Density of universal classes of series-parallel graphs

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Abstract

A class of graphs $\mathcal{C}$ ordered by the homomorphism relation is *universal* if every countable partial order can be embedded in $\mathcal{C}$. It was shown in [1] that the class $\mathcal{C}_k$ of k-colorable graphs, for any fixed $k \geq 3$, induces a universal partial order. In [6], a surprisingly small subclass of $\mathcal{C}_3$ which is a proper subclass of K_4-minor-free graphs ($\mathcal{G}/K_4$) is shown to be universal. We have shown in [8] that if we assume the class of graphs are closed under the minor relation, then the results of [6] is a minimal set. On another note, a density result was given in [10], that for each rational number $a/b \in [2, 8/3] \cup \{3\}$, there is a K_4-minor-free graph with circular chromatic number equal to a/b. In this note we show for each rational number a/b within this interval the class $\mathcal{K}_{a/b}$ of K_4-minor-free graphs with circular chromatic number a/b is universal if and only if $a/b \neq 2$, $5/2$ or 3. This shows yet another surprising richness of the K_4-minor-free class that it contains universal classes as dense as the rational numbers.

1 Introduction

We assume graphs are finite and simple (with no loops and parallel edges unless specified otherwise). Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency. That is, $\{f(u), f(v)\} \in E(G')$ whenever $\{u, v\} \in E(G)$. We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G < G'$ means $G \leq G' \not\leq G$, whereas $G \sim G'$ means $G \leq G' \leq G$. If $G \sim G'$, we say G and G' are *hom-equivalent.* The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G. This will be denoted by $H \subseteq G$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two classes of graphs. We also write $\mathcal{C} \sim \mathcal{C}'$ if for each graph $G \in \mathcal{C}$ there exists a $G' \in \mathcal{C}'$ such that $G \sim G'$ and vice versa. See [4] for introduction to graphs and their homomorphisms.

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For two integers $k \geq d \geq 1$, a (k,d)*-coloring* of a graph G is a coloring c of the vertices of G with colors $0, 1, 2, \ldots, k-1$ such that $d \leq |c(x) - c(y)| \leq k - d$, whenever $\{x, y\}$ is an edge of G. The infimum of the ratio k/d for which there exists a (k,d)-coloring is denoted by $\chi_c(G)$ and we say $\chi_c(G)$ is the *circular chromatic number* of G. Note that a $(k,1)$-coloring is the usual vertex coloring problem. An equivalent definition of $\chi_c(G)$ is the smallest k/d such that $G \leq K_{k/d}$, where $K_{k/d}$ is the *circular graph* with $K_{k/d} = (V, E)$, $V = \{0, 1, 2, \ldots, k-1\}$ and $E = \{\{i, j\} : d \leq |i - j| \leq k - d\}$.

Let $\mathcal{K}_{a/b}$ denote the class of graphs in $\mathcal{G}/K_4$ with circular-chromatic number a/b. It is trivial to see the following:

Theorem 1 $\mathcal{K}_2 \sim \{K_2\}$.

It is well known that graphs in $\mathcal{G}/K_4$ are 3-colorable. Hell and Zhu have shown that triangle-free graphs in $\mathcal{G}/K_4$ have circular chromatic number at most 8/3 in [5]. Hence no graph in $\mathcal{G}/K_4$ has circular chromatic number in the interval (8/3,3). Hence, we have:

Theorem 2 $\mathcal{K}_3 \sim \{C_3\}$.

The main results of this note are the following two theorems establishing nice dichotomy between universality and homomorphism finiteness of the class $\mathcal{K}_{a/b}$:

Theorem 3 $\mathcal{K}_{5/2} \sim \{C_5\}$.

Theorem 4 *$\mathcal{K}_{a/b}$ is universal if $a/b \in (2, 5/2) \cup (5/2, 8/3]$.*

In section 2 we prove Theorem 3 using a folding lemma. In section 3, we prove Theorem 4.

2 $\mathcal{K}_{5/2}$ is equivalent to $\{C_5\}$

Let G be a graph. A *thread* in G is a path $P \subseteq G$ such that the two endpoints of P have degree at least 3 and all internal vertices of P are degree 2 in G. We shall often use the fact that if P and P' are two edge-disjoint paths and if the lengths of P and P' have same parity such that P is a thread and has at least equal length as P', then there is a homomorphism that maps P to P' sending the two ends of P to the two ends of P'. Such a homomorphism is said to *fold* P to P'. Let G be a graph and let G^s denote the graph we obtain from G by "smoothing" all degree 2 vertices of G. For each edge e of G^s, let P_e denote the thread of G represented by e in G^s, and let l_e denote the length of P_e.

Lemma 5 (**Edge folding lemma**) *Let $G \in \mathcal{G}/\{K_4\}$ be of odd-girth $2k+1$ and let e, e' be parallel edges in G^s, with common end vertices x, y. If G is not homomorphic to a strictly smaller graph of the same odd girth, then $l_e + l_{e'} = 2k+1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k+1$ containing both x and y.*

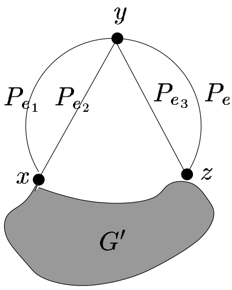

Figure 1: Unavoidable configuration of G with odd-girth $2k+1$ and $l_{e_1}+l_{e_2} = l_{e_3}+l_{e_4} = 2k+1$.

Proof. Assume $l_e \leq l_{e'}$. If l_e and l'_e have same parity, then $P_{e'}$ can be folded to P_e to obtain a strictly smaller graph H of same odd girth hom-equivalent to G, contrary to assumption. Hence $P_e \cup P_{e'}$ is an odd cycle of length $l_e + l_{e'} \geq 2k+1$. Suppose $l_e + l_{e'} > 2k+1$. Let x_1, x_2, x_3 be three consecutive vertices of $P_{e'}$, and let G' be obtained by identifying x_1 and x_3. By the choice of G, G' must have odd-girth less than $2k+1$, because $G \leq G'$ and $|V(G)| > |V(G')|$. This implies $P_{e'}$ is contained in a cycle of length $2k+1$. Hence there is a path P of G connecting x and y with length $2k+1-l_{e'}$. But then, P and P_e have same parity and so P_e can be folded to P, contrary to G being minimal. So $l_e + l_{e'} = 2k+1$.
We also note that, if there is another cycle C of length $2k+1$ containing both x and y, then there is a path P distinct from $P_e, P_{e'}$, of length l_e or $l_{e'}$ connecting x and y, where the length of P is l_e or $l_{e'}$. Hence P_e or $P_{e'}$ can be folded to P, a contradiction. The result follows. □

We prove now for any K_4-minor-free graph G with odd-girth at least 7, we have a natural number $m \geq 0$ such that $\chi_c(G) \leq (7+5m)/(3+2m) < 5/2$. For short, we use a notation K^m for $K_{(7+5m)/(3+2m)}$. In Figure 2, we draw a Hamilton cycle of K^m by positioning the $7+5m$ vertices in clockwise order as $\{0, a, 2a, 3a, \ldots, (7+5m-1)a, 0\}$. Since $\gcd(7+5m, 3+2m) = 1$, we let $a = 3+2m$ which is a generator of the additive group $\mathbb{Z}_{7+5m}$. This gives us a Hamilton cycle (we denote by C^m) that is suitable for our purpose.

Lemma 6 *Let $G \in \mathcal{G}/\{K_4\}$ be of odd-girth at least 7. Then $\chi_c(G) < 5/2$.*

Proof. Let $G \in \mathcal{G}/\{K_4\}$ be of odd-girth $g \geq 7$. To prove the lemma, we show $G \leq K^m$ for some $m \geq 0$. Suppose the contrary that there is a minimal counterexample G such that $G \not\leq K^m$, for all $m \geq 0$. Then, G is not homomorphic to a strictly smaller graph of same odd-girth. Since G is minimal, we also have $g = 7$. Let G^* be obtained from G^s by identifying the parallel edges of G^s. Since G is not hom-equivalent to a cycle, we observe that $|V(G^*)| > 2$. Hence G^* is simple and 2-connected. Also note that $G^* \in \mathcal{G}/K_4$ and so let y be a degree 2-vertex in G^* ($d_{G^*}(y) = 2$). We deduce that $d_G(y) = d_{G^s}(y) = 3$ or 4. Assume $d_G(y) = d_{G^s}(y) = 3$ and the 3 edges incident to y are e_1, e_2, e_3, where e_1, e_2 are parallel edges and $l_{e_1} > l_{e_2}$. Assume $\{y, w\} = e_3 \in E(G)$. By Lemma 5, e_3 is not in any cycle of length g in G. Then, we identify w with w', where w' is a neighbor of y

in $V(P_{e_1})$, and obtain a G' of odd girth g, such that $G \leq G'$ and $|V(G)| > |V(G')|$, contrary to the choice of G. Hence, $d_{G^s}(y) = d_G(y) = 4$. By Lemma 5 we have a configuration of two cycles $P_{e_1} \cup P_{e_2}$ and $P_{e_3} \cup P_{e_4}$ as shown in Figure 1.

Then, we delete the threads $P_{e_i}, i = 1, 2, 3, 4$ (without deleting the attachment vertices x and z) to obtain a smaller graph G'. By minimality of G, we find an $m \geq 0$, such that $G' \leq K^m$ by a homomorphism f. We shall extend f to f^* and show that $G \leq K^{m+1}$.

We may assume $f(x) = 0$. Let $P_d \subset K^m$ be the shortest path of length $d = dist_{K^m}(f(x), f(z))$ from $f(x)$ to $f(z)$. If $E(P_d) \not\subseteq E(C^m)$, then we show that $G \leq K^m$ as follows: (See Figure 2).

Note that for every vertex $i \in V(K^m)$ there is a 7 cycle containing both 0 and i. Hence, $d \leq 3$. We map the deleted subgraph $\bigcup_{i=1}^4 P_{e_i}$ to K^m, first by letting $f^*(x) = f(x)$ and $f^*(z) = f(z)$. Now if $d = 1$ or 0, clearly we can fold the four threads to each other to form a 7-cycle C and identify it to the 7-cycle of K^m containing $f(x)$ and $f(z)$. Moreover, (assuming l_{e_1} and l_{e_3} are odd), if $d = 2$, then $l_{e_1} = l_{e_3}$, and if $d = 3$ then either $l_{e_1} = l_{e_3} = 3$ or $|l_{e_1} - l_{e_3}| = 2$, or else we could once more fold the four threads in to a 7-cycle C and identify it with a 7-cycle of K^m. Recall that by Lemma 5, we have $l_{e_1} + l_{e_2} = l_{e_3} + l_{e_4} = 7$. Since G is minimal, none of the four threads have length 1. That is, G has no clique-cut. Hence, $l_{e_1}, l_{e_3} \in \{3, 5\}$. If $d = 2$, then assume $f(z) = 4m + 6 + i, 1 \leq i \leq m + 1$ (the case $f(z) = i$ is symmetric, see the edges shown by dashed lines in Figure 2). Now, if $l_{e_1} = l_{e_3} = 5$, then let $f^*(y) = 4m + 6 + (i - 1)$ and if $l_{e_1} = l_{e_3} = 3$, then let $f^*(y) = m + 2 + (i - 1)$. On the other hand, if $d = 3$, then assume $f(z) = m + 2 + i, 1 \leq i \leq m + 1$ ($f(z) = 3m + 4 + i$ is symmetric). Now, if $l_{e_1} = l_{e_3} = 3$, let $f^*(y) = 3m + 4 + i$ and if $l_{e_1} = 5$ and $l_{e_3} = 3$, let $f^*(y) = i$. It is clear now each $P_{e_i}, i = 1, 2, 3, 4$ can be folded to the paths of the 7 cycles of K^m shown by thick lines in the figure.

Finally, suppose $E(P_d) \subset E(C^m)$, then we can extend K^m to K^{m+1} so that $E(P_d) \not\subset E(C^{m+1})$ (See Figure 3). It follows by what we just proved that $E(P_d) \not\subseteq E(C^{m+1})$ implies $G \leq K^{m+1}$. □

Proof of Theorem 3: Let $G \in \mathcal{G}/K_4$ be of odd-girth g. Assume first $\chi_c(G) = 5/2$. This implies $G \leq C_5$. By Lemma 6, we deduce $g \leq 5$. By Theorem 2, $g > 3$. Hence $g = 5$ and so $C_5 \leq G \leq C_5$. We have $G \sim C_5$. The converse is obvious since $\chi_c(C_5) = 5/2$.

3 Universal sets of $\mathcal{G}/K_4$ are dense in $(2, 5/2) \cup (5/2, 8/3]$

In this section we shall show that we obtain a universal class $\mathcal{K}_{p/q} \subset \mathcal{G}/K_4$ for arbitrary $p/q \in (2, 5/2) \cup (5/2, 8/3]$. We use a graph $G_{p/q}$ with $\chi_c(G) = p/q$ as a generator of $\mathcal{K}_{p/q}$. We assume $G_{p/q}$ has the following two properties:

(P1) $G_{p/q}$ is not hom-equivalent to a cycle nor to a vertex.
(P2) if $G' \in \mathcal{G}/K_4$ satisfies (P1) and $\chi_c(G') = p/q$, then $|V(G')| \geq |V(G)|$.

Lemma 7 *Let $G \in \mathcal{G}/K_4$ have properties (P1) and (P2). Then, G is 2-connected. Moreover, G is a core and it is not vertex-transitive.*

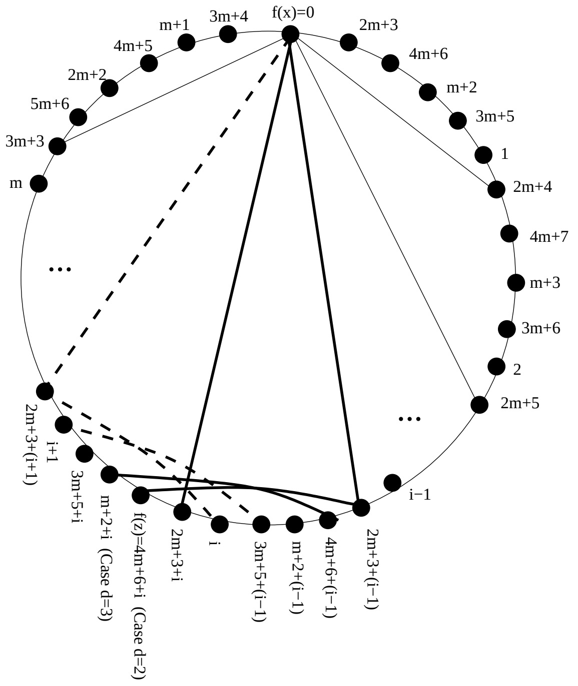

Figure 2: The subgraph of K^m with the Hamilton cycle $\{0, a, 2a, \ldots, (5m+6)a\}$ where $a = 2m+3$ and all of the $m+2$ edges adjacent to 0.

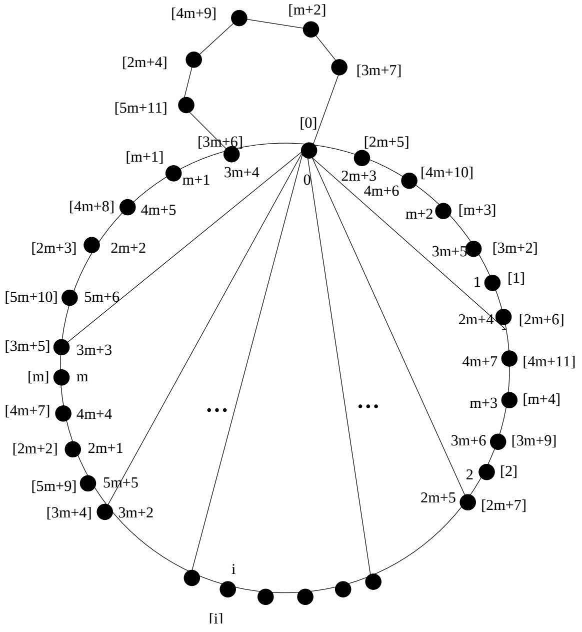

Figure 3: Extending K^m to K^{m+1}, at the edge $\{3m+4, 0\}$ of K_m. The vertices of K^{m+1} are distinguished from vertices of K^m by the "[ ]" symbol. To check the validity of the extension, it suffices to see that $5i(3+2m)$ modulo $(7+5m) = 5i(3+2(m+1))$ modulo $(7+5(m+1)) = i, 1 \leq i \leq m+1$.

Proof. Since $\chi_c(G) = \max_i(\chi_c(H_i)), 1 \leq i \leq p$, where each H_i is a 2-connected component of G. Here, (P2) implies that $p = 1$ and so G is 2-connected. Next, note that any graph $G \in \mathcal{G}/K_4$ is vertex-transitive if and only if G is an odd cycle or K_1 or K_2. This is because all other 2-connected graphs in $\mathcal{G}/K_4$ have at least one degree-2 vertex and one non-degree-2 vertex. Hence by (P1), G is not vertex-transitive. Moreover, by (P1) we deduce that G is not hom-equivalent to any vertex-transitive graph $H \in \mathcal{G}/K_4$. Hence, the core of G also is not vertex-transitive. By minimality property (P2), we deduce G is a core. □

For any rational number $p/q \in (2, 8/3]$, Pan and Zhu have shown in [10] a recursive method of constructing a 2-connected graph G with $\chi_c(G) = p/q$. If p/q is not of the form $(2k+1)/k$ then it is easy to see the graph they constructed satisfy (P1). If $p/q = (2k+1)/k$, the graph given in [10] is the cycle C_{2k+1} which is the natural candidate. Since we need (P1), we introduce a graph denoted by $G_{(k+1)/k}$ of odd girth $2k+3$ (depicted in Figure 4) such that:

Lemma 8 *$\chi_c(G_{(k+1)/k}) = (2k+1)/k$ for all $k \geq 3$.*

Proof. We can see that $G_{(k+1)/k} \leq C_{2k+1}$, by identifying y to a vertex y' of P_{2k-1} with $dist(x, y') = k$. Hence $\chi_c(G_{(k+1)/k}) \leq (2k+1)/k$. It is also easy to check that $G \not\leq C_{2k+3}$. We have $(2k+3)/(2k+1) < \chi_c(G_{(k+1)/k}) \leq (2k+1)/k$. Suppose $\chi_c(G_{(k+1)/k}) = a/b$ and $a/b < (2k+1)/k$. Note that $\gcd(4k+4, 2k+1) = 1$. From basic number theory [9], using what is known as the *Farey sequence*, we can see that any rational strictly between $(2k+3)/(2k+1)$ and $(2k+1)/k$ has numerator $a \geq 4k+4$. But then, if $k \geq 3$ the circumference of $G_{(k+1)/k}$ is $4k+3$. It is well known [12] that the numerator a of a circular chromatic number a/b of a graph G is at most its circumference. We deduce $\chi_c(G_{(k+1)/k}) = (2k+1)/k$. □

Lemma 9 *For every rational number $p/q \in (2, 5/2) \cup (5/2, 8/3]$ there is a graph $G_{p/q}$ satisfying (P1) and (P2).*

Proof. Let $p/q \in (2, 5/2) \cup (5/2, 8/3]$ be given. If $p/q = ((2k+1)/k$, then by Theorem 2 and 3, we have $k \geq 3$ and so we choose $G_{(k+1)/k}$. Otherwise let $G_{p/q}$ be as given in [10]. Clearly (P1) is satisfied. Note that $G_{(2k+1)/k}$ is a minimal graph with respect to (P1). That is, by Lemma 5 any minimal graph G of odd girth $g > 3$ has the configuration in Figure 1. Moreover the rest of the graph depicted as G' in Figure 1, can not be bipartite or else we have $G \sim C_g$, contrary to (P1). Hence for any graph G', if $\chi_c(G') = (2k+1)/k$ then either G' is at least as large as $G_{(2k+1)/k}$ (with odd girth at least $2k+3$) or $G' \sim C_{2k+1}$, (contrary to (P1)). For $p/q \neq (2k+1)/k$, by definition it is clear that we can choose a graph so that (P2) is satisfied, without losing (P1). □

Next we prove that $\mathcal{K}_{p/q}$ inherits universality from the class of directed paths [7]. Note that the circular graphs are vertex-transitive. So, if we take several copies of a fixed graph $G_{p/q}$ and construct a tree like graph G' where every 2-connected component is an isomorphic copy of $G_{p/q}$, then $\chi_c(G') = p/q$. We call such a construction *tree-concatenation.*

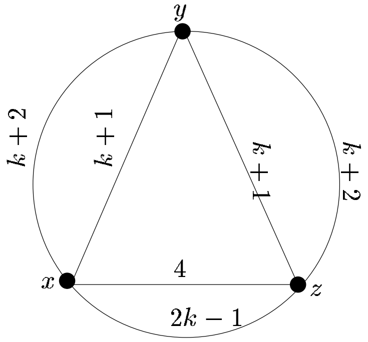

Figure 4: A graph $G_{(k+1)/k}$ of odd-girth $2k+3, k \geq 3$, $\chi_c(G_{(k+1)/k}) = (2k+1)/k$. The integers on edges indicate length of the corresponding thread.

Without going into technical details we give an outline of this association of a K_1-concatenation of a graph G (such as $G_{p/q}$) to a directed path P as follows:
For each oriented path $P \in \mathcal{P}$ of length $n \geq 1$, we define a concatenation of length n denoted by $P * (G, a, b)$ where $a, b \in V(G)$. We assume there is no automorphism sending a to b or b to a. By (P1), we know there exists such a pair. We take n isomorphic copies $H_1, H_2, \ldots, H_n$, of G and let a_i, b_i be the vertices of H_i corresponding to a and b. Then according to the orientations of the edges of P, we choose either a_i or b_i to be identified with either a_{i+1} or b_{i+1}.

Lemma 10 *Let $G_{p/q} \in \mathcal{G}/K_4$ satisfy (P1), (P2). Then, $\mathcal{K}_{p/q}$ is universal.*

Proof. Since the class of oriented paths $\mathcal{P}$ is universal we show for every $P, P' \in \mathcal{P}$, we have $P \leq P'$ if and only if $P * (G, a, b) \leq P' * (G, a, b)$. This proves the lemma. One direction is obvious. That is if $P \leq P'$, then we can mimic the mapping of P to P' and for each 2-connected component use the identity map to deduce that $P(G_{p/q}, a, b) \leq P'(G_{p/q}, a, b)$. To prove the converse, we show that if f maps $P(G_{p/q}, a, b)$ to $P'(G_{p/q}, a, b)$ then f restricted to a copy of $G_{p/q}$ is an automorphism of $G_{p/q}$, sending a to a and b to b. By Lemma 7, $G_{p/q}$ is a core. Since $G_{p/q}$ is not vertex-transitive, if f is not as claimed, then f maps a copy of $G_{p/q}$ to some connected (but not 2-connected) subgraph H of $P'(G_{p/q}, a, b)$. But then $\chi_c(H) = \max_i(\chi_c(H_i))$, where H_i is a 2-connected component of H. If $\chi_c(H_i) = p/q$ we contradict (P2) because $H_i \subset G_{p/q}$. If $\chi_c(H_i) < \chi_c(G_{p/q})$, then $G_{p/q} \not\leq H$. This contradiction proves that the restriction of f for each copy of $G_{p/q}$ is an automorphism h of $G_{p/q}$. By the choice of a and b, there is no automorphism sending a to b and b to a. If $h(a) \neq a$ then, we can see f is a trivial mapping of $P(G, a, b)$ to $G_{p/q}$. Otherwise, as claimed the restriction of f to $G_{p/q}$ is automorphism sending a to a and b to b. Hence, we can deduce $P \leq P'$, by following the positioning of a_i and b_i as "indicators" of how the directed edges of P should be folded to P'. The result follows. □

Proof of Theorem 4: Let $a/b \in (2, 5/2) \cup (5/2, 8/3]$. By Lemma 9, there is a graph $G_{a/b}$ with properties (P1),(P2). By Lemma 10, $\mathcal{K}_{a/b}$ is universal. This concludes our result.

References

[1] Z. Hedrlín, On universal partly ordered sets and classes, J. Algebra 11(1969),503-509.

[2] R. Fraïssé, Theory of Relations, North-Holland, Amsterdam, 1953.

[3] P.Hell, J. Nešetřil, The core of a graph, Discrete Mathematics 109 (1992)117-126.

[4] P.Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, 2004.

[5] P.Hell, X. Zhu, The circular chromatic number of series-parallel graphs, J. Graph Theory, 33(2000),14-24.

[6] J. Hubička, J. Nešetřil, Universal Partial Order Represented by Means of Trees and Other Simple Graphs, KAM-DIMATIA Series 2003-620, Charles University, 2003. (to appear in Europe J. Comb.)

[7] J. Hubička, J. Nešetřil , Finite Paths are Universal, ITI Series 2003-129, Charles University, 2003 (to appear in Order)

[8] J. Nešetřil, Y. Nigussie, Minimal universal and dense minor closed classes (to appear in Europe J. Combin.)

[9] I. Niven, H. Zuckerman and H. Montgomery, An introduction to the Theory of Numbers, Hojn Wiley & Sons, Inc., Fifth Edition, 1991.

[10] Z. Pan, X. Zhu, Density of the circular chromatic number series-parallel graphs, J. Graph Theory, 46(2004), 57-68.

[11] Z. Pan, X. Zhu, The circular chromatic number series-parallel graph of large odd-girth, Discrete Mathematics, 245(2002), 235-246.

[12] X. Zhu, Circular Chromatic Number, a survey. Discrete Mathematics, Vol.229 (1-3)(2001),371-410