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On a new reformulation of Hadwiger's conjecture

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Abstract

Assuming that every proper minor closed class of graphs contains a maximum with respect to the homomorphism order, we prove that such a maximum must be homomorphically equivalent to a complete graph. This proves that Hadwiger's conjecture is equivalent to saying that every minor closed class of graphs contains a maximum with respect to homomorphism order. Let $\mathscr{F}$ be a finite set of 2-connected graphs, and let $\mathscr{C}$ be the class of graphs with no minor from $\mathscr{F}$. We prove that if $\mathscr{C}$ has a maximum, then any maximum of $\mathscr{C}$ must be homomorphically equivalent to a complete graph. This is a special case of a conjecture of Nešetřil and Ossona de Mendez.

1. Introduction

Given two graphs, G and H, we say H is a minor of G if H can be obtained from G by a series of operations: contracting edges, deleting isolated vertices and deleting edges. The following conjecture, a generalization of the four colour conjecture introduced by Hadwiger, is one of the most outstanding open problems in graph theory.

Hadwiger's conjecture (*Hadwiger [2]*). *Every k-chromatic graph G contains the complete graph K_k as a minor.*

The conjecture is almost trivial for $k=1, 2$ and 3. For $k=4$, it is proved by Hadwiger and also by Dirac in [1]. For $k=5$, it is stronger than the four colour theorem. It was proved by Wagner that this case is actually equivalent to the four colour conjecture [9]. The case $k=6$ has also been proved to be equivalent to the four colour theorem by Robertson et al. in [8]. It remains open for $k \geqslant 7$ and has been a fruitful research area.

We say a class $\mathscr{C}$ of graphs is *minor closed* if for every graph G in $\mathscr{C}$ and every minor H of G, H is also in C. A minor closed class of graphs that consists of only a graph H and all of its minors is called *principal ideal* and will be denoted by $[H]$. We say $\mathscr{C}$ is a *proper* minor closed class of graphs if it is not the class of all graphs. The following theorem of Halin (and also Wagner) was one of the first advances toward Hadwiger's conjecture.

Theorem 1 (*Halin [3] and Wagner [10]*). *For every proper minor closed class $\mathscr{C}$ of graphs, there is an integer k such that each graph in $\mathscr{C}$ is k-colourable.*

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A reformulation of the Hadwiger's conjecture in terms of graph homomorphism and homomorphism order of graphs has been recently given in [6]. Given two graphs G and H, a *homomorphism* of G to H is an edge preserving mapping $f : V(G) \to V(H)$. That is to say, for every edge xy of G, $f(x)f(y)$ is an edge of H. The existence of a homomorphism of G to H is denoted by $G \to H$. This notation captures the classical vertex colouring problem of graphs because a graph G is k-colourable if and only if it admits a homomorphism to the complete graph K_k.

Graphs G and H are said to be homomorphically equivalent provided that each admits a homomorphism to the other. If G and H are homomorphically equivalent, then we write $G \sim H$. Using the notation of homomorphisms, we can define an order on the class of graphs by

$$G \preccurlyeq H \quad \text{if and only if } G \to H.$$

This homomorphism order, which is also called *colouring order*, is a quasi-order on the class of all graphs and is a partial order on a class of graphs for which no two members are homomorphically equivalent. An arbitrary class of graphs $\mathscr{C}$ is said to be *bounded* by a graph H if, for every graph G in the class $\mathscr{C}$, $G \preccurlyeq H$. Moreover, if H is also in $\mathscr{C}$, then H is called a *maximum* of $\mathscr{C}$.

Now using Theorem 1, one can see that Hadwiger's conjecture is equivalent to the conjunction of the following two conjectures. These two conjectures were introduced by Nešetřil and Ossona de Mendez in [6].

Conjecture 2. *Any bounded minor closed class of graphs has a maximum.*

Conjecture 3. *If a minor closed class of graphs $\mathscr{C}$ admits a maximum, then the maximum is homomorphically equivalent to a complete graph.*

Here we prove that Conjecture 3 is implied by Conjecture 2. In other words, we show that if every proper minor closed class of graphs contains a maximum, then the maximum of every minor closed class of graphs is homomorphically equivalent to a complete graph. This implies that Conjecture 2 is equivalent to Hadwiger's conjecture. We also prove Conjecture 3 for a special class of minor closed classes.

2. On the maximum of a bounded minor closed class

The next theorem shows that, in fact, a weaker form of Conjecture 2 implies Conjecture 3.

Theorem 4. *Suppose every principal ideal contains a maximum. Let $\mathscr{C}$ be any minor closed class of graphs with a maximum element H. Then H must be homomorphically equivalent to a complete graph.*

Proof. We will prove this by contradiction. Assume this is not true for some minor closed classes. Let $\mathscr{G}/K_k$ be the class of all graphs which do not contain K_k as a minor. By Lemma 1, any proper minor closed class is contained in some $\mathscr{G}/K_k$. Let k be the smallest integer such that $\mathscr{G}/K_k$ contains a minor closed subclass $\mathscr{C}$ for which the statement of the theorem does not hold. Note that $k \geqslant 4$.

Let H be a maximum of $\mathscr{C}$. Then $[H]$ is a finite minor closed class of graphs for which the statement of the theorem also does not hold. Because of this finiteness, we may assume $\mathscr{C}$ is a minimal subclass of $\mathscr{G}/K_k$ with respect to having a maximum H which is not homomorphically equivalent to a complete graph. This minimality then implies that $\mathscr{C} = [H]$.

By the choice of k, we also know K_{k-1} is in $\mathscr{C}$. For otherwise $\mathscr{C} \subseteq \mathscr{G}/K_{k-1}$ and we are done. Since H is a maximum of $\mathscr{C}$ and K_{k-1} is an element of $\mathscr{C}$, we have $K_{k-1} \to H$. Thus, K_{k-1} is also a subgraph K of H.

We first claim that every vertex of K is adjacent to a vertex of H that is not in K. To see this, suppose there is a vertex x of K that is adjacent only to the $k-2$ vertices of $V(K)\backslash x$. By the minimality of $\mathscr{C}$, the graph H_x obtained from H by deleting the vertex x must be $(k-1)$-colorable. For otherwise $[H_x]$ is a minor closed subclass of $\mathscr{C}$ for which the maximum is not homomorphically equivalent to a complete graph. Since x is adjacent to $k-2$ vertices, any $k-1$ coloring of H_x can be extended to a $k-1$ coloring of H. This implies that H is homomorphically equivalent to K_{k-1}, which is a contradiction.

Our next claim is that the induced subgraph H' of H on $V(H)\backslash V(K)$ is connected. Again, by contradiction, assume it has parts H_1' and H_2' with no edges from H_1' to H_2'. Then by a similar argument as before, each of the subgraphs induced on $V(H_1') \cup V(K)$ and $V(H_2') \cup V(K)$ must be $(k-1)$-colorable. But then just a permutation of colors will produce a $k-1$ coloring of H. Thus H must be homomorphically equivalent to K_{k-1}.

To complete the proof, note that because H' is connected one can contract all the edges in H' to obtain a single vertex that is adjacent to all the vertices of K. Therefore, K_k is a minor of H, but this contradicts the choice of $\mathscr{C}$ and H. □

This theorem proves that the validity of Hadwiger's conjecture for all graphs is equivalent to the validity of Conjecture 2 for all minor closed classes. For the sake of completeness, we give a proof of this equivalence in the following theorem.

Theorem 5. *The following two statements are equivalent*:

(a) *Every graph G with $\chi(G) = k$ contains K_k as a minor*.
(b) *Every proper minor closed class of graphs contains a maximum with respect to homomorphism order*.

Proof. Suppose (a) is true and let $\mathscr{C}$ be a proper minor closed class of graphs. Then by Theorem 1, the chromatic number of the graphs in $\mathscr{C}$ is bounded. Let k be the maximum chromatic number of the graphs in $\mathscr{C}$ and let G be a graph in $\mathscr{C}$ with the chromatic number equal to k. Then by (a), G contains K_k as a minor so K_k is in the class and, therefore, $\mathscr{C}$ contains a maximum.

For the other implication, assume (b) is true and let G be a graph with $\chi(G) = k$. Then $[G]$ has a maximum. By Theorem 4, such a maximum can be chosen to be a complete graph K_r. But since $G \to K_r$, $r \geqslant \chi(G)$ and, therefore, G contains K_k as a minor. □

We believe that it should not be very difficult to prove Conjecture 3 independently. In fact, we present a proof of this conjecture for a large class of minor closed classes. To prove this, we will need a lemma on vertex transitive graphs. This lemma, which seems to be a folklore lemma, was formulated by Hell and the proof we are presenting was suggested by Robertson.

Lemma 6. *If G is a vertex transitive graph*, *then G does not contain a clique cut set.*

Proof. By contradiction, let K be a clique cut and B a component of G_K, the induced subgraph by $V(G)\backslash V(K)$. Moreover, assume B has the smallest size over all possible K.

Let B' be another component of G_K and let x be a vertex in K that is adjacent to a vertex in B'. Let b be a vertex in B and t an arbitrary vertex in K. Since G is vertex transitive, there is an automorphism φ of G that maps t to b. The image of K under φ is a clique K' containing b. Therefore, K' cannot intersect any component of G_K other than B.

Let $G_{K'}$ be the subgraph induced by $V(G)\backslash V(K')$. We claim that B' is also a component of $G_{K'}$. In fact, there are three possibilities for a component of $G_{K'}$: (1) it is a subgraph of B, (2) it contains $V(K)\backslash V(K')$ or (3) it is also a component of G_K (being connected only to $V(K) \cap V(K')$). Note that the first case cannot happen because of the minimality of B. There is only one component of the second type. Since G_K and $G_{K'}$ have the same number of components, every other component of G_K (i.e., a component other than B) must be a component of $G_{K'}$ as well. This proves our claim.

Since B' is a component of $G_{K'}$, the vertex x of K, which is connected to a vertex in B, must be a vertex of K', as well. In particular, this implies that x is adjacent to b. However, since the choice of b was arbitrary, x must be adjacent to every vertex in B. This is a contradiction because x is adjacent to every possible neighbour of b (i.e., all the vertices in K and B) plus, at least, to one more vertex in B'. □

Corollary 7. *Let G be a connected vertex transitive graph that contains K_k as a proper subgraph. Then G must contain K_{k+1} as a minor.*

Proof. Let K_k be a subgraph of G. Since G is connected, there must be an edge connecting a vertex in K_k to a vertex not in K_k. Because G is vertex transitive, every vertex of K_k must be adjacent to a vertex not in K_k. On the other hand, by Lemma 6, the subgraph induced by $V(G)\backslash V(K_k)$ is connected. Thus, if we contract all the edges in this subgraph, we will obtain K_{k+1} as a minor of G. □

Theorem 8. *Let $\mathscr{F}$ be a set of* 2*-connected graphs and let $\mathscr{C}$ be the class of all graphs with no minor in $\mathscr{F}$. If $\mathscr{C}$ contains a maximum, then every maximum must be homomorphically equivalent to a complete graph.*

Proof. Let H be a maximum of $\mathscr{C}$ with the minimum number of vertices (therefore H is connected). It will be enough to prove that H is a complete graph because, then, any other maximum must be homomorphically equivalent to H. We first claim H must be vertex transitive. Let H_1 and H_2 be two disjoint copies of H and let x and y be two distinct vertices of H. Form a new graph H' from H_1 and H_2 by identifying the copy of x in H_1 and the copy of y in H_2.

Note that H' is also in $\mathscr{C}$ because it does not contain any member of $\mathscr{F}$ as a minor (if $F \in \mathscr{F}$ is a minor of H', then F, which is a 2-connected graph, is either a minor of H_1 or H_2). Now, since H is a maximum of $\mathscr{C}$, there is a homomorphism $f : H' \to H$. By the minimality of H, the restriction f_1 of f to H_1 (and, similarly, the restriction f_2 of f to H_2) is an isomorphism. Obviously, $f_1(x) = f_2(y)$. Hence $f_1^{-1} f_2$ is an isomorphism that maps y to x.

Now, let k be the order of the largest complete graph in $\mathscr{C}$. Since H is a maximum of $\mathscr{C}$, $K_k \to H$ and, therefore H, contains K_k as a subgraph. If H is not isomorphic to K_k, then it contains K_k as a proper subgraph. Then, by Corollary 7, H (and, therefore, $\mathscr{C}$) contains K_{k+1} as a minor, which is a contradiction. □

We should remark that this theorem, in particular, implies (without using the four colour theorem) that if the class $\mathscr{P}$ of planar graphs contains a maximum, then K_4 is a maximum of $\mathscr{P}$. This is an old result of Hell and our methods in proving Theorem 8 are an extension of the methods of [4].

We would like to end this article by introducing yet another reformulation of Hadwiger's conjecture.

Conjecture 9. *Let G be a graph and let H_1 and H_2 be two minors of G. Then there is a graph $H \in [G]$ that bounds* $\{H_1, H_2\}$.

It is clear that Hadwiger's conjecture implies Conjecture 9. To see that this conjecture also implies Hadwiger's conjecture, note that it assures the existence of a maximum for every principal ideal and, therefore, we can use Theorem 4.

Remark. We have just been informed that Theorem 5 has been discovered independently (with a similar proof) by Nešetřil and Ossona de Mendez.

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References

[1] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[2] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrsscgrift der Naturforschenden Gesellschaft Zürich 88 (1943) 133–142.
[3] R. Halin, Über trennende Eckenmengen in Graphen und den Mengerschen Satz, Math. Ann. 157 (1964) 34–41.
[4] P. Hell, Absolute planar retracts and the four colour conjecture, J. Combin. Theory, Ser. B 48 (1990) 92–110.
[6] J. Nešetřil, P. Ossona de Mendez, Cuts and bounds, KAM-DIMATIA Series, vol. 592, 2002.
[8] N. Robertson, P. Seymour, R. Thomas, Hadwiger's conjecture for K_6-free graphs, Combinatorica 13 (3) (1993) 279–361.
[9] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 144 (1937) 570–590.
[10] K. Wagner, Beweis einer Abschwächung der Hadwiger-Vermutung, Math. Ann. 153 (1964) 139–141.

Further reading

[5] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, Oxford, 2004.
[7] J. Nešetřil, P. Ossona de Mendez, Colouring and homomorphisms of minor-closed classes, in: B. Aronov, S. Basu, J. Pach, M. Sharir (Eds.), The Goodman–Pollack Festschrift, Discrete & Computational Geometry, Series: Algorithms and Combinatorics, vol. 25, 2003, pp. 651–664.