On the chromatic number of the $\alpha-$overlap graphs

Debra Knisley[1], Yared Nigussie[1], Attila Pór[2] *
[1]Department of Mathematics
East Tennessee State University
[2]Department of Mathematics
Western Kentucky University

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Abstract

The generalized deBruijn graph $dB(a,k)$ is the directed graph with a^k vertices and edges between vertices $x = a_1, a_2, \cdots a_k$ and $y = b_1, b_2, \cdots b_k$ precisely when $a_2, \cdots a_k = b_1, b_2, \cdots b_{k-1}$. The deBruijn graphs can be further generalized by introducing an overlap variable $t \leq k-1$ where the number of digits by which the vertex labels (sequences) overlap is t. The α-overlap graph is the underlying simple graph of the generalized deBruijn digraph with vertex label overlap $0 < t \leq k-1$.We denote the α-overlap graph by $G_\alpha = G(a,k,t)$ and the parameters a, k and t are positive integers such that $a \geq 2$ and $k > t > 0$. Thus $dB(a,k) = G(a,k,k-1)$. In this paper, we show that every α-overlap graph is 3-colorable for any a if k is sufficiently large. We also determine bounds on the chromatic number of the α-overlap graphs if a is much larger than k.

1 Introduction

The binary deBruijn graph $dB(2,k)$ is the directed graph with 2^k vertices and edges between vertices $x = a_1, a_2, \cdots a_k$ and $y = b_1, b_2, \cdots b_k$ precisely when $a_2, \cdots a_k = b_1, b_2, \cdots b_{k-1}$. The arcs are labeled by $a_1, a_2, \cdots a_{k-1} b_{k-1}$ or by $a_1 b_1, b_2, \cdots b_{k-1}$. Many properties of the binary deBruijn graph are easily generalized for an arbitrary alphabet A of size a denoted by $dB(a,k)$. The deBruijn graphs can be further generalized by introducing an overlap variable $0 < t \leq k-1$ where the number of digits by which the vertex labels (sequences) overlap is t. We denote the α-overlap graph by $G_\alpha = G(a,k,t)$ and the parameters a, k and t are positive integers such that $a \geq 2$ and $k > t > 0$. Thus $dB(a,k) = G(a,k,k-1)$. These graphs were first introduced in [1].

A drawing of the binary deBruijn digraph for $n = 3$, the hypercube $Q^3 = K_2 \times K_2 \times K_2$ and the $G(2,3,2)$ is shown in figure 1.

We note that overlap graphs have been previously defined and are distinct from the α-overlap graphs. *Overlap graphs* are graphs whose vertices correspond with a collection of intervals on a line such that two vertices are adjacent if and only if the corresponding intervals intersect but neither contains the others. Gyarfas proved that $\chi(G) \leq 2^\omega \omega^2(\omega - 1)$ where ω denotes

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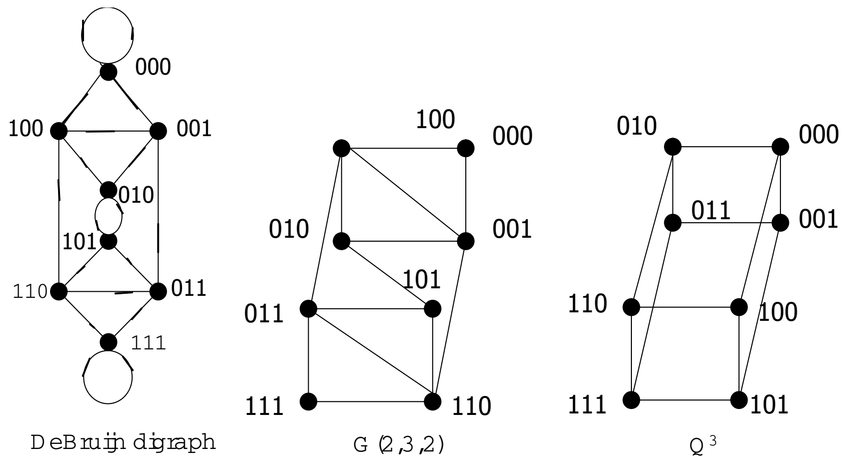

Figure 1: The DeBruijn digraph, G(2,3,2) and Q3

the clique number of G, holds for overlap graphs [6]. Gyarfas and Lehel also investigated the coloring properties of a number of interval families of graphs, primarily focusing on the relationships between the clique number and the chromatic number[7]. Another investigation on the chromatic number of cube-like graphs was carried out by Payan in [10]. A *cube-like* graph is a graph whose vertices are all 2^n subsets of a set E of cardinality n in which two vertices are adjacent if their symmetric difference is a member of a given specified collection of subsets of E. Many authors were interested in the chromatic number of these graphs and thought it was always a power of 2. Payan proved this to be false by exibiting a cube-like graph of chromatic number 7 and he further showed that there is no cube-like graph with chromatic number 3.

There are many applications for the deBruijn graphs. Since any code with block length n over the finite alphabet A may be viewed as a set of edges in the associated deBruijn digraph, the deBruijn graphs have many applications in coding theory. Relationships between the generalized deBruijn graph and the hypercube were investigated in [2] for coding theory purposes. More recently, deBruijn graph has been utilized to align protein sequences or align DNA fragments[11]. There are also many applications in communication network design theory. For example, in [12] it is shown that the maximum number of arc-disjoint spanning trees in a deBruijn network rooted at an arbitrary vertex with small depth can be used in the design of effecient store-and-forward communiction networks. A network of concurrent processors is called an *alternator* network if one processor executes a step, then none of its neighbors executes the same step at the same time. Designing a network that can simultanealy excecute k steps is equivalent to designing a k-chromatic graph. The chromatic number of the deBruijn networks were investigated in [9] in view of alternator networks.

Thus the chromatic number of these graphs are of much interest, both in the theoretical sense and for application purposes. In this work we investigate the chromatic number of the α-overlap graph for $t = k - d$. In [9], it was shown that $\chi(G(a, k, k-1) \leq c + 1$ where c is the least positive integer such that

$$a \leq \left(\begin{array}{c} c \\ \lfloor \frac{c}{2} \rfloor \end{array} \right)$$

We prove the surprising result that $\chi(G(a, k, k-1) = 3$ if $k \geq 5$ and $a \leq k$. We also show that for any a and an integer $d \geq 0$, there exists k_0 such that $\chi(G(a, k, k-d)) = 3$ for all $k \geq k_0$. In section 3, we consider the case where a is not bounded above by k. In contrast, we show the chromatic number of α-overlap can be bounded above by $2\lfloor\log^{(k-1)} a\rfloor$ when a is very large and $a > k$.

2 3-Color Theorem for long words

Turning our attention to when $a = 3$, Figure 2 below shows a four coloring of $G(3, 2, 1)$. Note that $G(3, 2, 1)$ contains a W_4 and thus $\chi(3, 2, 1) = 4$.

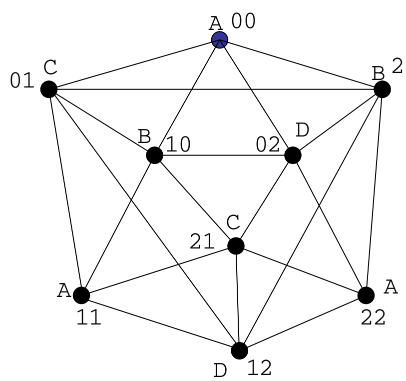

Figure 2:

It is also straightforward to argue that $\chi(G(3, 3, 2)) = 4$. One might prematurely speculate that $\chi(G(3, k, k-1)) = 4$. The next case $G(3, 4, 3)$ is quite nontrivial as it contains eighty one vertices. We invite the reader to attempt coloring it minimally before reading on. We are able to 3-color it and the coloring is explained in the proof. What makes it more interesting is that determining $\chi(G(3, 4, 3))$ allows us to determine $G(3, k, k-1)$ for all $k \geq 4$ as follows:

Proposition 1 *Let $k \geq 4$ be a positive integer. Then $\chi(G(3, k, k-1)) = 3$ for all k.*

Proof. We first give a 3-coloring of $G(3, 4, 3)$. Note that the set of 26 vertices,

$$\{*10*, *12*, 0000, 2222, 200(*\backslash 0), 022(*\backslash 2), (*\backslash 1)111\}$$

is an independent set, where $*$ denotes any symbol from $\{0, 1, 2\}$ and $(*\backslash i)$ denotes any symbol except i. Deleting these vertices from $G(3, 4, 3)$ obtains a bipartite graph as easily can be checked. Since $G(3, k, k-1)$ for all $k \geq 2$ contains a triangle, we deduce that $\chi(3, 4, 3) = 3$.

We show now a 3-coloring of $G(3, 4, 3)$ can be used to 3-color $G(3, k, k-1)$ for any $k \geq 5$. Take each of the three color classes (that is, an independent set of $G(3, 4, 3)$) and extend it to an independent set of $G(3, 5, 4)$ by adding to each word w each symbol 0, 1, and 2 as a suffix, to obtain 3 new words of length 5. As exceptions, we remove the resulting constant words $w = iiiii$, where $i = 0, 1$, or 2, since $iiiij$ and $iiiii$ would be adjacent within the color class. This is almost a 3-coloring of $G(3, 5, 4)$, except that the three constant words are to be colored yet. Since the original 3-coloring of $G(3, 4, 3)$ is a proper coloring, for each constant word $iiiii$, we can find a color class that does not have an edge to $iiiii$. This process can be repeated as many times as needed recursively. We have $3 = \chi(3, 4, 3) \geq \chi(3, k, k-1) \geq 3$, for all $k \geq 4$ and the result follows. $\square$

To obtain a theorem which generalizes the above proposition for any alphabet size a, we need the following lemma.

Lemma 2 *If $a \leq k$ and $k \geq 3$ then there exists a homomorphism $\phi : G(a,k,k-1) \rightarrow G(3,k-1,k-2)$.*

Proof. Let $w = a_1 a_2 \ldots a_k$ be a word in $G(a,k,k-1)$ and define $\mu(a_i, a_{i+1}) = sign(a_{i+1} - a_i) \mod 3, i = 1,2,\ldots,k-1$. That is, let

$$\mu(a_i, a_{i+1}) = \begin{cases} 1 & \text{if } a_{i+1} > a_i \\ 0 & \text{if } a_{i+1} = a_i \\ 2 & \text{if } a_{i+1} < a_i \end{cases}$$

Note that if $w_1 = a_1, a_2 \cdots a_k$ and $w_2 = a'_1 a'_2 \cdots a'_k$ are two adjacent words in $G(a,k,k-1)$ then $\phi(w_1)$ and $\phi(w_2)$ are also adjacent in $G(3,k-1,k-2)$, since $a_i = a'_{i+1}$ for all i, $1 \leq i < k$ implies $a_i - a_{i-1} = a'_{i+1} - a'_i$. It remains to show that if $\phi(w_1) = \phi(w_2) = w$ and w is a constant word $ii...i$, then $w_1 = w_2$. It is necessary to check this since by definition the constant word is adjacent to itself without this specific restriction . If $\phi(w_1) = \phi(w_2) = 0$, then both words are constant and so $w_1 = w_2$. If $\phi(w_1) = \phi(w_2) = 1...1$, then $w_1 = a_1, a_2 \cdots a_k$ and $a_1 < a_2 < .... < a_k$. Therefore $a_k - a_1 \geq k-1$. The assumption of the lemma is that $a \leq k$. Note that $a < k$ is impossible hence there can be no word w_1 such that $\phi(\mathrm{w}_1)$ is the constant 111...1 word. If $a = k$, then the only possibility is that $w_1 = 12...k$. But the same is true for w_2 so $w_1 = w_2$. Similar arguments hold if $\phi(w_1) = \phi(w_2) = 2...2$.□

The theorem which applies to any a follows:

Theorem 3 *Let a and k be positive integers such that $k \geq 5$. Then $\chi(G(a,k,k-1)) = 3$, for all $k \geq a$.*

Proof. The proof follows easily from the above lemma: Any given $G(a,k,k-1)$ contains a triangle and hence $\chi(G(a,k,k-1)) \geq 3$. On the other hand, if $a \leq k$, then by Lemma , we obtain $\chi(G(a,k,k-1)) \leq \chi(G(3,k-1,k-2)) \leq 3$, by Proposition 1. □

We now show how we obtain a 3-Color theorem for $\chi(G(a,k,t))$ for any a,k,t assuming k is large enough. Consider the lemma below.

Lemma 4 *Let a,k,l and t be integers such that $a > 2$, $k > t$ and $l > 0$. The graph $G(a,kl,tl)$ is isomorphic to $G(a^l,k,t)$.*

Proof. Let $w = a_1 \ldots a_{kl} \in G(a,kl,tl)$ be an arbitrary word. Let $v_i = a_{(i-1)l+1} \ldots a_{il}$ for $1 \leq i \leq k$ then we have $w = v_1 \circ \ldots \circ v_k$. Let A' be the set of all words of length l over the alphabet of size a. Take each word in A' as a new symbol and note that $|A| = a^l$. Let us define the mapping ϕ as follows, $\phi : G(a,kl,tl) \rightarrow G(A',k,t)$ where $\phi(w) = v_1 \ldots v_k$. Clearly this defines an isomorphism. □

Theorem 5 *For any finite alphabets a and an integer $d \geq 0$, there exists k_0 such that $\chi(G(a,k,k-d)) = 3$, for all $k \geq k_0$.*

Proof. Assume first that $k = md$, for some $m > 1$. Then $t = (m-1)d$. By Lemma 4, we note that $G(a, md, (m-1)d)$ is isomorphic to $G(a^d, m, m-1)$. If we let $a^d \leq m$ so that $k_0 \geq da^d$, then by Theorem 3, we obtain the desired result. The remaining case is where $k = md + i$, $1 \leq i \leq d-1$ (the case where $k-d$ does not divide k). But then, by Lemma 1, we know $G(a, md+i, (m-1)d+i)$ is homomorphic to $G(3, md+(i-1), (m-1)d+(i-1))$, assuming $a \leq md+i$. Next we have, $G(3, md+i-1, (m-1)d+i-1)$ is homomorphic to $G(3, md+(i-2), (m-1)d+(i-2))$ and so on. Hence, we arrive at the case of $G(3, md, (m-1)d) = G(3^d, m, (m-1))$, and so the result follows for $k_0 \geq \max(3^d, da^d)$. □

This concludes our first set of results where $a \leq k$. We proceed to the next case, where the alphabet size is allowed to grow unbounded.

3 An upper bound for the chromatic number when a is large

We have shown that for any a and sufficiently large k we can construct a 3 colorable α-overlap graph. Since the triangle 0000...0, 100...0 and 000...01 is always a subgraph of G, this is in some sense best possible. In this section, we consider a different problem. That is, not when can we find a 3 colorable α-overlap graph for any $a < k$, but what is the chromatic number if a exceeds k. In this case we find good bounds on the chromatic number. We first consider the case when $k = 2$ and find both an upper and lower bound for $\chi(G(a, 2, 1))$.

Theorem 6 *Let* $f(c) = \binom{c}{\lfloor \frac{c}{2} \rfloor}$. *If* $f(c-1) < a \leq f(c)$ *then we have* $c \leq \chi(a, 2, 1) \leq c+1$.

Proof. First we show the lower bound. Let $c = \chi(a, 2, 1)$. Let $C \subset V(G)$ be a colour class of a colouring with c colours. Let $F_C = \{x \in A | \exists y \neq x \text{ such that } xy \in C\}$. Obviously if $x \neq y$ and $xy \in C$ then $x \in F_C$. If $y \in F_C$ then there would be an edge inside that colour class therefore we have to have $y \not\in F_C$. Let S be the set of colors, $|S| = c$. For each letter in the alphabet, $x \in A$, we define a subset of S as follows, $S_x = \{C \in S | x \in F_C\}$. Let $x \neq y$ be two different letters of the alphabet. Let C_1 be the color of the vertex xy and C_2 be the colour of the vertex yx. Now we have $C_1 \in S_x$, $C_1 \not\in S_y$, $C_2 \in S_y$ and $C_2 \not\in S_x$. That means that our subsets S_x, $x \in A$ form a Sperner family and thus there are at most $f(c)$. Therefore $a \leq f(c)$.

To show the upper bound we first color all constant words, $(1,1), \ldots, (a,a)$ with the same color and no other vertices will get this color. Now choose a Sperner family of size a on the subsets, $S_x \subset [c], x \in A$. If $x \neq y$ then S_x and S_y are not containing each other and therefore there is a color $C \in [c]$ such that $C \in S_x$ and $C \not\in S_y$. Colour the vertex xy with colour C. It is easy to see that this is a coloring with $c+1$ colors. □

In [9] a coloring algorithm was given for the $dB(2, k)$ that resulted in the upper bound of the theorem above, but not the lower bound. Their proof showed that in fact $\chi G(a, 2, 1) = c+1$ when $a = f(c)$. They also extended the coloring algorithm for $dB(a, k)$ and showed that $\chi(dB(a, k)) = \chi(G(a, k, k-1) \leq c+1$ for all a and k. By requiring that a and k be sufficiently large, we improve on this bound.

Before proceeding to cases where $k \geq 3$, we clarify our notation and provide a few needed definitions. We define the *monotone overlap digraph* by $\overline{M}(a, k, k-1) = D(V, A)$ with vertex set $V = \{(a_1 \dots a_k) \in V(G(a,k,k-1)) | a_1 < \dots < a_k\}$ and the arcs are $(a_1 \dots a_k, a_2 \dots a_{k+1})$ for any sequence $a_1 < \dots < a_{k+1}$. Let $M(a,k,k-1)$ denote the underlying simple graph of $\overline{M}(a,k,k-1)$ which we call the *monotone overlap graph* and denote that graph by $M(a,k,k-1)$. We observe that it is the induced subgraph of $G(a,k,k-1)$ on the vertices $w = a_1 \dots a_k$ where $a_1 < \dots < a_k$.

Lemma 7 *The chromatic number of the Monotone overlap graph $M(a,2,1)$ is $\lceil \log a \rceil$.*

Proof. Let C be the set of k colors and let $\phi : M(a,2,1) \to C$ be a coloring. For any $c \in C$ let define $S_c = \{x \in [a] \mid \exists y \in [a], \phi(xy) = c\} \subset [a]$. If $2^k < a$ then there exists two distinct letters $x < y \in [a]$ such that for any $c \in C$ we have $x \in S_c$ iff $y \in S_c$. Let $c = \phi(xy)$ and observe that $x \in S_c$. Therefore $y \in S_c$ so there exists a $z \in [a]$ such that $\phi(yz) = c$ which is a contradiction since ϕ was a proper coloring. So $2^k \geq a$.Now let $2^k \geq a$ and we show that there exists a coloring $\phi : M(a,2,1) \to [k]$. Write each number $x \in [a]$ in its binary form. Let $x < y$ be two arbitrary letters from $[a]$ and let $\phi(xy) = i$ if x and y are the same on the first $i-1$ digits and differ on the i-th digit. To show that this is a proper coloring let $x < y < z$ be arbitrary letters from $[a]$.Let assume that the edges xy and yzgot the same color i.That means that x and y are the same on the first $i-1$ digits and so are y and z. But $x$and y differ on the i-th digit and y and z also differ on the i-th digit. But then x and z have to agree on the i-th digit and therefore y is either larger or smaller then both x and z. But this cannot be since $x < y < z$ and therefore ϕ is a proper coloring, showing that the chromatic number of $M(a,2,1)$ is $\lceil \log a \rceil$. □

Given a directed graph $D = (V, A)$, the *shift graph* δD of D has vertex set A and the arcs are all ordered pairs $((a,b),(b,c))$ such that $(a,b),(b,c) \in A$ (taking D to be loopless). Shift graphs have been utilized by various authors [3, 4, 4] to obtain coloring and Ramsey type results. We require the following lemma on the shift graph of the Monotone graph $\delta\overline{M}(a,k,k-1)$.

Lemma 8 *If $a > k$ then the shift graph $\delta\overline{M}(a,k,k-1)$ is isomorphic to $\overline{M}(a,k+1,k)$.*

Proof. The vertices of $\delta\overline{M}(a,k,k-1)$ are the arcs $(a_1 \dots a_k, a_2 \dots a_{k+1})$ which can be identified with the vertex $a_1 \dots a_{k+1}$ in the graph $\overline{M}(a,k+1,k)$. The arcs are the pairs $((a_1 \dots a_k, a_2 \dots a_{k+1}), (a_2 \dots a_{k+1}, a_3 \dots a_{k+2}))$ which are exactly the arcs $a_1 \dots a_{k+1}, a_2 \dots a_{k+2})$ in the graph $\overline{M}(a,k+1,k)$. □

We use the base 2 logarithm and therefore we will denote it simply by log. For any positive integer i we use the notation $log^i a$ defined recursively by $log^{i+1} a = log(log^i a)$. We now prove a lemma that yields a lower bound for the chromatic number of $M(a,k,k-1)$

Lemma 9 *For any positive integer a and k, $\chi(M(a,k,k-1))$ is at least $\log^{(k-1)} a$*

Proof. We prove the Lemma by induction on k. If $k = 2$ then by Lemma 9 we know that $\chi(M(a,2,1)) \geq \log a = \log^1 a$. If $k > 2$ then by Lemma 10 we know that $\chi(M(a,k,k-1)) = \chi(\delta(M(a,k-1,k-2))) \geq \log(\chi(M(a,k-1,k-2))) \geq \log(\log^{(k-2)} a) = \log^{(k-1)} a$. □

We will use the following lemma that is due to Erdős and Hajnal [4].

Lemma 10 *For a directed graph D,*

$$min\{t|\chi(D) \leq 2^t\} \leq \chi(\delta D)) \leq min\left\{t|\chi(D) \leq \binom{t}{\lfloor\frac{t}{2}\rfloor}\right\}$$

We now have the following upper bound for $\chi(M(a,k,k-1))$.

Lemma 11 *For any positive integer a and k we have that* $\chi(M(a,k,k-1)) \leq \log^{(k-1)} a + 2\log^{(k-2)} a$ *if* $\log^{(k-2)} a \geq 4$.

Proof. Observe that Lemma 10 implies that if $\chi(D) = a$, then we have $\chi(\delta D) \leq \lceil loga + log(loga) + 1\rceil$. Thus, the lemma now follows by induction. □

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Surprisingly the above lemmas provide bounds that are not far from being tight as the following theorem shows.

Theorem 12 *If* $k > 2$ *and* $\log^{(k-2)} a \geq 4$, *then* $\chi(G(a,k,k-1) \geq \chi(M(a,k,k-1)) \geq \chi(G(a,k,k-1)) - 4$.

Proof. Let $M^- = M^-(a,k,k-1)$ be the induced subgraph of $G(a,k,k-1)$ containing a vertex $w = a_1 \ldots a_k$ if and only if $a_1 > \ldots > a_k$. Obviously M is isomorphic to M^- and since $k > 2$ there are no edges between them. So we can color M with $\chi = \chi(M(a,k,k-1))$ colors and with the same χ colors we can color M^- as well.

Let $w = a_1 \ldots a_k \in V(G) \setminus (V(M) \cup V(M^-))$. We have three cases: $a_1 = a_2$, $a_1 < a_2$ and $a_1 > a_2$. In each of these cases we are looking for the largest i such that the first i letters of w are all equal or strictly monotone. That is, $a_1 = \ldots = a_i \neq a_{i+1}$, $a_1 < \ldots < a_i \not< a_{i+1}$ or $a_1 > \ldots > a_i \not> a_{i+1}$. Now we color w with one of the following four colors: $C_o, C_=, C_<$ or $C_>$. If i is odd then we colour w with C_o. If i is even then we color w according to the relation of a_1 and a_2, that is if $a_1 > a_2$ then we color it $C >$ if $a_1 = a_2$ then we color it $C_=$ and if $a_1 < a_2$ then we color it $C_<$. Now its easy to see that this is a good coloring of $G(a,k,k-1)$ with $\chi(M(a,k,k-1)) + 4$ colors. Therefore $\chi(M(a,k,k-1)) + 4 \geq \chi(G(a,k,k-1))$. □

Now applying Lemma 6 and the above results, we obtain an upper bound on the chromatic number of $G(a,k,k-1)$

Theorem 13 *Let a and k be positive integers such that* $a \geq 2$, *let* $G = G_\alpha(a,k,k-1)$ *be an α-overlap graph and let* $\chi(a,k,k-1)$ *denote the chromatic number of G. Then*

$$\chi(a,k,k-1) \leq 2\lfloor\log^{(k-1)} a\rfloor + 10$$

if $\log^{(k-2)} a \geq 1$.

4 CONCLUSION

As shown in the second section if the length k of the words is sufficiently long, then 3-colors are sufficient to color $G(a,k,k-d)$, for any fixed a, d. There is an interesting question, however; what are the chromatic numbers before 3-coloring is attained? This is apparently a nontrivial pursuit since the solution is already difficult for the case $a = 2$. For example, the case of $\chi(G(2, k, k-1))$ is straightforward [9] and we know that 3-coloring is attained for all $k \geq 3$. The next case $\chi(G(2, k, k-2))$ was studied in [5]. It is shown that $\chi(G(2, k, k-2)) = 3$ for $k \geq 12$, and for the case $k = 3, 4, 5, 6, 10$, the chromatic numbers are shown to be 4, 5, 4, 4 and 4, respectively. The cases $k = 7, 8, 9$ and 11 are remain open, although we know that for each of these unsolved cases, the chromatic number is either 3 or 4. We find the determination of the behavior of these finite sequences as d gets larger to be an interesting problem.

We would like to remark that it is possible to strengthen Theorem 5, by mapping $G(a, k, k-d)$ to a different graph to show that the minimum length of words k_0 could be reduced to a number much less than what is shown in the proof of the theorem. However, the best possible case is still undetermined.

From the last theorem, we see that it is possible to construct arbitrarily large α-overlap graphs by letting a be a tower of twos and selecting the appropriate k such that the chromatic number is bounded above by the constant 10. Of course, we know that by allowing k to be as large as a, then by the results in the second section, the chromatic number equals 3. What happens in between is not known.

There is very little in the literature regarding the underlying simple graphs of the deBruijn graphs and thus little is known about the structure and properties of the $G_\alpha(a, k, t)$ graphs in general. There is a substantial amount of literature on the chromatic number of similar graphs such as cube-like graphs and interval graphs, with this work we extend the results on colorability to include the α-overlap graphs and we present many open related problems for future exploration.

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