega}$ is the set of finite sequences of Q and for $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_m)$, $A, B \in Q^{<\omega}$, $A \leq B$ if there is an increasing map $f : \{0, 1, \ldots, n\} \to \{0, 1, , \ldots, m\}$, such that $a_i \leq_Q b_{f(i)}$ for $i = 1, 2, \ldots, n$.*

Lemma 2 *The set $\mathcal{T}_h$ of all directed trees of height at most $h \geq 0$ is wqo in the subtree order $\subseteq$.*

Proof. We prove a stronger result by assuming that trees are rooted. Clearly, there is no infinite strictly descending chain of finite subtrees by subtree inclusion. We show there is no infinite anti-chain either. For $h = 0$, we only have a root vertex and the inclusion $\subseteq$ preserves root. We proceed by induction. Let $h > 0$ be minimal such that $X = \{T_i\}_{i=1}^{\infty}$ is an infinite anti-chain in $\mathcal{T}_h$. For each i, delete the root of T_i and let $F_i = [(T_1^i, x_1^i), (T_2^i, x_2^i), \ldots, (T_{k_i}^i, x_{k_i}^i)]$ be the forest of rooted trees T_j^i we obtain in $\mathcal{T}_{h-1}$ and $x_i \in \{in, out\}$ indicating whether an in or out edge connects the root of T_i with the root of T_j^i. By induction assumption and by Lemma **??**, there exist $i, j, i < j$, such that $F_i \subseteq^{<\omega} F_j$ That is, for some $i < j$, there is an injective function $f : \{1, 2, \ldots, k_i\} \to \{1, 2, \ldots, k_j\}$ such that for all $k = 1, 2, \ldots, k_i$, $T_k^i \subseteq T_{f(k)}^j$ and $x_k^i = x_{f(k)}^j$. This implies $T_i \subseteq T_j$, a contradiction. □

Lemma 3 *The set $\mathcal{T}_h$ of all directed trees of height at most $h \geq 0$ contains no infinite strictly increasing sequence in the homomorphism order $\leq$.*

Proof. Assume the contrary and let $h > 0$ be minimal such that $Y = \{T_i\}_{i=1}^{\infty}$ is an infinite strictly increasing sequence of core trees in $\mathcal{T}_h$. With same notation as in Lemma **??**, let $F_i = [(T_1^i, x_1^i), (T_2^i, x_2^i), \ldots, (T_{k_i}^i, x_{k_i}^i)]$ be what we obtain by removing the root of T_i. We may assume Y is monotone under $\subseteq$, by Lemma **??**. Moreover, we can assume for all i, j, $x_k^i = x_k^j$, $1 \leq k \leq k_i$. By induction assumption, there is no infinite strictly increasing sequence under $\leq$ in $\{T_j^i\}_{i=1}^{\infty}$, since $T_j^i \in \mathcal{T}_{h-1}$. Hence, for all $i \geq 1$ and $k = 1, 2, \ldots, k_i$, T_k^i is isomorphic to T_k^j, for all $j > i$, because T_j is a core. It follows that for all $j > 1$ and for $k > k_{(j-1)}$, $T_k^j \not\leq T_l^i$ or $x_k^j \neq x_l^i$ if $i < j, 1 \leq l \leq k_i$, for otherwise $T_k^j \leq T_l^j$ and $x_k^j = x_l^j$, and so T_j maps to it proper subtree, contrary to T_j being a core. Note that we can assume $\{k_i\}_{i=1}^{\infty}$ is strictly increasing. Consider the diagonal sequence $\{T_{k_i}^i, x_{k_i}^i\}_{i=1}^{\infty}$. By Lemma **??**, $\{T_{k_i}^i\}_{i=1}^{\infty} = \mathbb{T}$ can be assumed to be monotone increasing under $\subseteq$. Then $\mathbb{T}$ is also monotone increasing under $\leq$. We may also choose a subsequence such that $x_{k_i}^i = x_{k_j}^j$ for all $i \neq j$. By induction, $\mathbb{T}$ can not be strictly increasing under $\leq$. Hence we find $i < j$ such that $T_{k_i}^i \sim T_{k_j}^j$ and $x_{k_i}^i = x_{k_j}^j$, a contradiction. □

We note that without the condition of bounded height, the sequence we obtain from the set of all finite paths with all edges directed forward is an infinite strictly increasing sequence under $\leq$ inclusion. It is also not hard to construct infinite anti-chains under $\subseteq$ and even under $\leq$ if we have unbounded height tree sequence.

We give a formal definition of a directed path (an element of $\mathcal{P}$) and of a 'somewhere dense' set before proving the next theorem. We say P is a *directed path* if P is a directed graph (V, E) with $V = \{v_0, v_1, \ldots, v_m\}$ where for every $i = 1, 2, \ldots, m$ either $v_{i-1}v_i \in E$ or $v_i v_{i-1} \in E$ (but not both), and there are no other edges. We say an edge $v_{i-1}v_i$ is *directed forward* and $v_i v_{i-1}$ is *directed backward*. The *length* of P is the number of edges in P. For any directed tree T, let *height*(T) denote the maximum length of a directed path-subtree in T.

A class of graphs $\mathcal{C}$ is *dense* if $G, G' \in C$ and $G < G'$ implies there exists a $G'' \in \mathcal{C}$, such that $G < G'' < G'$. A class of graphs $\mathcal{K}$, is said to be *somewhere dense* if it contains a dense subset. Note that by definition if $\mathcal{K}$ is somewhere dense, then it is infinite.

Theorem 4 *Let $\mathcal{K}$ be a minor closed class of directed trees and let $\mathcal{P}$ be the set of all directed paths. Then, the following are equivalent:*
(i) $\mathcal{P} \subseteq \mathcal{K}$,
(ii) $\mathcal{K}$ is universal,
(iii) $\mathcal{K}$ is somewhere dense,
(iv) $\mathcal{K}$ is not wqo.

Proof. We note that $(i) \to (ii)$ and $(i) \to (iii)$ follow from [8] and [9], respectively. $(ii) \to (iv)$ is obvious. Hence $(i) \to (ii), (iii)$ and (iv). Conversely, assume $\mathcal{P} \not\subseteq \mathcal{K}$. Let P_r be a directed path of length $r \geq 0$ such that $P_r \notin \mathcal{K}$. We show each statement (ii), (iii) and (iv) fails.
To prove that $\mathcal{K}$ is not universal it suffices to show $\mathcal{K}$ contains no infinite anti-chain of directed trees under $\leq$. Suppose $X = \{T_i\}_{i=1}^{\infty}$, is an infinite anti-chain of directed trees. Since $T_1 \not\leq T_i$, for all $i > 1$, there exists a natural number M, $1 \leq M < height(T_1)$ such that any path-subtree P in T_i with all edges directed forward or all directed backward has length at most M. (For otherwise, for some i there is a path-subtree P of T_i with more than $height(T_1)$ one-direction edges, and we get $T_1 \leq P \leq T_i$, a contradiction.) Also there exists a natural number N, $1 \leq N < 2r$ such that, for all i, any path-subtree P in T_i has number of changes in direction of its edges, that is, from forward to backward and vice versa at most N times. Otherwise by contracting P at most r times we obtain $P_r \preceq P \preceq T_i$, contrary to assumption that $P_r \notin \mathcal{K}$. Hence, for all i, we have $height(T_i) \leq MN$. By Lemma **??**, $T_i \subseteq T_j$ for some $i < j$. This implies $T_i \leq T_j$, a contradiction. Hence no infinite anti-chain exists and so $\mathcal{K}$ is not universal.

Next, we show the stronger implication that $\mathcal{K}$ is wqo, by showing there is no infinite strictly decreasing sequence either. Indeed, if we have an infinite strictly descending chain $Y = \{T_i\}_{i=1}^{\infty}$, then we deduce Y is a bounded height tree sequence because $T_1 \not\leq T_i$ for all $i > 1$ and $P_r \notin \mathcal{K}$. By Lemma **??**, Y is wqo by $\subseteq$. Hence, we find $i < j$, such that $T_i \sim T_j$, a contradiction.

To see that $\mathcal{K}$ is nowhere dense, assume the contrary and for some $\mathcal{S} \subseteq \mathcal{K}$ pick $T, T' \in \mathcal{S}$, where $T' < T$. Then we can find an infinite chain $T' < T_1 < T_2 < \ldots < T$ in S. Since $T \not\leq T_i$ for all $i > 1$ and since $P_r \notin \mathcal{K}$, we deduce similarly we have a bounded height tree sequence, contrary to Lemma **??**. □

3 A total order by forbidding $3\mathbb{P}$ as a minor

For the remaining part of this paper we examine undirected graphs of SPG. Several universal classes that are proper subclasses of SPG were found in [7] using K_1-concatenations of core graphs. The core-graphs are constructed by subdividing the

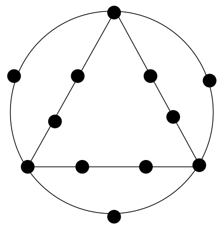

Figure 1: The 3-pentagon, $3\mathbb{P}$.

edges of the dual of $K_{2,3}$ a finite number of times. There is exactly one minor minimal core-graph which we call *3-pentagon* that is constructed by this procedure. The *3-pentagon*, denoted by $3\mathbb{P}$, is depicted in Figure 1. Since we are interested in finding the minimal universal minor closed classes, we consider in particular this unique minor-minimal graph. Interestingly any minor closed class in SPG that forbids it has a linear order by homomorphism. We prove this fact in this section.

It follows that any subclass of SPG forbidding $3\mathbb{P}$ is far from being universal, since it can not represent any two incomparable elements of a given partial order. However, we can use clique-concatenations of $3\mathbb{P}$ to construct a universal class. In Section 6 we describe in greater detail how a universal class of undirected graphs is obtained by clique concatenations of core-graphs such as $3\mathbb{P}$. One may ask why, if $3\mathbb{P}$ is a unique minor-minimal core-graph that can be used to attain universality, do we not obtain a unique minimal universal class of SPG as in the case of directed trees. This is due to the fact that there is more than one way of concatenating $3\mathbb{P}$ and each of them gives us a universal class which is incomparable to the others by the subset order. Therefore we know we have several minimal universal classes.
Are there minimal universal classes other than classes obtained by clique concatenation? The result of this and the next sections are basic structural results that can be used as tools to answer this question.
For convenience, we introduce a few more definition and notation. Let $\mathcal{K}$ be a minor closed class of graphs, and let $\mathcal{F}$ be a finite set of graphs. We write $\mathcal{K}/\mathcal{F}$ to denote the set of all graphs in $\mathcal{K}$ not containing any element of $\mathcal{F}$ as a minor. We denote the set of all graphs by $\mathcal{G}$. The following is a restatement of a well known fact that all SPG graphs are 3-colorable.

Theorem 5 *If $G \in \mathcal{G}/\{K_4\}$, then $G \leq K_3$.*

Thus, any graph in SPG that contains K_3 subgraph is hom-equivalent to K_3.

Let G be a graph. A *thread* in G, is a path $P \subseteq G$ such that the two endpoints of P have degree at least 3 and all other vertices of P are degree 2 in G. We shall often use the fact that if P and P' are two edge-disjoint paths in G with common end points such that P is a thread and if the lengths of P and P' have same parity such that P has at least the same length as P', then there is a homomorphism that maps P to P' sending the two ends of P to the two ends of P'. Such a homomorphism is said to *fold* P to P'. Assuming G is 2-connected and simple, let G^s denote the graph we obtain from G by “smoothing” all degree 2 vertices of G. Thus, G^s is

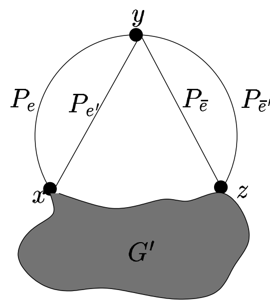

Figure 2: Unavoidable configuration for a minimal counterexample G

2-connected and has minimum degree 3, if G is not a cycle. For each edge e of G^s, let P_e denote the thread of G represented by e in G^s, and let l_e denote the length of P_e.

Lemma 6 *(***Edge folding lemma***) Let $G \in \mathcal{G}/\{K_4\}$ be of odd-girth $2k+1$ and let e, e' be parallel edges in G^s, with common end vertices x, y. If G is not homomorphic to a strictly smaller graph of the same odd girth, then $l_e + l_{e'} = 2k+1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k+1$ containing both x and y.*

Proof. Assume $l_e \leq l_{e'}$. If l_e and l_e' have same parity, then $P_{e'}$ can be folded to P_e to obtain a strictly smaller graph H of same odd girth hom-equivalent to G, contrary to assumption. Hence $P_e \cup P_{e'}$ is an odd cycle of length $l_e + l_{e'} \geq 2k+1$. Suppose $l_e + l_{e'} > 2k+1$. Let x_1, x_2, x_3 be three consecutive vertices of $P_{e'}$, and let G' be obtained by identifying x_1 and x_3. By the choice of G, G' must have odd-girth less than $2k+1$, because $G \leq G'$ and $|V(G)| > |V(G')|$. This implies $P_{e'}$ is contained in a cycle of length $2k+1$. Hence there is a path P of G connecting x and y with length $2k+1-l_{e'}$. But then, P and P_e have same parity and so P_e can be folded to P, contrary to G being minimal. So $l_e + l_{e'} = 2k+1$.
We also note that, if there is another cycle C of length $2k+1$ containing both x and y, then there is a path P distinct from $P_e, P_{e'}$, of length l_e or $l_{e'}$ connecting x and y, where there length of P is l_e or $l_{e'}$. Hence P_e or $P_{e'}$ can be folded to P, a contradiction. The result follows. □

Theorem 7 *Let $G \in \mathcal{G}/\{K_4, 3\mathbb{P}\}$. Then G is hom-equivalent to its smallest odd cycle.*

Proof. Let $G \in \mathcal{G}/\{K_4, 3\mathbb{P}\}$ be of odd-girth g and assume to the contrary that $G \not\leq C_g$ is a minimal counterexample. By Theorem **??**, $g > 3$. Note that G has no clique-separation, for otherwise we can separate G into strictly smaller graphs G_1 and G_2 and the mapping of each G_i to C_g can be extended to mapping of G to C_g. The graph G^s has parallel edges of multiplicity at most two. Let G^* be the graph obtained from G^s by identifying each parallel edge pair e, e'. Since G is not hom-equivalent to a cycle, we observe that $|V(G^*)| > 2$. Hence G^* is simple and 2-connected. Also note that $G^* \in \mathcal{G}/K_4$ and thus let v be a vertex of degree 2 in G^* ($d_{G^*}(v) = 2$). We deduce that $d_G(v) = d_{G^s}(v) = 3$ or 4. Assume first $d_G(v) = d_{G^s}(v) = 3$ and the 3 edges incident to v are $e, e', \bar{e}$, where e, e' are parallel edges. Assume $\{v,w\} = \bar{e} \in E(G)$. By Lemma **??**, $\bar{e}$ is not in any cycle of length g in G. Then, we identify w with w', where w' is a neighbor of v in $V(P_{e'})$ and obtain a graph G' of odd girth g, such that $G \le G'$ and $|V(G)| > |V(G')|$, contrary to the choice of G. Secondly assume $d_G(v) = d_{G^s}(v) = 4$. Then v in G^s is incident with two pairs of parallel edges e, e' and $\bar{e}, \bar{e}'$. The structure of G with the desired configuration is depicted in Figure 2.

Now the graph G' contains an odd cycle C, for otherwise G' maps to K_2, and hence G maps to C_g, a contradiction. Since $g > 3$, we note $C_5 \preceq C$. Since G is 2-connected there are two edge disjoint paths P and P' connecting C with x and z. Since G has no clique-cut it is clear now that $3\mathbb{P}$ is a minor of G, contrary to assumption, and the result follows. □

Corollary 8 *Let $\mathcal{K} \subseteq SPG$ be a minor closed class. If $\mathcal{K}$ forbids any minor of $3\mathbb{P}$, then $\mathcal{K}$ is totally ordered and hence non-universal.*

4 Homomorphism anti-chains of ear-faces

In this section we prove that the class of graphs we call ear-faces has no infinite anti-chain. We deduce that the class of ear-faces and their minors do not contain any universal class.

Let $C_{2,k}$, for $k \ge 3$ denote the dual graph of the complete-bipartite graph $K_{2,k}$. We say that a graph G is a *k-ear-face* if G is obtained from $C_{2,k}$ by subdividing some of the edges of $C_{2,k}$ a finite number of times and by folding some parallel threads of same parity. In other words, we obtain a k-ear-face G from a cycle C_k by doubling some (at least one) edges of C_k and by subdividing the edges. Hence any k-ear-face has at least three threads. We say G is an *ear-face* when the integer k is irrelevant. An *ear* of an ear-face consists of one or two threads. Recall that a thread has exactly two vertices of degree at least 3.
An example of a 3-ear-face is $3\mathbb{P}$ in Figure 1. We can represent $3\mathbb{P}$ by a vector $([2,3],[2,3],[2,3])$, indicating the even and odd lengths of the three ears of $3\mathbb{P}$. In general, for any $k \ge 2$ we may represent an ear-face G by $(E_1, E_2, \ldots, E_k)$, where each $E_i = [\epsilon_i, \delta_i]$ is an ear of G consisting of an even and odd thread of length ϵ_i and δ_i, respectively. The integer ϵ_i is the minimum even length of the threads in E_i. We let $\epsilon_i = \infty$ if E_i has no even length thread. The integer δ_i is defined similarly for odd length. If $\epsilon_i = \infty$ (or $\delta_i = \infty$), then E_i, is called a *diminished-ear*. For example if we delete a thread of length 2 from $3\mathbb{P}$ we obtain a 3-ear-face $G' = ([\infty,3],[2,3],[2,3])$. Instead if we delete a thread of length 3 we obtain a 3-ear-face $G'' = ([2,\infty],[2.3],[2,3])$. The diminished-ear of G' has odd length whereas G'' has even.

One reason we study ear-faces is their simplicity. Ear-faces are the simplest form of graphs in SPG where we find a homomorphism anti-chain. A second and a fundamental reason is that any 2-connected graph in SPG is in fact a 'recursive ear-face'. We give a formal definition of this concept in the next section.

We also investigate an algorithmic aspect of ear-face graphs. Recognizing a core of a graph in a class of k-colorable graph, for $k \geq 3$ is shown to be an NP-complete problem in [4]. SPG is a proper subset of the class of 3-colorable graphs. It is not known if the same problem becomes easy for SPG. However, for an ear-face G we show that there is a linear time algorithm deciding if G is a core or not.
The following lemma shows that the length of an anti-chain of ear-faces can grow exponential in the size of the graphs in the anti-chain.

Lemma 9 *For any integer $N \geq 2$, and odd composite g, $g \neq 9$, there exists an anti-chain of length 2^N, consisting of odd-girth g ear-faces of size $O(N)$.*

Proof. Let $g = pq$ be an odd composite, $3 \leq p \leq q$. If $g \neq 9$, then $g - 2p \neq p$. Let $X = G_1, G_2, \ldots, G_{2^{N+1}}$, where for all i, the vector representation of G_i is $V_i = (E_1^i, E_2^i, \ldots, E_{N+2}^i)$, $E_1^i = [\infty, 1]$ and $E_j^i \in \{[g-p, p], [2p, g-2p]\}, j = 2, 3, \ldots, N+2$. Note that under any homomorphism a cycle maps to a cycle and so if $G_i \leq G_j$, then no edge of an ear of G_i is mapped to the edge of E_1^j and so the diminished-ear E_1^i must be mapped to the diminished-ear E_1^j. Each G_i and G_j are $(N+2)$-ear-faces, and so the ears of G_i must map to ears of G_j injectively. Therefore, if $G_i \leq G_j$, then either $V_i = V_j$ or $V_j = (E_{N+2}^i, E_{N+1}^i, \ldots, E_1^i)$ (the ears of V_i listed in reverse order). Hence we have a $(2^{N+1})/2$ length anti-chain. □

Note also that for any $\alpha \geq 2$, we can find an anti-chain of length α^N if g is large enough. Despite of this result we show (see Theorem 17) that ear-faces form no infinite anti-chain. This is easy to see if the number of ears in a sequence is bounded.

Lemma 10 *Let $X = G_1, G_2, \ldots$ be an infinite sequence of ear-faces such that, for all i, G_i is a k_i-ear-face, and $\{k_i\}_{i=1}^{\infty}$ is bounded. Then $G_i \leq G_j$, for some $i \neq j$.*

Proof. Let $V = \{V_i\}_{i=1}^{\infty}$, where $V_i = ([\epsilon_1^i, \delta_1^i], [\epsilon_2^i, \delta_2^i], \ldots, [\epsilon_k^i, \delta_{k_i}^i])$ is the vector representation of G_i. Since $\{k_i\}_{i=1}^{\infty}$ is bounded, by taking subsequence, we may assume $k_i = k$, for all i. Integers are wqo under the usual integer inequality and so is the finite Cartesian product of integers and the k-Cartesian product of the products, ordered by co-ordinatewise corresponding inequality. Hence for some i, j where $i < j$, we have $V_i \leq V_j$. This implies $G_j \leq G_i$, as needed. □

Non-existence of infinite anti-chains holds even if the ear sequence is unbounded. First we show that in any anti-chain of ear-faces the odd-girth is bounded by constant, using a corollary of the next lemma.

Lemma 11 (Ear-folding lemma) *Let G be an ear-face of odd-girth $g > 3$ with k ears, $k > 3$. Then there exists an ear-face G' of same odd-girth and $k-1$ ears such that $G \leq G'$.*

Proof. Let $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], \ldots, [\epsilon_k, \delta_k])$ be a minimal counterexample ear-face of odd-girth g, with ears $E_i = [\epsilon_i, \delta_i], k \geq 4$ and $v_{i(i+1)} \in V(E_i) \cap V(E_{i+1})$ (assuming $k + 1 = 1$). Then by the choice of G, we deduce G has no diminished ears and that for all $i, \epsilon_i + \delta_i = g$. Suppose that for some i_0 (without loss of generality, we may

choose δ_{i_0} or ϵ_{i_0}) $\epsilon_{i_0} \leq \gamma_i$ for all i, where $\gamma_i \in \{\epsilon_i, \delta_i\}$ for $i = 1, 2, \ldots, k$. In addition we can assume that among the minimal even lengths, ϵ_{i_0} is chosen so that either ϵ_{i_0+1} or ϵ_{i_0-1} is as large as possible. By relabeling if necessary we can assume $i_0 = 1$. Taking the largest from $\delta_2, \epsilon_2, \delta_k$, and ϵ_k, we have by symmetry two cases:
Case 1: δ_2 is largest. Then $G' = ([\epsilon_2 + \epsilon_1, \delta_2 - \epsilon_1], [\epsilon_3, \delta_3], [\epsilon_4, \delta_4] \ldots, [\epsilon_k, \delta_k])$.
Case 2: ϵ_2 is largest. Then $G'' = ([\epsilon_2 - \epsilon_1, \delta_2 + \epsilon_1], [\epsilon_3, \delta_3], [\epsilon_4, \delta_4], \ldots, [\epsilon_k, \delta_k])$.
To see that $G \leq G'$, identify v_{k1} to the vertex $v \in V(E_2)$ that is of distance ϵ_1 from v_{12} and of distance $\delta_2 - \epsilon_1$ from v_{23}. To see that odd-girth(G')=g, we observe that every ear of G' has length g and that (since $k \geq 4$ and $\epsilon_1 \leq \gamma_3$), $\delta_2 - \epsilon_1 + \gamma_3 + \gamma_4 + \ldots + \gamma_k \geq \delta_2 + \gamma_k \geq \bar{\gamma}_k + \gamma_k = g$. Similar proof works for G''. Note also by the choice of ϵ_1, if case 2 holds, then we have $\epsilon 2 - \epsilon_1 > 0$. Hence both G' and G'' are (k-1)-ear-faces with the desired property. □

Corollary 12 *Let G be an ear-face of odd-girth $g > 3$. Then $G \leq C_{g'}$ where g' is odd and $g' \geq (g+1)/2$.*

Proof. By the Ear-folding lemma, we may assume $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], [\epsilon_3, \delta_3])$. If $\epsilon_i \leq (g-1)/2$ for some i, say $i = 1$, then $\delta_1 \geq (g+1)/2$. Let G' be the graph obtained by identifying v_{k1} with v_{12}. Clearly, $G \leq G'$, and from the odd-girth constraint for G we have odd-girth$(G') = g' = \min(\delta_1, \epsilon_2 + \delta_3, \delta_2 + \epsilon_3) \geq (g+1)/2$. On the other hand, suppose $\epsilon_i > (g-1)/2$, for all i. Say $\delta_1 \leq \delta_3$. We have $G \leq G'$ where $G' = ([\delta_1 + \delta_2, \epsilon_2 - \delta_1], [\epsilon_3, \delta_3])$. Note that $g' = \epsilon_2 - \delta_1 + \delta_3 \geq \epsilon_2 \geq (g+1)/2$. By Theorem **??**, $G' \sim C_{g'}$. □

It follows from Corollary **??**, every anti-chain has a bounded odd-girth sequence. Another interesting corollary of Ear-folding lemma is the fact that $3\mathbb{P}$ is a maximum of the class of all triangle-free ear-faces.

Corollary 13 *$G \leq 3\mathbb{P}$ for any triangle-free ear-face G.*

Before proving non-existence of an infinite anti-chain for the general case, we give necessary and sufficient conditions for a graph to be a core-ear-face. This allows us to give a linear time algorithm that decides, even computes the core of an ear-face. The algorithm is implicitly used in the proof of the main result of this section, Theorem **??**.

Lemma 14 *Let G be a k-ear-face of odd-girth g. Then G is a core if and only if every ear of G is an odd cycle and $G \not\leq C_g$.*

Proof. If G is a core, then clearly G has no ear of even length and $G \not\leq C_g$. Conversely, assume the two conditions hold. Suppose $G' \subseteq G$ and $G \leq G'$ and that G' is the core of G. By transitivity of $\leq$, we deduce $G' \not\leq C_g$, and so G' is a k'-ear-face, for k' satisfying $3 \leq k' \leq k$. Since G' is an induced subgraph it follows that $k = k'$ and so the ears of G are mapped to ears of G' injectively. Since every ear of G is an odd cycle, the homomorphism from G to G' restricted to each ear is an isomorphism, (for otherwise G' would have a shorter ear, which is not an induced subgraph). It follows that G is isomorphic to G'. □

Using Lemma **??** and the following observations, we give a linear time "labeling" algorithm *Core-Ear-Face* (CEF) to decide whether a k-ear-face is core or not. If g is an odd-girth of a core-ear-face of G then it follows that G has the following properties:

- For each ear $E = [\epsilon, \delta]$ of G, $\epsilon + \delta \geq g$, while for some ear we have equality. (*)
- For any ear $E = [\epsilon, \delta]$ of G at least one of ϵ or δ is less than g (if $\epsilon \geq g$ and $\delta \geq g$, then $G \leq C_g$). (**)
- Any odd cycle C of G which is not an ear has length larger than g. (***)

Algorithm 15 Core-Ear-Face (CEF)
Input: k-ear-face $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], \ldots, [\epsilon_k, \delta_k])$, $k \geq 3$, $\epsilon_1 + \delta_1 = g$, $\epsilon_i + \delta_i \neq 0$ for all i:
1. verify (*) and (**)
2. verify (***) in two ways:
(2.1) let $s = \Sigma_{i=1}^{k} \gamma_i$, where $\gamma_i = \min(\epsilon_i, \delta_i)$. If s is odd and $s \leq g$ then G is NOT-CORE
(2.2) for $i = 1, 2, \ldots, k$ let $s_i = \zeta_i + \Sigma_{j \neq i} \gamma_i$, where $\zeta_i = \max(\epsilon_i, \delta_i)$ and $\gamma_j = \min(\epsilon_j, \delta_j)$. If $s_i \leq g$, then G is NOT-CORE. (Note that for any two maximums $\zeta_i, \zeta_j, i \neq j$, we have $\zeta_i + \zeta_j + \gamma_l > g$, any $l, 1 \leq l \leq k$
3. label v_{k1} by $L_0 = \{0\}$.
4. for $i = 1, \ldots, k-1$, label $v_{i(i+1)}$ by $L_i = L_{i-1} + \epsilon_i \cup L_{i-1} + \delta_i$, (where $L + x = \{l + x$ (modulo g) : $l \in L\}$)
5. if $0 \in L_k = L_{k-1} + \epsilon_k \cup L_{k-1} + \delta_k$ then G is NOT-CORE. In all other instances G is a CORE.

The correctness of CEF follows form Lemma **??**, and one can easily verify that CEF runs in linear time in the size of G.
The following lemma on ordering of finite sequences of a finite set is due to A. Pór. We use it in proving non-existence of infinite anti-chains of ear-faces. Let $S_p \subseteq S = \{x_1, x_2, \ldots, x_g\}$ be a p-element subset of a finite set S, where $|S| = g$, and $1 \leq p \leq g$. Let $W = (w_1, w_2, \ldots, w_n)$ and $W' = (w'_1, w'_2, \ldots, w'_{n'})$ be finite sequences of S_p. Let $W \circ W'$ denote the sequence $(w_1, w_2, \ldots, w_n, w'_1, w'_2, \ldots, w'_{n'})$, the concatenation of W and W'. We write $W \leq_g W'$ if $W' = W'_1 \circ w_1 \circ W'_2 \circ w_2 \circ \ldots W'_n \circ w_n \circ W'_{n+1}$, where for each i and for any $y \in S_p$, y appears in W'_i at least g times or zero times. Intuitively, $W \leq_g W'$ means, W is a subsequence of W' with many or no occurrence of each symbol. It can be seen that $\leq_g$ is a quasi-order. Denote by $(\mathbb{F}, \leq_g)$, the set of all finite sets ordered by $\leq_g$.

Lemma 16 *$(\mathbb{F}, \leq_g)$ is wqo.*

Proof. Let an infinite sequence $X = W_1, W_2, \ldots$ of words of $S_p \subseteq \{x_1, x_2, \ldots, x_g\}$ be given. Clearly there is no infinite strictly descending chain in X under $\leq_g$. To prove $W_i \leq_g W_j$, for some $i < j$, we induct on $|S_p|$. Suppose first $p = 1$. Then

each $W_i = (x_t, x_t, \ldots, x_t)$, repeated n_i times where $1 \leq t \leq g$. We can find $i < j$, such that $W_i = W_j$ if $\{n_i\}$ is bounded , and $n_j - n_i \geq g$ if unbounded. We have $W_i \leq_g W_j$.
Suppose next $p > 1$. Partition each word $W_i = P_1^i \circ x_{i_1} P_2^i \circ x_{i_2} \circ \ldots P_{m_i}^i \circ x_{i_{m_i}} \circ P_{r_i}^i$, such that each $y \in S_p$ appears at least g times, x_{i_j} appears exactly $g-1$ times and there exists $y \in S_p$ such that y appears in $P_{r_i}^i$ less than g times. If $m_i \geq 2m_1 + 2$ for some i, then $W_1 \leq_g W_i$. Hence we can assume that $\{m_i\}$ is bounded and by taking subsequence we let $m_i = m$ for all i.
Further partition P_j^i as $P_j^i = P_{j_1}^i \circ x_{i_j} \circ P_{j_2}^i \circ x_{i_j} \ldots \circ x_{i_j} \circ P_{j_g}^i$, where $j = 1, 2, \ldots, m$. Each $P_{j_l}^i$ is an S_{p-1}-element word and hence by induction we can recursively take subsequence to construct a monotone subsequence of X. □

Theorem 17 *There is no infinite anti-chain of ear-faces.*

Proof. Suppose $X = G_1, G_2, \ldots$ is an infinite anti-chain of core-ear-faces. For each $i \geq 1$, let $V_i = ([\epsilon_1^i, \delta_1^i], [\epsilon_2^i, \delta_2^i], \ldots, [\epsilon_{k_i}^i, \delta_{k_i}^i])$ be the vector representation of G_i and let g_i be the odd-girth of G_i. By Corollary **??**, $\{g_i\}$ is bounded and so we may assume $g_i = g$, for all i. By Lemma **??**, if $K = \{k_i\}$ is bounded we are done, so assume K is a strictly increasing sequence.
Let d_i be the number of diminished ears of G_i. Then $d_i < g - 1$, or else $G_i \leq C_g$, contrary to G_i being a core. Let c_i be the number of ears of G_i with length bigger than g. Once more $c_i < g - 1$, or else $G_i \leq C_g$. Hence $\{c_i\}$ and $\{d_i\}$ are bounded. Let c and d denote the common values by taking subsequence.
Partition each G_i in to $c+d+1$ parts $P_1, P_2, \ldots, P_{c+d+1}$, such that between each part there is a diminished-ear or a long-ear. There are finitely many ways of arranging these finite 'irregular-ears'. Hence may assume, y taking subsequence that for all i, if we contract all P_j^i of G_i to a vertex, then we obtain G_i', such that $G_1', G_2', \ldots$ are all isomorphic.
Now consider P_j^i. Each ear is of length g. We may represent each ear by $\epsilon_{j_l}^i$ since $\delta_{j_l}^i = g - \epsilon_{j_l}^i$. So P_j^i is a finite set of integers. Now we apply Lemma **??** for each $\{P_j^i\}_{i=1}^{\infty}$ $j = 1, 2, \ldots, c+d+1$ to obtain inclusion by $\leq_g$. This and the fact that G_i's are all isomorphic allows us to deduce $V_i \leq_g V_j$. We show now this implies $G_j \leq G_i$ by defining a map f from $V(G_j)$ to $V(G_i)$. Let f restricted to $V(G_j')$ preserve the isomorphism.
We use commutativity and associativity property of addition of integers modulo g. It suffices that each number appears in P_j^i either zero or more than $g-1$ times. We have the ears of G_i injectively contained by a mapping h as a subsequence in G_j. Between the ears $E_{h(i)}^j$ and $E_{h(i+1)}^j$, every ear that appears has length g. If an ear that appears is represented by ϵ, then there at least $m \geq g$ ears represented by ϵ. We can map all these ears to C_g by adding modulo g such that the vertices $v_{h(i)(h(i)+1)}$ and $v_{(h(i+1)-1)h(i+1)}$ are identified to a vertex. It follows $G_j \leq G_i$. □

Corollary 18 *Let $\mathcal{K} \subseteq SPG$ be a minor closed class consisting of ear-faces and their minors. Then $\mathcal{K}$ is non-universal.*

5 2-connected SPG core-graphs are recursive ear-faces.

We prove in this section that every 2-connected graph G in SPG can be constructed by a simple recursive procedure. We call this constructive procedure *ear-face recursion.* We define a *recursive ear-face* to be a graph $G = G_m, m \geq 1$, that can be obtained from a sequence of graphs $G_0, G_1, \ldots, G_m$. G_0 is a cycle and every edge of G_0 is *newly-created.* For $i > 0$, G_i is obtained from G_{i-1} by doubling at least one edges of the the newly-created-outer-cycle edges of G_{i-1} and by subdividing each parallel pair a finite number of times. For each i, the newly-created edges of G_i are defined to be exactly the outer-cycle edges obtained by the subdivisions of the parallel edges in the construction of G_i.
A typical structure of an arbitrary 2-connected graph G embedded in a plane is depicted schematically in Figure 3. In constructing G we start with the subgraph H that is highlighted by thick dark line in the figure. Each label "(i)" $i = 1, 2, \ldots, k$, on the outer cycle corresponds to a thread of G on the outer cycle. Note that H is an ear-face. We emphasize that the subgraph H is chosen so that the number of threads k on the outer cycle is as large as possible. Note that H is hom-equivalent to a subdivision of $C_{2,k}$. In other words, we take the graph $C_{2,k}$ and subdivide each parallel pair to be isomorphic to the corresponding ear in H. If the corresponding ear is a diminished-ear, we identify the parallel edges and subdivide. We obtain a graph isomorphic to H. From H, we can obtain G as follows:

- Let $u \in V(H)$ be such that u belongs to exactly one ear in H. (See the label u in Figure 3 inside the subgraph labeled by G_i^1). We claim we need not have any operation on u to construct G: Indeed if we add new edges to connect u to to a vertex $v \notin V(P)$, then we obtain a K_4-minor, contrary to assumption. If $v \in V(P), v \neq u$ we obtain an embedding of G with $k + 1$ ears on the outer-cycle, contrary to the choice of k. If $u = v$, then G is not 2-connected, a contradiction. The claim follows and hence no operation is required on vertices of H that belong to one ear.

- Let $v \in V(H)$ be such v a vertex of two ears. By the same argument we note that newly added edges can only connect v to another vertex $v' \in V(H)$ that belongs to two ears. Moreover, we claim a path of newly added edges from v to v' traverses at least two ears of H. For example in Figure 3, $m_1 \geq 2$. Otherwise both v and v' belong to the same ear E of H. Since k is maximal the newly added edges from v to v' for a thread P'. Then P' can be folded to either the even or the odd thread of E. Hence such operation is trivial and the claim follows.

- By the above two observations, we deduce $k_2 \leq k/2$, and $k_3 \leq k/4$ and so on. Hence $d \leq \log_2(k)$. It can easily be proved by induction now that by taking $G_0 = C_d$ we can arrive at a recursive ear-face G in $\log_2(k)$ steps of replacing outer-cycles edges by cycles.

From these considerations we obtain the following:

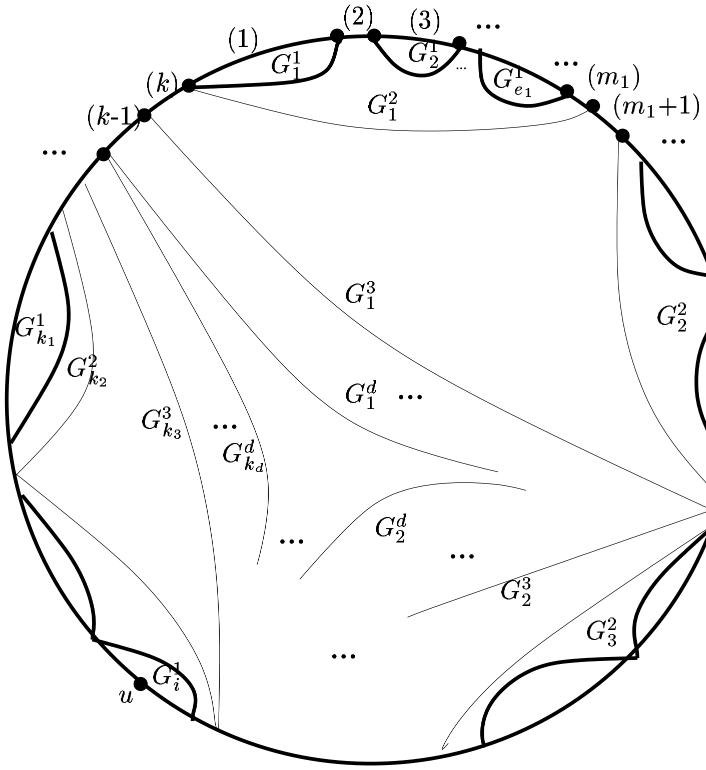

Figure 3: A general structure of a 2-connected graph in SPG

Theorem 19 *For every 2-connected graph $G \in SPG$, there is a recursive ear-face G' hom-equivalent to G. G' is constructible in $log_2(k)$ steps of replacing edges by cycles, where k is maximal for which H is a k-ear-face and $H \preceq G$.*

We remark here that if $G \in SPG$ is 1-separable, then G has a tree structure of its 2-connected components.

6 Concluding remarks and problems

Our goal is to determine all minimal universal minor-closed classes in SPG. In Section 4, we showed ear-faces can not form a universal set. We hope the structural description in Section 5, will help us to obtain a larger class of graphs which contains no infinite anti-chain and thus non-universal. Perhaps we may able to prove the necessity of unbounded concatenation using structural arguments.
Adopting the method in [8], we show in a forthcoming paper that clique-concatenation of core ear-faces (in particular $3\mathbb{P}$), we obtain several minimal universal minor-closed subclasses of SPG. The minimality of such subclasses of SPG is proved using the result of Section 2, and by associating each K_1-concatenation of the $3\mathbb{P}$ to a particular oriented path. We list all possible minimal ones we can construct, because we want to determine the remaining minor closed classes that do not contain any one in our list.

Without going in to technical details we give an outline of this association of a

K_1-concatenation of a core-ear-face G (for instance $3\mathbb{P}$) to a path P, that is shown [7] as follows: For each oriented path $P \in \mathcal{P}$ of length $n \geq 1$, we define a G-concatenation of length n (denoted by $P * (G, a, b)$ where $a, b \in V(G)$ in two steps. First, we take n isomorphic copies $G_1, G_2, \ldots, G_n$, of G and let a_i, b_i be the vertices of G_i corresponding to a and b. Then according to the orientations of the edges of P, we chose either a_i or b_i to be identified with either a_{i+1} or b_{i+1}.
Such a construction will allow us to deduce for any two paths P and P', we have $P \leq P'$ if and only if $P*(G, a, b) \leq P'*(G, a, b)$. Then the universality and minimality of the minor-closed class formed by the concatenation follows from universality and minimality of the class of directed paths $\mathcal{P}$ (see Theorem **??**). The property of somewhere density also can be proved similarly. Aside from this method, one may also ask is there another way of constructing a universal minor-closed class? Along this work, we present the questions below:

Problem 20 *Can we determine the set of all minimal universal minor-closed classes of SPG?*

Problem 21 *We have shown in Section 3, that forbidding K_4 and $3\mathbb{P}$ as minors induces a total order under homomorphism. Does there exist a finite set F of core-graphs such that forbidding $K_5, K_{3,3}$ and elements of $\mathcal{F}$ as minors results a total order under homomorphism? (Note that to avoid triviality, we must assume that if $G \in \mathcal{F}$ then G is minor-incomparable to K_5 and $K_{3,3}$. We may also assume no element of $\mathcal{F}$ is a cycle.)*

Problem 22 *Is there an efficient algorithm that decides if a graph is a core or not in SPG?*

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