ON STRUCTURAL DESCRIPTIONS OF LOWER IDEALS OF TREES

Yared Nigussie*and Neil Robertson†
Department of Mathematics
The Ohio State University
{robertso,yared}@math.ohio-state.edu

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Abstract: A finite structural description for proper lower ideals of trees was found, in [5], by Robertson, Seymour and Thomas. They have constructively proved existence of a specific description satisfying three axioms. We present our result starting from this fundamental result found in [5]. In an attempt to make the description unique and efficient, they introduced a fourth axiom, which conflicts with one of the prior axioms. In this paper we make use of their valid constructive result and prove the desired uniqueness and efficiency using a different set of axioms.

1. INTRODUCTION

Trees in this paper are finite, rooted, and directed away from the root. Formally, a *tree* is a triple $T = (V, E, r)$, where V is the set of vertices, E, the set of edges, is a subset of $V \times V$, and r, the *root* of T, is an element of V, such that for every $t \in V$ there is a unique directed walk from r to t and is denoted by $r = root(T)$. (A sequence $t_0, t_1, \ldots, t_n$ is a *directed walk* from t_0 to t_n if $(t_{i-1}, t_i) \in E$ for all $i, i = 1, 2, \ldots, n$.) The *height* of T is the maximum number of edges in a directed walk in T. The *out-degree* of a vertex $t \in V$ is the number of edges directed away from t. For $s, t \in V(T)$, let $s \wedge t$ denote the last vertex of the directed walk from $root(T)$ to s which belongs to the directed walk from $root(T)$ to t. Given trees T_1, T_2, we say that T_2 *topologically contains* T_1, if there exists a 1-to-1 mapping $f : V(T_1) \to V(T_2)$, called a tree embedding, such that $f(s \wedge t) = f(s) \wedge f(t)$, for every pair $s, t \in V(T_1)$. Let $\mathcal{I}$ be a set of trees. We say $\mathcal{I}$ is a *lower ideal* of trees, or an *ideal* for short, if $\mathcal{I}$ is closed under topological embedding. (i.e.,

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if $T \in \mathcal{I}$ and T topologically contains S then $S \in \mathcal{I}$.) We denote the set of all trees by $\mathcal{F}$. An ideal $\mathcal{I}$ is *proper* if $\mathcal{I} \subset \mathcal{F}$. An ideal $\mathcal{I}$ is *coherent* if $\mathcal{I} \neq \emptyset$ and for every $T_1, T_2 \in \mathcal{I}$, there exists $T \in \mathcal{I}$ such that T topologically contains both T_1 and T_2. Note that $\mathcal{F}$ is a coherent ideal.

The object of this paper is to present an "exact structural description" for any proper lower ideal $\mathcal{I}$ of finite rooted trees under topological inclusion. In [5] a method was developed, associating a finite set $\mathcal{B}$ of "bits" with $\mathcal{I}$, a bit B_0 expressed as a finite sequence of proper subideals of $\mathcal{I}$ and a natural number k. Recursively, these ideals can be expressed in the same way, leading to finitely many proper subideals of the proper ideals, etc. The well-quasi-ordering of finite trees [1] means ideals of trees are well-founded under set inclusion, thus by König's infinity lemma this finite branching process cannot repeat indefinitely. Thus a finite rooted tree with vertices ideals alternating with bits can finitely express the ideal $\mathcal{I}$. The finite set $\mathcal{B}_I$ determines how the trees in $\mathcal{I}$ can be constructed, given that the subideals in the bits are already specified.

In the larger scheme of things our description could be taken as a model of an exact structural description for graph ideals of bounded tree-width under minor inclusion. In the graph minor series of papers an "approximate structural description" is given of any proper ideal $\mathcal{I}$ of graphs under minor inclusion by exactly describing an ideal $\mathcal{J}$, that contains $\mathcal{I}$ but does not contain all graphs, in terms of "tree-structure" and "surface structure". See the excluded minor theorem for cliques developed in [7]. It would be of great interest to refine the approximate structure to an exact structure for $\mathcal{I}$ as it would lay out explicitly the features one could expect of any minor-closed class under investigation. For example, it would tell (perhaps by successive approximations) what are the fine structural features of excluding a (k+1)-clique as a minor of a graph G. The better the structure of G is known the better that coloring techniques could then be applied to prove Hadwiger's conjecture that any such G is k-colorable. There is credible evidence that an exact description can be given, with the features of the approximate description, for any minor ideal $\mathcal{I}$, but the result of this paper is as far along this avenue that research has progressed to date.

Another consequence of exact structure is that it would allow an approach to showing minor ideals are well-quasi-ordered by subset inclusion along the lines of the theorem for finite graphs (which is a consequence of a theorem [9] about labeled hypergraphs under an inclusion that reduces to minor inclusion when the hypergraphs are graphs). This paper concentrates on topological inclusion for trees rather than minor inclusion for two reasons; because minor ideals are topological ideals and so the theorem is stronger, and because topological inclusions between structured trees are used in the graph minor series [6] to prove bounded tree-width graphs are well-quasi-ordered under minors. An extension of the main result of [5], (a finite description for ideals of finite trees labeled by a wqo label set Q) is given in [4]. As further motivation to this approach, if ideals are finitely described, so that they can be proven to be well-quasi-ordered, then iterated ideals would likely also have this property, and this would establish the better-quasi-ordering of finite graphs [2]. It is conjectured by Thomas [9] that countable graphs are better-quasi-ordered and so proving this property for finite graphs would be a precondition to establishing that conjecture.

Structural descriptions can be viewed as dual to the characterization of ideals by obstructions. For every ideal $\mathcal{I}$, by Joseph Kruskal's well known tree theorem, there is up to isomorphism a unique finite set $\mathcal{O}_{\mathcal{I}}$ of minimal forbidden trees, the *obstruction set.* We can also regard $\mathcal{O}_{\mathcal{I}}$ as a finite description of $\mathcal{I}$. However, note that such a description is a description from without. That is to say, we recognize an element of $\mathcal{I}$ by checking that none of the forbidden trees are embedded in it as a topological minor. For example, in the case of graph minor ideals, planarity testing can be done by searching for K_5 and $K_{3,3}$ minors. The characterization of ideals by structural description, on the other hand, is a description of tree ideals from within. It constructs the element itself starting from a vertex, using a certain finite well-defined set of rules, without regard to what is not in the ideal. An important point is the relationship between the dual descriptions via structures and via obstructions. We believe such an analysis may be useful from the point of view of complexity of algorithms. The duality between the structural description and the obstacle description of a tree ideal is made explicit in [3],[4] by an recursive process showing how to obtain one such type of description from the other.

As a notational convenience Γ is defined to be the *null-tree*, where $V(\Gamma) = E(\Gamma) = \emptyset$. Clearly, Γ is not a rooted tree. If $T_1, T_2, \ldots, T_n, n \geq 0$, are pairwise vertex disjoint trees or Γ we define a new tree, $Tree(T_1, T_2, \ldots, T_n)$, called the *tree sum* of $T_1, \ldots, T_n$. Its set of vertices is $V(T_1) \cup V(T_2) \cup \ldots \cup V(T_n) \cup \{t_0\}$, where t_0 is a new vertex, and its set of edges is $E(T_1) \cup E(T_2) \cup \ldots \cup E(T_n) \cup \{(t_0, root(T_i)) : 1 \leq i \leq n, T_i \neq \Gamma\}$. We say that $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)$ is a *bit* if $n, k \geq 0$ are integers, $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are coherent ideals and $\mathcal{I}_0$ is an ideal. The integer k is called the *width* of B, and is denoted by $k(B)$. For all $i = 0, 1, \ldots, n$, we write $\mathcal{I}_i @ B$ and alternatively we say $\mathcal{I}_i$ is a *component* of B. We also say $\mathcal{I}_i$ is the *right component* of B and write $\mathcal{I}_i @_R B$, if $i = 0$ and a *left component* of B and write $\mathcal{I}_i @_L B$, otherwise. If $k = 0, n = 1$, and $\mathcal{I}_0 = \emptyset$, (i.e., $B = (\mathcal{I}_1; 0; \emptyset)$), we call B *atomic.* In order to avoid redundancy of bits, we assume two bits B and B' to be equal if they differ only by a permutation of their left components.

Definition 1.1 Let $\mathcal{B}$ be a set of bits. We give a constructive definition of $I(\mathcal{B})$. For each B, $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$, let $J_0(B) = \bigcup_{i=0}^{n} \mathcal{I}_i$ and $\mathcal{J}_0(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} J_0(B)$. Clearly, $\mathcal{J}_0(\mathcal{B})$ is an ideal. We call $\mathcal{J}_0(\mathcal{B})$ the *base* of $I(\mathcal{B})$. Inductively we define $\mathcal{J}_j(\mathcal{B})$, for all $j \geq 1$:
$J_j(B) = \{T : T = Tree(T_1, \ldots, T_n, \ldots, T_{n+k}, \ldots, T_{n+k+m})$, for some integer $m \geq 0$ and $T_i \in \mathcal{I}_i \cup \{\Gamma\}$, for $i = 1, \ldots, n; T_{n+i} \in \mathcal{J}_{j-1}(\mathcal{B}) \cup \{\Gamma\}$, for $i = 1, \ldots, k$ and $T_{n+k+i} \in \mathcal{I}_0$, for $i = 1, \ldots, m\}$, $J_j(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} J_j(B)$, and $\mathcal{J}_j(\mathcal{B}) = J_j(\mathcal{B}) \cup \mathcal{J}_{j-1}(\mathcal{B})$.
It can be seen that each $\mathcal{J}_j(\mathcal{B})$ is an ideal. We let $I(\mathcal{B}) = \bigcup_{j \geq 0} \mathcal{J}_j(\mathcal{B})$.

In [5] the ideal $I(\mathcal{B})$ is defined to be the intersection of all ideals $\mathcal{I}$ satisfying (*) if $(\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}, T_i \in \mathcal{I}_i$, for $i = 1, \ldots, n; T_{n+i} \in \mathcal{I} \cup \{\Gamma\}$, for $i = 1, \ldots, k, j \geq 0, T_{n+k+1}, \ldots, T_{n+k+j} \in \mathcal{I}_0$, then $Tree(T_1, \ldots, T_{k+n+j})$ belongs to $\mathcal{I}$.
Clearly, our ideal $I(\mathcal{B})$ satisfies (*), and is contained in any ideal satisfying (*)

by construction, and so the definitions are equivalent. We repeat an important definition.

Definition 1.2 Let T be a tree and $\mathcal{B}$ be a set of bits. We say T *conforms* to $\mathcal{B}$ if there exists a bit $B = (\mathcal{I}_1, \dots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$ such that either:

(1) $T \in \mathcal{I}_0 \cup \mathcal{I}_1 \cup ... \cup \mathcal{I}_n$, or

(2) there exists an integer $m \geq 0$ and trees $T_1, \dots, T_{n+k+m}$ such that $T = Tree(T_1, \dots, T_n, \dots, T_{n+k}, \dots, T_{n+k+m})$, where $T_i \in \mathcal{I}_i \cup \{\Gamma\}$ for $i = 1, \dots, n$; $T_{n+i} \in I(\mathcal{B}) \cup \{\Gamma\}$ for $i = 1, \dots, k$ and $T_{n+k+i} \in \mathcal{I}_0$ for $i = 1, \dots, m$.

Note that in (2) at most k trees in the tree sum are in $I(\mathcal{B}) - J_0(B)$ and in (1) $T \in J_0(B)$. For the sake of clarity we sometimes use the term T *conforms to* B *by (1)*, if (1) holds, and *conforms to* B *by (2) in* $\mathcal{B}$, if (2) holds. The following theorem is obvious from (1.1).

Theorem 1.3 *(Theorem 2.1 of [5]) Let $\mathcal{B}$ be a set of bits, and let T be a tree. Then $T \in I(\mathcal{B})$ if and only if T conforms to $\mathcal{B}$.* ∎

A bit $B = (\mathcal{I}_1, \dots, \mathcal{I}_n; k; \mathcal{I}_0)$ is said to be *reflexive* if $k = 0$ and $I(\{B\}) \subseteq \mathcal{I}_j$ for some $j, 0 \leq j \leq n$. We note that $I(\{B\}) \subseteq \mathcal{I}_0$ implies $I(\{B\}) = \mathcal{F}$. It can be seen that $I(\{B\}) = \mathcal{F}$ if and only if $\mathcal{F}@B$. Recall that $\mathcal{F}@B$ means $\mathcal{F}$ is a component of B. We note also that $I(\{B\}) \subseteq \mathcal{I}_j, 1 \leq j \leq n$, implies $I(\{B\}) = \mathcal{I}_j$, since $\mathcal{I}_j \subseteq I(\{B\})$. We emphasize that in [5], the construction of a finite set of bits $\mathcal{B}_0$ satisfying three axioms, "(N1) (N2) and (N3)", is a valid result which we use in our paper. In order to make (N3) clear we also present the definition of "$\mathcal{I}$-domination". It can easily be seen that the relation $\leq_{\mathcal{I}}$ induces a partial order in sets of bits.

Definition 1.4 Let $B = (\mathcal{I}_1, \dots, \mathcal{I}_n; k; \mathcal{I}_0), B' = (\mathcal{I}'_1, \dots, \mathcal{I}'_{n'}; k'; \mathcal{I}'_0)$ be bits and let $\mathcal{I}$ be a tree ideal. We say B is *$\mathcal{I}$-dominated* by B' if there exists a set $S \subseteq \{1, 2, \dots, n\}$ and a mapping $f : \{1, 2, \dots, n\} - S \to \{0, 1, \dots, n'\}$ such that:
(D1) $|S| \leq k' - k$ (and hence $k \leq k'$),
(D2) $\mathcal{I}_i \subseteq \mathcal{I}$ for every $i \in S$,
(D3) for $i, j \in \{1, 2, \dots, n\} - S$, if $f(i) = f(j) > 0$ then $i = j$, and
(D4) $\mathcal{I}_0 \subseteq \mathcal{I}'_0$, and $\mathcal{I}_i \subseteq \mathcal{I}'_{f(i)}$ for every $i \in \{1, 2, \dots, n\} - S$.

The following are the axioms given in [5] as (N1)-(N4). Let $\mathcal{I}$ be a tree ideal and $\mathcal{B}$ be a finite set of bits. Then:
(N1) if $(\mathcal{I}_1, \dots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$ then $\mathcal{I}_i \not\subseteq \mathcal{I}_0$ for all $i = 1, 2, \dots, n$ and $\mathcal{I}_i$ is a proper subset of $\mathcal{I}$ for all $i = 0, 1, \dots, n$,
(N2) $I(\mathcal{B}) = \mathcal{I}$,
(N3) no bit of $\mathcal{B}$ is $\mathcal{I}-$dominated by a bit of $\mathcal{B}$ other than itself, and
(N4) if $B \in \mathcal{B}$ has width zero and $I(\{B\}) \subseteq \mathcal{I}'$ for a component ideal $\mathcal{I}'$ of a bit $B' \in \mathcal{B}$, then $B = B' = (\mathcal{I}'; 0; \emptyset)$.

For any proper ideal $\mathcal{I}$, what is constructed as $\mathcal{B}_0$ in [5] using a well-quasi-ordered set (wqo) called the set of all "germs of $\mathcal{I}$" does lead to a finite description of $\mathcal{I}$ as a finite structured tree, as outlined in that paper. The set of bits $\mathcal{B}_0$ satisfies (N1)-(N3) but it contains redundant bits, namely the reflexive bits can appear in various forms in $\mathcal{B}_0$. To remove the redundant bits without losing uniqueness, (N4) was introduced. However, satisfying (N4) can lead to violation of (N3). After stating a few theorems, we offer examples at the end of this section to show that what is called in [5] a "name" with axioms (N1)-(N4) does not always exist, and when it does, the resulting name need not be unique as claimed.

Theorem 1.5 *Let Q be a wqo and let I be a non-empty ideal in Q. Then:*
(i) there exists a unique finite set $\{I_1, I_2, \dots, I_m\}$ of coherent ideals such that $I = \bigcup_{i=1}^m I_i$ and $I_i \not\subseteq I_j$ for all $i, j = 1, 2, \dots, m$ with $i \neq j$, and
(ii) if I is coherent and $\bigcup_{i=1}^m I_i$ is any finite union of ideals, then $I \subseteq \bigcup_{i=1}^m I_i$ if and only if $I \subseteq I_i$, for some $i, 1 \leq i \leq m$.

Proof. We obtain (i) from (1.4) of [5]. To prove (ii), let any finite union $\bigcup_{i=1}^m I_i$, of ideals be given. If $I \subseteq I_i$, for some i, then obviously $I \subseteq \bigcup_{i=1}^m I_i$. Conversely, assume $I \not\subseteq I_i, \forall i$ and for each i choose $a_i \in I - I_i$. Let $b_1 = a_1$ and for $i > 1$ let b_i be an element of I containing b_{i-1} and a_i. Then $b_m \in I - \bigcup_{i=1}^m I_i$. Hence $I \not\subseteq \bigcup_{i=1}^m I_i$. ∎

For a set of bits $\mathcal{B}$ we denote the set of positive width bits of $\mathcal{B}$ by $\mathcal{B}^+$, the set of width-zero bits by $\mathcal{B}^0$, the set of reflexive bits by $\mathcal{B}^{ref}$, and the set of atomic bits by $\mathcal{B}^{atm}$. Note that $\mathcal{B} = \mathcal{B}^0 \cup \mathcal{B}^+$, $\mathcal{B}^{atm} \subseteq \mathcal{B}^0$ and $\mathcal{B}^{ref} \subseteq \mathcal{B}^0$. A finite set of bits $\mathcal{B}$ is defined in [5] to be *coherent* if $\mathcal{B}$ is non-empty and if there is a bit $B \in \mathcal{B}$ with $k(B) \geq 2$ or if $|\mathcal{B}^0| \leq 1$. (i.e., $\exists$ at most one width-zero bit in $\mathcal{B}$.)

Corollary 1.6 *((2.2) of [5]) Let $\mathcal{B}$ be a set of bits. If $\mathcal{B}$ is coherent then $I(\mathcal{B})$ is a coherent ideal.* ∎

In (1.7) we present a generalization of (2.3) of [5]. Its proof is given in section 3 and can be read at this point independently from what follows.

Theorem 1.7 *Let $\mathcal{B}$ and $\mathcal{B}'$ be finite sets of bits. Then $I(\mathcal{B}) \subseteq I(\mathcal{B}')$ if and only if either:*
(i) $I(\mathcal{B}) \subseteq \mathcal{J}_0(\mathcal{B}')$ *or*
(ii) $\forall B \in \mathcal{B}, \exists B' \in \mathcal{B}'$ *such that either* $B \leq_{I(\mathcal{B}')} B'$, *or* $k(B) = 0$ *and* $I(\{B\}) \subseteq J_0(B')$. ∎

We now present the examples mentioned earlier in this section. Denote the ideal of all finite paths of length at most l by $\mathcal{P}_l$. Let $B_1 = (\mathcal{P}_1; 1; \emptyset)$, $B_2 = (\mathcal{P}_0, \mathcal{P}_0, \mathcal{P}_0; 0; \emptyset)$, $B_3 = (\mathcal{P}_2, \mathcal{P}_2; 0; \emptyset)$ and $B_4 = (; 1; \mathcal{P}_0)$.

Example 1. Let $\mathcal{I}_1 = I(\{B_1, B_2\})$, $\mathcal{I}_2 = I(\{B_1, B_3\})$ and let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. Then $\{B_1, B_2, B_3\}$ and $\{(\mathcal{I}_1; 0; \emptyset), (\mathcal{I}_2; 0; \emptyset)\}$ are both "names" of $\mathcal{I}$.

Example 2. Let $B^{rf} = (I(\{B_4\}); 0; \mathcal{P}_0)$, $\mathcal{B}_0 = \{B_1, B^{rf}\}$ and let $\mathcal{I} = I(\mathcal{B}_0)$. Then $\mathcal{B}_0$ is not a "name" of $\mathcal{I}$, because (N4) is not satisfied. If we satisfy (N4) by replacing B^{rf} by $(I(\{B_4\}); 0; \emptyset)$, then (N3) is violated. Note that instead of using B^{rf} in $\mathcal{B}_0$, we could use another reflexive bit such as $(I(\{B_4\}), \mathcal{P}_0, \mathcal{P}_0; 0; \emptyset)$, or $(I(\{B_4\}), \mathcal{P}_0, \mathcal{P}_0, \mathcal{P}_0; 0; \emptyset)$, etc. and still satisfy (N1)-(N3).

Proposition 1.8 *The ideal $\mathcal{I}$ of example 2 has no "name".*

Proof. We show every finite set of bits $\mathcal{B}$ satisfying (N1)-(N3) for $\mathcal{I}$ cannot satisfy (N4). Indeed, let $\mathcal{B}$ be given and let $\mathcal{B}_0$ be as in example 2. By (1.6), $I(\mathcal{B}_0)$ is coherent. By (N2), $I(\mathcal{B}_0) = I(\mathcal{B})$. By (1.5)(ii) and (N1), $I(\mathcal{B}_0) \not\subseteq \mathcal{J}_0(\mathcal{B})$ and $I(\mathcal{B}) \not\subseteq \mathcal{J}_0(\mathcal{B}_0)$. So (1.7)(i) does not hold. By (1.7)(ii), $B_1 \leq_{\mathcal{I}} B^+ \leq_{\mathcal{I}} B_1$, for some $B^+ \in \mathcal{B}$. We have $B_1 = B^+$. By (1.7) and by (N3), we have $\mathcal{B}^+ = \{B_1\}$.

Now we show that a reflexive bit B^0 must exist in $\mathcal{B}$. Since $B^{rf} \not\leq_{I(\mathcal{B})} B^+$ and $I(\{B^{rf}\}) = I(\{B_4\}) \not\subseteq \mathcal{P}_1 @_L B^+$, it follows that, $\exists B^0 \in \mathcal{B}$ of width zero for which (1.7)(ii) holds. In either case of (1.7)(ii), we have $I(\{B_4\}) \subseteq J_0(B^0)$. We apply (1.7)(ii) again to $I(\mathcal{B}) \subseteq I(\mathcal{B}_0)$. In either case we have $I(\{B^0\}) \subseteq I(\{B_4\})$ and so B^0 is reflexive. By (N3), B^0 is not atomic. Hence (N4) is not satisfied.∎

2. A NEW NAME FOR PROPER IDEALS

Definition 2.1 Let $\mathcal{I}$ be a proper ideal and let $\mathcal{B}$ be a finite set of bits. We say $\mathcal{B}$ is a *proper description* of $\mathcal{I}$ if $I(\mathcal{B}) = \mathcal{I}$ and if for every bit $B \in \mathcal{B}$ and for every $\mathcal{I}' @ B$, we have $\mathcal{I}' \subset \mathcal{I}$. When we want to be explicit about this property we denote any finite set satisfying these two conditions by $\mathcal{B}_{\mathcal{I}}$. We note that our definition is not vacuous, by the construction shown in [5]. Note that $\mathcal{B}_\emptyset = \emptyset$ is the unique proper description of the null ideal $\emptyset$.

Let Q be a wqo and let I be a non-null ideal in Q. Let $U = \{I_1, \ldots, I_m\}$ be the unique finite set of maximal coherent subideals of I as given in (1.5)(i). Note that, by (1.5)(ii), for any $i, 1 \leq i \leq m$, if $I_i \subseteq I' \subseteq I$, and I' is coherent then $I' = I_i$. We call U the *maximal coherent decomposition* or *mcd* of I.

Theorem 2.2 *Let $\mathcal{I}$ be a proper ideal and $\{\mathcal{I}_1, \ldots \mathcal{I}_m\}$, for $m \geq 1$ be the mcd of $\mathcal{I}$ and $\mathcal{B}_{\mathcal{I}}$ be a proper description of $\mathcal{I}$. Then $I(\mathcal{B}_{\mathcal{I}}) = \mathcal{J}_0(\mathcal{B}_{\mathcal{I}})$ if and only if $m \geq 2$, and there are m distinct reflexive bits $B_1, \ldots, B_m$ of $\mathcal{B}_{\mathcal{I}}$ such that $\mathcal{I}_i @_L B_i$, $I(\{B_i\}) = \mathcal{I}_i$ and $\mathcal{I}_i \not@ B_j$, for all $i, j = 1, \ldots, m$ with $i \neq j$.*

Proof. Let a proper description $\mathcal{B}_{\mathcal{I}}$ be given. Note that the sufficiency is obvious, since $\mathcal{I} = \bigcup_{i=1}^{m} \mathcal{I}_i \subseteq \mathcal{J}_0(\mathcal{B}_{\mathcal{I}}) \subseteq I(\mathcal{B}_{\mathcal{I}}) = \mathcal{I}$.

To prove the necessity assume $I(\mathcal{B}_{\mathcal{I}}) = \mathcal{J}_0(\mathcal{B}_{\mathcal{I}})$. Then, by (1.5)(ii), $I(\mathcal{B}_{\mathcal{I}})$ is not coherent and so $m \geq 2$. Next, for each i, $\mathcal{I}_i @ B_i$, for some $B_i \in \mathcal{B}_{\mathcal{I}}$, since otherwise, by (1.5)(ii), we obtain $\mathcal{I}_i \not\subseteq \mathcal{J}_0(\mathcal{B}_{\mathcal{I}}) = \mathcal{I}$, a contradiction. By (1.6),

$I(\{B_i\})$ is coherent. As $\mathcal{I}_i \subseteq I(\{B_i\}) \subseteq I(\mathcal{B}_\mathcal{I})$ and $\mathcal{I}_i$ is maximal, we deduce $\mathcal{I}_i = I(\{B_i\})$. It follows, by (1.5)(i), $\mathcal{I}_j \not@ B_i$, since $\mathcal{I}_j \not\subseteq \mathcal{I}_i\ \forall j \neq i$. Next, since $\mathcal{I}_i \subset \mathcal{F}$, we have $\mathcal{I}_i \not@_R B_i$. Hence $\mathcal{I}_i @_L B_i$. Note that $\forall i$, $\exists T_i \in \mathcal{I}_i - \bigcup_{j \neq i} \mathcal{I}_j$. Now if $k(B_i) \neq 0$ then, for any $j \neq i$, $Tree(T_i, T_j) \in I(\mathcal{B}_\mathcal{I}) - \mathcal{I}$, a contradiction. Hence $k(B_i) = 0$. It follows that $B_1, B_2, \ldots, B_m$ are distinct reflexive bits of $\mathcal{B}_\mathcal{I}$ with $I(\{B_i\}) = \mathcal{I}_i$, for $i = 1, \ldots, m$. ∎

Definition 2.3 Let $\mathcal{B}$ be a set of bits and let $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$. Let $\mathcal{S}_B = \{(\mathcal{I}'_{i_1}, \ldots, \mathcal{I}'_{i_j}; k'; \mathcal{I}'_0) : \{i_1, \ldots, i_j\} \subseteq \{1, 2, \ldots, n\}, k' \leq k, \mathcal{I}'_i \subseteq \mathcal{I}_i, i = 0, i_1, \ldots, i_j\}$. We say B is *shellable in* $\mathcal{B}$, if $I(\mathcal{B}') = I(\mathcal{B})$, where $\mathcal{B}' = (\mathcal{B} - \{B\}) \cup \mathcal{S}$, for some $\mathcal{S} \subset \mathcal{S}_B$, such that $B \notin \mathcal{S}$ and $\mathcal{S} = \emptyset$ if $B \in \mathcal{B}^{atm}$. For a set of bits $\mathcal{B}$, we write $shell(\mathcal{B})$ to denote the set of all bits $B \in \mathcal{B}$ shellable in $\mathcal{B}$. If $shell(\mathcal{B}) = \emptyset$ we say $\mathcal{B}$ is *unshellable*. Otherwise we say $\mathcal{B}$ is *shellable*.
Note that we always have $I(\mathcal{B}') \subseteq I(\mathcal{B})$ and if B is not shellable in $\mathcal{B}$, then $I(\mathcal{B}') \subset I(\mathcal{B})$. By definition, an atomic bit B is shellable if and only if $I(\mathcal{B} - \{B\}) = I(\mathcal{B})$.

Proposition 2.4 *Let $\mathcal{B}$ be a finite set of bits such that $I(\mathcal{B})$ is a proper ideal. Then $\mathcal{B}$ is shellable if and only if there exists a bit $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}$ such that:*
(i) $\mathcal{I}_i \subseteq \mathcal{I}_0$ for some $i, i = 1, 2, \ldots, n$, or
(ii) $k(B) > 0$ and $I(\mathcal{B}) @_L B$, or
(iii) $B \in \mathcal{B}^{ref} - \mathcal{B}^{atm}$, or
(iv) $I(\mathcal{B}) = I(\mathcal{B} - \{B\})$, or
(v) $B \in \mathcal{B} - \mathcal{B}^{atm}$ and $\exists B' \in \mathcal{B}, k(B') > k(B)$ such that $B \leq_{I(\mathcal{B})} B'$.

Proof. The proof of sufficiency when one of (i)-(v) holds is straightforward. Namely, let $\mathcal{S} = \{(\mathcal{I}_1, \ldots, \mathcal{I}_{i-1}, \mathcal{I}_{i+1}, \ldots, \mathcal{I}_n; k; \mathcal{I}_0)\}$ if (i) holds, $\mathcal{S} = \{(\mathcal{I}_1, \ldots, \mathcal{I}_n; 0; \mathcal{I}_0)\}$ if (ii) holds, $\mathcal{S} = \{(I(\{B\}); 0; \emptyset)\}$ if (iii) holds, $\mathcal{S} = \emptyset$ if (iv) holds, and let $\mathcal{S} = \{(\mathcal{I}_i; 0; \emptyset) : 1 \leq i \leq n\}$ if (v) holds. We have $B \in shell(\mathcal{B})$.

To prove necessity suppose $B \in shell(\mathcal{B})$. By definition $I(\mathcal{B}) = I((\mathcal{B} - \{B\}) \cup \mathcal{S})$ for some set $\mathcal{S} \subset \mathcal{S}_B, B \notin \mathcal{S}$. Let $\mathcal{B}' = (\mathcal{B} - \{B\}) \cup \mathcal{S}$. First suppose that $B \in \mathcal{B}^{atm}$. Then by definition $\mathcal{S} = \emptyset$. Hence $I(\mathcal{B}) = I(\mathcal{B} - \{B\})$, which is (iv).

Next suppose that $B \notin \mathcal{B}^{atm}$ and assume none of (i)-(iv) hold for any bit in $\mathcal{B}$. In particular, since (iii) does not hold we have $\mathcal{B}^{ref} \subseteq \mathcal{B}^{atm}$, and so $B \notin \mathcal{B}^{ref}$. We show (v) must hold.

If $I(\mathcal{B}) @_R B$ then $I(\mathcal{B}) = \mathcal{F}$, contrary to assumption. If $I(\mathcal{B}) @_L B$ then $\mathcal{B} = \{B\}$, since otherwise (iv) holds. If $k(B) > 0$ then (ii) holds, a contradiction. Hence $k(B) = 0$. But then we get $B \in \mathcal{B}^{ref}$, a contradiction. Hence $\mathcal{B}$ is a proper description. Also if $I(\mathcal{B}) = \mathcal{J}_0(\mathcal{B})$ then by (2.2), for $B \in \mathcal{B} - \mathcal{B}^{ref}$ (iv) holds, a contradiction. Hence $I(\mathcal{B}) \supset \mathcal{J}_0(\mathcal{B})$. It follows that $I(\mathcal{B}) \supset \mathcal{J}_0(\mathcal{B}')$ because $\mathcal{J}_0(\mathcal{B}) \supseteq \mathcal{J}_0(\mathcal{B}')$. Apply (1.7) to the inclusion $I(\mathcal{B}) \subseteq I(\mathcal{B}')$. As (1.7)(i) does not hold, (1.7)(ii) must hold. We note that if $k(B) = 0$, then $I(\{B\}) \not\subseteq J_0(B_s), \forall B_s \in \mathcal{S}$, because $B \notin \mathcal{B}^{ref}$. It follows that $I(\{B\}) \not\subseteq J_0(B'), \forall B' \in \mathcal{B}'$, since (iv) does not hold. We deduce that there exists a $B' \in \mathcal{B}'$ such that $B \leq_{I(\mathcal{B})} B'$. By definition of $\mathcal{S}$ and since (i) does not hold, no bit of $\mathcal{S}$ can

$I(\mathcal{B})$-dominate B. We deduce that $B' \in \mathcal{B} - \{B\}$. In addition, if $k(B') = k(B)$, then (iv) holds, contrary to assumption. Hence (v) holds. ∎

Let $\mathcal{B}$ be a finite set of bits. We use the notation (2.4)$\neg(j)$, for $j \in \{i, ii, iii, iv, v\}$, to mean (2.4)$(j)$ does not hold for any bit $B \in \mathcal{B}$. The following are remarkable corollaries of (1.7) and (2.4).

Corollary 2.5 *Let $\{B\}$ and $\{B'\}$ be unshellable. If $k(B) = k(B') = 0$ then $I(\{B\}) = I(\{B'\})$ if and only if $B = B'$.*

Proof. If B and B' both are reflexive, then the result follows from (2.4)$\neg(iii)$. If both are non-reflexive then $I(\{B\}) = I(\{B'\})$ if and only if $B \leq_{\mathcal{I}} B' \leq_{\mathcal{I}} B$ if and only if $B = B'$. If B is reflexive and B' is non-reflexive or vice versa, by (1.7)(ii), neither of the equalities hold in the conclusion of (2.5). ∎

Corollary 2.6 *Let $\mathcal{B}$ and $\mathcal{B}'$ be unshellable sets of bits such that $I(\mathcal{B}) = I(\mathcal{B}')$, $I(\mathcal{B}) \supset \mathcal{J}_0(\mathcal{B})$, and $I(\mathcal{B}') \supset \mathcal{J}_0(\mathcal{B}')$. Then:*
(i) if $\mathcal{B}^+ = \emptyset$, then $\mathcal{B} = \mathcal{B}'$, otherwise
(ii) $\mathcal{B}^+ = \mathcal{B}'^+$,
(iii) $\forall B \in \mathcal{B}^0$ and $\forall B' \in \mathcal{B}'^{atm}$, if $I(\{B\}) \subseteq J_0(B')$, then $B' \in \mathcal{B}'^{atm}$, and
(iv) if $\mathcal{B}^{atm} = \mathcal{B}'^{atm}$ then $\mathcal{B} = \mathcal{B}'$.

Proof. If $\mathcal{B}^+ = \emptyset$ then, by (1.7)(ii), $\mathcal{B}'^+ = \emptyset$. Hence, $I(\mathcal{B}) = \{I(\{B\}) : B \in \mathcal{B}^0\}$ and $I(\mathcal{B}') = \{I(\{B'\}) : B' \in \mathcal{B}'^0\}$, and so (i) follows from (2.4)$\neg(iv)$, (1.5)(i) and (2.5). To get (ii), let $\mathcal{I} = I(\mathcal{B})$. From the given conditions we observe that $I(\mathcal{B}) \not\subseteq \mathcal{J}_0(\mathcal{B}')$ and $I(\mathcal{B}') \not\subseteq \mathcal{J}_0(\mathcal{B})$. Hence (1.7)(ii) holds to the inclusions $I(\mathcal{B}) \subseteq I(\mathcal{B}')$ and to $I(\mathcal{B}') \subseteq I(\mathcal{B})$. Therefore, $\forall B \in \mathcal{B}^+, \exists B' \in \mathcal{B}'^+$ and $B'' \in \mathcal{B}^+$ such that $B \leq_{\mathcal{I}} B' \leq_{\mathcal{I}} B''$. By transitivity of $\mathcal{I}$-domination and by (2.4)$\neg(v)$, $B = B''$. By anti-symmetry $B = B'$. Hence $\mathcal{B}^+ \subseteq \mathcal{B}'^+$. Similarly $\mathcal{B}'^+ \subseteq \mathcal{B}^+$ and so (ii) follows.

To get (iii) assume the contrary that $\exists B \in \mathcal{B}^0$, such that $I(\{B\}) \subseteq J_0(B')$ and $B' \in \mathcal{B}' - \mathcal{B}'^{atm}$. Since $\mathcal{B}^+ = \mathcal{B}'^+$, we have by (2.4)$\neg(iv)$, $k(B') = 0$. By (1.7)(ii) we find either $B' \leq_{\mathcal{I}} B''$ or $I(\{B'\}) \subseteq J_0(B''), B'' \in \mathcal{B}$. By (2.4)$\neg(iv)$, (2.4)$\neg(v)$ and (2.6)(ii), we deduce $k(B'') = 0$. It follows $I(\{B\}) \subseteq J_0(B'')$, and so $B = B''$ and $B \in \mathcal{B}^{atm}$. Hence, $B' \in \mathcal{B}^{atm}$, a contradiction.

To get (iv), let $B \in \mathcal{B}^0 - \mathcal{B}^{atm}$. By (iii), and by assumption, we deduce $I(\{B\}) \not\subseteq J_0(B'), \forall B' \in \mathcal{B}'$, or else (2.4)$(iv)$ holds. Hence the proof of the equality is the same as in (ii). ∎

Corollary 2.7 *Let $\mathcal{I}$ be a proper ideal and let $\mathcal{B}_{\mathcal{I}}$ be unshellable. Then $I(\mathcal{B}_{\mathcal{I}}) = \mathcal{J}_0(\mathcal{B}_{\mathcal{I}})$, if and only if $\mathcal{B}_{\mathcal{I}} = \mathcal{B}_{\mathcal{I}}^{ref} = \{(\mathcal{I}_j; 0; \emptyset) : 1 \leq j \leq m\}$, where $\{\mathcal{I}_1, \ldots, \mathcal{I}_m\}$, is the mcd of $\mathcal{I}$, and $m \geq 2$.*

Proof. The result follows directly from (2.2) and (2.4)$\neg(iii)$ and $\neg(iv)$. ∎

Lemma 2.8 *Let $B = (\mathcal{I}; 0; \emptyset)$ be a bit. Then B is reflexive if and only if $\mathcal{B}_{\mathcal{I}}^+ \neq \emptyset$.*

Proof. By (1.5)(ii), $\mathcal{I} \not\subseteq \mathcal{J}_0(\mathcal{B}_\mathcal{I})$. Apply (1.7)(ii) to $I(\{B\}) \subseteq I(\mathcal{B}_\mathcal{I})$. ∎

Definition 2.9 Let $\mathcal{B}$ be a set of bits and let $B = (\mathcal{I}_1, 0; \emptyset) \in \mathcal{B}$. Let $\mathcal{B}' = (\mathcal{B} - \{B\}) \cup \mathcal{B}_{\mathcal{I}_1}$. We say B is *splittable in* $\mathcal{B}$, if $I(\mathcal{B}') = I(\mathcal{B})$, where $\mathcal{B}_{\mathcal{I}_1}$ is any proper description of $\mathcal{I}_1$. We denote the set of all bits $B \in \mathcal{B}$ splittable in $\mathcal{B}$ by $split(\mathcal{B})$. If $split(\mathcal{B}) = \emptyset$ we say $\mathcal{B}$ is *unsplittable.* Otherwise we say $\mathcal{B}$ is *splittable.*

Note that if B is reflexive, then we always have $I(\mathcal{B}) \subseteq I(\mathcal{B}')$ and if B is not splittable in $\mathcal{B}$, then $I(\mathcal{B}) \subset I(\mathcal{B}')$. On the other hand, if B is non-reflexive, then $I(\mathcal{B}') \subseteq I(\mathcal{B})$ (since $I(\mathcal{B}_{\mathcal{I}_1}) \subset I(\{B\})$) and if B is not splittable in $\mathcal{B}$, then $I(\mathcal{B}') \subset I(\mathcal{B})$. We will shortly see that if $B \in \mathcal{B}^{atm} - \mathcal{B}^{ref}$, then $B \in split(\mathcal{B})$ if and only if $\mathcal{B}^+ \neq \emptyset$.

Proposition 2.10 *Let $\mathcal{B}$ be an unshellable set of bits such that $I(\mathcal{B})$ is proper. Then $\mathcal{B}$ is splittable if and only if $\exists B = (\mathcal{I}_1; 0; \emptyset) \in \mathcal{B}$ such that one and only one of the following holds.*
(i) $\mathcal{B} = \mathcal{B}^{ref} = \{(\mathcal{I}_1; 0; \emptyset)\}$,
(ii) $\mathcal{B} = \mathcal{B}^{ref} = \{(\mathcal{I}_i; 0; \emptyset) : 1 \leq i \leq m\}, m \geq 2, \forall B^+ \in \mathcal{B}^+_{\mathcal{I}_1}, k(B^+) = 1$ *and* $\forall j \neq 1, \exists B_j \in \mathcal{B}_{\mathcal{I}_j}$ *such that* $B^+ \leq_{\mathcal{I}_j} B_j$,
(iii) $\mathcal{B}^+ \neq \emptyset$, *and* $\forall B^+ \in \mathcal{B}^+_{\mathcal{I}_1} \exists B' \in \mathcal{B}^+$ *such that* $B^+ \leq_{I(\mathcal{B})} B'$.

Proof. We prove the if and the only if parts under three disjoint cases:
case 1: Suppose $I(\mathcal{B})@B \in \mathcal{B}$. First if $I(\mathcal{B})@_R B$, then $I(\mathcal{B}) = \mathcal{F}$, a contradiction. Next if, $I(\mathcal{B})@_L B \in \mathcal{B}$ then, by (2.4)$\neg(ii)$, we have $k(B) = 0$. By $\neg(iii)$ we have $B = (I(\mathcal{B}); 0; \emptyset)$, and by $\neg(iv), \mathcal{B} = \{(I(\mathcal{B}); 0; \emptyset)\}$. Clearly, $B \in split(\mathcal{B})$ if and only if $B \in \mathcal{B}^{ref}$ if and only if (i) holds.
case 2: Suppose $I(\mathcal{B}) \not@ B', \forall B' \in \mathcal{B}$ and $I(\mathcal{B}) = \mathcal{J}_0(\mathcal{B})$. By (2.7), $\mathcal{B} = \mathcal{B}^{ref} = \mathcal{B}^{atm}$. Then $I(\mathcal{B})$ is not coherent and $I(\mathcal{B}) = \bigcup_{j=1}^m \mathcal{I}_j = \bigcup_{j=1}^m I(\mathcal{B}_{\mathcal{I}_j})$, with $m \geq 2$. Let $\mathcal{B}' = (\mathcal{B} - \{(\mathcal{I}_1; 0; \emptyset)\}) \cup \mathcal{B}_{\mathcal{I}_1}$. Note that if $\exists B^+ \in \mathcal{B}_{\mathcal{I}_1}$ with $k(B^+) > 1$, then $\mathcal{B}'$ is coherent. By (1.6), $I(\mathcal{B}')$ is coherent. Hence $I(\mathcal{B}') \neq I(\mathcal{B})$, and so $B \notin split(\mathcal{B})$. Assume $\forall B^+ \in \mathcal{B}^+_{\mathcal{I}_1}, k(B^+) = 1$. Let $\mathcal{B}_1 = \mathcal{B}_{\mathcal{I}_1}$ and $\mathcal{B}_j = \mathcal{B}^+_{\mathcal{I}_1} \cup \{(\mathcal{I}_j; 0; \emptyset)\}$ if $j \geq 2$. Note that $I(\mathcal{B}') \supseteq \bigcup_{j=1}^m I(\mathcal{B}_j)$. Since we have $\forall B^+ \in \mathcal{B}^+_{\mathcal{I}_1}, k(B^+) = 1$, it can be seen by induction on height of trees of $I(\mathcal{B}')$ that $I(\mathcal{B}') \subseteq \bigcup_{j=1}^m I(\mathcal{B}_j)$ and hence $I(\mathcal{B}') = \bigcup_{j=1}^m I(\mathcal{B}_j)$. Next we show that $I(\mathcal{B}') = I(\mathcal{B})$ if and only if (ii) holds. Note that $\forall j \geq 2$, $\mathcal{B}_j$ is coherent. By (1.6), $I(\mathcal{B}_j)$ is coherent. By (1.5)(ii), $I(\mathcal{B}_j) \not\subseteq \mathcal{J}_0(\mathcal{B}_{\mathcal{I}_j})$. For $j = 2, \ldots, m$, apply (1.7)(ii) to $I(\mathcal{B}_j) \subseteq I(\mathcal{B}_{\mathcal{I}_j})$. Note also that by (2.8), $\mathcal{B}^+_{\mathcal{I}_j} \neq \emptyset$. Hence $(\mathcal{I}_j; 0; \emptyset) \leq_{\mathcal{I}_j} B^j$, for any bit $B^j \in \mathcal{B}^+_{\mathcal{I}_j}$. It follows $B \in split(\mathcal{B})$ if and only if (ii) holds.
case 3: Suppose $I(\mathcal{B}) \supset \mathcal{J}_0(\mathcal{B})$. Note that if $\mathcal{B}^+ = \emptyset$, then by (2.6)(i), $split(\mathcal{B}) = \emptyset$. Now, by (1.7)(ii), $I((\mathcal{B} - \{B\}) \cup \mathcal{B}_{\mathcal{I}_1}) \subseteq I(\mathcal{B})$ if and only if (iii) holds, because $\forall B^0 \in \mathcal{B}^0_{\mathcal{I}_1}$ we know (1.7)(ii) holds (i.e. $I(\{B^0\}) \subseteq J_0(B)$). Note that if B is non-reflexive then by (2.8), $\mathcal{B}^+_{\mathcal{I}_1} = \emptyset$ and so $B \in split(\mathcal{B})$ whenever $\mathcal{B}^+ \neq \emptyset$. ∎

Corollary 2.11 *Let $\mathcal{B} = \mathcal{B}^{ref} = \bigcup_{j=1}^{m}\{B_j\}$, where $B_j = (\mathcal{I}_j; 0; \emptyset)$, and $m \geq 2$. For some p, $0 \leq p \leq m$, let $split(\mathcal{B}) = \bigcup_{j=1}^{p}\{B_j\}$. Let $\mathcal{B}' = (\mathcal{B} - split(\mathcal{B})) \cup \{\mathcal{B}_{\mathcal{I}_j} : 1 \leq j \leq p\}$. If $shell(\mathcal{B}_{\mathcal{I}_j}) = split(\mathcal{B}_{\mathcal{I}_j}) = \emptyset, \forall j$ then $shell(\mathcal{B}') = split(\mathcal{B}') = \emptyset$.*

Proof. If $p = 1$, from the given conditions we can see that $B \in shell(\mathcal{B}')$ implies $B \in shell(\mathcal{B}_{\mathcal{I}_1})$ and also $B \in split(\mathcal{B}')$ implies $B \in split(\mathcal{B}_{\mathcal{I}_1})$, contrary to assumption. Hence the result holds for $p = 1$. Let $1 \leq i < j \leq p$ be given. Then $I((\mathcal{B} - \{B_i\}) \cup \mathcal{B}_{\mathcal{I}_i}) = I((\mathcal{B} - \{B_j\}) \cup \mathcal{B}_{\mathcal{I}_j})$ and so by (2.6)(ii), we have $\mathcal{B}^+_{\mathcal{I}_j} = \mathcal{B}^+_{\mathcal{I}_i}$. We deduce by symmetry that $\mathcal{B}^+_{\mathcal{I}_j} = \mathcal{B}'^+, \forall j, j = 1, \ldots, p$. By (2.4)$\neg(iv)$, and by coherence of $\mathcal{I}_j$, we note that $\mathcal{B}_{\mathcal{I}_j}$ contains exactly one width-zero bit B^0_j and that $I(\{B^0_j\}) \not\subseteq I(\{B^0_i\}), \forall i, j, i \neq j$. Otherwise we have $\mathcal{I}_i \subseteq \mathcal{I}_j$, a contradiction. Now (2.4)$\neg(i)$ $-$ (2.4)$\neg(v)$ can be easily checked to hold in $\mathcal{B}'$ and so $shell(\mathcal{B}') = \emptyset$. Moreover, since $\mathcal{B}'^+ = \mathcal{B}^+_{\mathcal{I}_j}$, by (2.10)(iii), we deduce $B^0_j \in split(\mathcal{B}')$ implies $B^0_j \in split(\mathcal{B}_{\mathcal{I}_j})$, contrary to assumption. Hence $split(\mathcal{B}') = \emptyset$. ∎

Definition 2.12 Let $\mathcal{I}$ be a proper lower ideal of trees. A *name* of $\mathcal{I}$ is a finite set of bits $\mathcal{B}$ such that,

(A1) $I(\mathcal{B}) = \mathcal{I}$
(A2) $shell(\mathcal{B}) = \emptyset$
(A3) $split(\mathcal{B}) = \emptyset$.

In section 4 we show that for every proper ideal $\mathcal{I}$ there is at least one name, and in the last section we prove there is at most one. We remark here that, using (2.4) and (5.1)(i), it is easy to see that (N1), (N2) and (N4) are implied by (A1), (A2) and (A3). We do not assume (N3).

3. CONTAINMENT OF IDEALS OF FINITE SETS OF BITS

We prove (1.7) in this section. Assuming (1.7)(i) does not hold, we prove (1.7)(ii) under two cases as shown in (3.1)(i) and (ii).

Theorem 3.1 *Let $\mathcal{B}$ and $\mathcal{B}'$ be finite sets of bits. Assume $I(\mathcal{B}) \not\subseteq \mathcal{J}_0(\mathcal{B}')$. Then $I(\mathcal{B}) \subseteq I(\mathcal{B}')$ if and only if:*
(i) $\forall B \in \mathcal{B}^+, \exists B' \in \mathcal{B}'$ *such that* $B \leq_{I(\mathcal{B}')} B'$ *and*
(ii) $\forall B \in \mathcal{B}^0, \exists B' \in \mathcal{B}'$ *such that* $B \leq_{I(\mathcal{B}')} B'$ *or* $I(\{B\}) \subseteq J_0(B')$.

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be finite sets of bits such that $I(\mathcal{B}) \not\subseteq \mathcal{J}_0(\mathcal{B}')$. Assume (i) and (ii) hold. Suppose, for a contradiction, $I(\mathcal{B}) \not\subseteq I(\mathcal{B}')$. Let T be a minimal height tree such that $T \in I(\mathcal{B}) - I(\mathcal{B}')$. Then T conforms to some bit B in $\mathcal{B}$. If $k(B) = 0$, and $I(\{B\}) \subseteq \mathcal{I}'@B' \in \mathcal{B}'$ then $T \in I(\{B\}) \subseteq \mathcal{I}' \subseteq I(\mathcal{B}')$, a contradiction. Otherwise $B \leq_{I(\mathcal{B}')} B'$, for some $B' \in \mathcal{B}'$. By the choice of T, it is a tree sum of trees in $I(\mathcal{B}')$.Using (D1)-(D4) it is routine to show that T conforms to B' in $\mathcal{B}'$, contrary to assumption. Hence $I(\mathcal{B}) \subseteq I(\mathcal{B}')$. Conversely, suppose that $I(\mathcal{B}) \subseteq I(\mathcal{B}')$. Let $\mathcal{B}' = \{B'_1, \ldots, B'_b\}$, and $B'_i = (\mathcal{I}^i_1, \ldots, \mathcal{I}^i_{n_i}; k_i; \mathcal{I}^i_0)$, for $i = 1, \ldots, b$. To show (3.1) (i) holds, suppose, for a contradiction, there is a bit $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; k; \mathcal{I}_0) \in \mathcal{B}^+$ such that $B \not\leq_{I(\mathcal{B}')} B'_i, \forall i$. By hypothesis, there is a tree $T \in I(\mathcal{B}) - \mathcal{J}_0(\mathcal{B}')$. Using T and B, we will construct a tree $T' \in I(\mathcal{B}) - I(\mathcal{B}')$, which leads to a contradiction.

claim: There exists a bit $B'_i \in \mathcal{B}'$ such that $k_i \geq k$ and $\mathcal{I}^i_0 \supseteq \mathcal{I}_0$.

For convenience, we write in the usual integer coefficient notation, mX, if a tree X appears m times in a tree sum. Let $T' = Tree(kT, (n_1 + k_1 + 1)T_1, \ldots, (n_b + k_b + 1)T_b)$, where $T_i = \Gamma$ if $\mathcal{I}^i_0 \supseteq \mathcal{I}_0$, and $T_i \in \mathcal{I}_0 - \mathcal{I}^i_0$, if $\mathcal{I}^i_0 \not\supseteq \mathcal{I}_0, i = 1, \ldots, b$. Then T' conforms to B in $\mathcal{B}$, clearly. Also, $T' \notin \mathcal{J}_0(\mathcal{B}')$ since T'contains T, and hence does not conform by (1) to $\mathcal{B}'$. Assume, contrary to assumption, that for each i, either $k_i < k$ or $\mathcal{I}^i_0 \not\supseteq \mathcal{I}_0$. Since T is repeated k times, T' conforms to no bit B'_i in $\mathcal{B}'$ of width $k_i < k$. If $\mathcal{I}^i_0 \not\supseteq \mathcal{I}_0$, then T_i has large enough multiplicity, that T' does not conform by (2) to $\mathcal{B}'$. Hence, T' conforms to no bit $B'_i \in \mathcal{B}'$. Thus $T' \in I(\mathcal{B}) - I(\mathcal{B}')$, contrary to assumption, and the claim follows. We may suppose (by reordering if necessary) that $\mathcal{S} = \{B_1, \ldots, B_s\} \subseteq \mathcal{B}'$, is the set of all bits of $\mathcal{B}'$ satisfying $k_i \geq k$ and $\mathcal{I}^i_0 \supseteq \mathcal{I}_0$, where $1 \leq s \leq b$. Fix i, and note that by assumption $B \not\leq_{I(\mathcal{B}')} B'_i$. Let $S^i = \{\mathcal{I}^i_1, \ldots, \mathcal{I}^i_{n_i}, I(\mathcal{B}'), \ldots, I(\mathcal{B}')\}$, with $k_i - k$ copies of $I(\mathcal{B}')$.

claim: There exist trees $T^i_1, \ldots, T^i_n$, from $\mathcal{I}_1 \cup \{\Gamma\}, \ldots, \cup \mathcal{I}_n \cup \{\Gamma\}$, respectively, such that there is no distinct representation of these trees from S^i, and none of the trees are contained in $\mathcal{I}^i_0$.

Let H_i be the set of indices j of left components of B, such that $\mathcal{I}_j \not\subseteq \mathcal{I}^i_0$. Since $B \not\leq_{I(\mathcal{B}')} B'_i$, we have $k_i - k < |H_i| \leq n$. Also, let $C_i = \{1, \ldots, n_i, n_i + 1, \ldots, n_i + (k_i - k)\}$. By assumption $B \not\leq_{I(\mathcal{B}')} B'_i$, therefore there is no 1-to-1 function from H_i to C_i that maps j to j' if and only if $\mathcal{I}_j \subseteq \mathcal{I}^i_{j'}$. Applying Hall's marriage theorem there is a subset $K_i \subseteq H_i$ with $|K_i| > |D_i|$, where $D_i = \{l \in C_i : \mathcal{I}_j \subseteq \mathcal{I}^i_l$ for some $j \in K_i\}$. Clearly $\{n_i + 1, \ldots, n_i + k_i - k\} \subseteq D_i \subseteq C_i$. Hence we can find trees $T^i_1 ..., T^i_{m_i}$ from K_i, where $m_i = |K_i|$, such that there is no distinct representation of these m_i trees from C_i. By the coherence of the ideals indexed by K_i, and the definition of H_i, we may assume these trees are not in $\mathcal{I}^i_0$ and the claim follows. Now, let i vary from 1 to s. Since $\mathcal{I}_j$ is a coherent ideal, we can find a tree $T'_j \in \mathcal{I}_j \cup \{\Gamma\}$ containing $T^1_j, T^2_j, ...,$ and T^s_j, where $1 \leq j \leq n$. We have trees $T'_1, \ldots, T'_n$ such that there is no distinct representation of $T'_1, \ldots, T'_n$ from the multi-set S^i for all $i, i = 1, 2, \ldots, s$. Then consider $T'' = Tree(T'_1, \ldots, T'_n, kT, (n_1 + k_1 + 1)T_1, \ldots, (n_b + k_b + 1)T_b)$. T'' conforms in $\mathcal{B}'$ neither to a bit of $\mathcal{B}' - \mathcal{S}$, nor to a bit of $\mathcal{S}$, but clearly conforms to B in $\mathcal{B}$, contrary to $I(\mathcal{B}) \subseteq I(\mathcal{B}')$. Hence (3.1) (i) must hold.

To establish (3.1) (ii), assume $B = (\mathcal{I}_1, \ldots, \mathcal{I}_n; 0; \mathcal{I}_0) \in \mathcal{B}^0$. If $I(\{B\}) \subseteq \mathcal{I}'@B'$ for some $B' \in \mathcal{B}'$, then (3.1) (ii) holds. Otherwise, since $I(\{B\})$ is a coherent ideal, $I(\{B\}) \not\subseteq \mathcal{J}_0(\mathcal{B}')$. For a contradiction, assume $B \not\leq_{I(\mathcal{B}')} B'_i$, for all $i, i = 1, \ldots, b$. We note that the argument for (3.1) (i) applies to show that T'' does not conform by (2) to $\mathcal{B}'$. However, since $k = 0$ (i.e. T is not contained in T'') it is possible that T'' conforms by (1) to some bit in

$\mathcal{B}'$. Let $T_B \in I(\{B\}) - \mathcal{J}_0(\mathcal{B}')$. If $T_B \in \mathcal{I}_0$, then $Tree(T_1', \ldots, T_n', (n_1 + k_1 + 1)T_1, \ldots, (n_b + k_b + 1)T_b, T_B) \in I(\mathcal{B}) - I(\mathcal{B}')$, a contradiction. If $T_B \in \mathcal{I}_i, 1 \leq i \leq n$, then $Tree(T_1', \ldots, T_{i-1}', T_i'', T_{i+1}' \ldots, T_n', (n_1 + k_1 + 1)T_1, \ldots, (n_b + k_b + 1)T_b) \in I(\mathcal{B}) - I(\mathcal{B}')$, where $T_i'' \in \mathcal{I}_i$ contains both T_B and T_i', a contradiction. Suppose $T_B = Tree(T_1^B, \ldots, T_n^B, T_{n+1}^B, \ldots, T_{n+m}^B)$.(i.e. T_B conforms by (2) to $\{B\}$.) Let $T_i'' \in \mathcal{I}_i \cup \Gamma$ be a tree containing both T_i' and T_i^B, for all $i = 1, \ldots, n$. Then $Tree(T_1'', \ldots, T_n'', (n_1 + k_1 + 1)T_1, \ldots, (n_b + k_b + 1)T_b, T_{n+1}^B, \ldots, T_{n+m}^B) \in I(\mathcal{B}) - I(\mathcal{B}')$, a contradiction. ∎

4. EXISTENCE

Theorem 4.1 *Every proper ideal $\mathcal{I}$ has at least one name.*

Proof. Let $\mathcal{I}$ be an ideal and let $\mathcal{B}_0$ be the finite set of bits constructed in [5] satisfying (N1), (N2) and (N3). We will construct a set satisfying (A1), (A2) and (A3). We use arrow notation for implication of axioms. The following are straightforward: (N2)⇔ (A1), (N1)⇒ (2.4)$\neg(i)$ and $\neg(ii)$, and (N3)⇒ (2.4)$\neg(v)$. To satisfy (2.4)$\neg(iii)$, we replace every $B' \in \mathcal{B}_0^{ref} - \mathcal{B}_0^{atm}$, by $(I(\{B'\}); 0; \emptyset)$. Next, since $\mathcal{B}_0$ is finite (2.4)$\neg(iv)$ can be attained in a finite number of steps. We have $\mathcal{B}$ satisfying (A1) and (A2).

Consider the case where $I(\mathcal{B}_0) \supset \mathcal{J}_0(\mathcal{B}_0)$. We show (N3) in $\mathcal{B}_0 \Rightarrow$ (A3) in $\mathcal{B}$. Suppose the contrary that $B = (\mathcal{I}_1; 0; \emptyset) \in split(\mathcal{B})$. If $B \in \mathcal{B}_0$ then, by (N3), $\mathcal{B}^+ = \emptyset$, contrary (2.10)(iii). Hence $B \notin \mathcal{B}_0$. Therefore there is a $B' \in \mathcal{B}_0^{ref} - \mathcal{B}_0^{atm}$, such that $B = (I(\{B'\}); 0; \emptyset)$. Apply (1.7)(ii) to $\{B'\}$ and $\mathcal{B}_{\mathcal{I}_1}$, so $B' \leq_{\mathcal{I}_1} B^+$, for some B^+ in $\mathcal{B}_{\mathcal{I}_1}^+$. By (2.10)(iii), $B^+ \leq_{\mathcal{I}} B''$, for some $B'' \in \mathcal{B}^+ \subseteq \mathcal{B}_0^+$. But then, $B' \leq_{\mathcal{I}_1} B^+ \leq_{\mathcal{I}} B''$ implies $B' \leq_{\mathcal{I}} B''$, contrary to (N3). It follows that $\mathcal{B}$ satisfies (A3). Hence existence of a name for coherent ideals is established at this point, because if $\mathcal{I}$ is coherent, then $I(\mathcal{B}_0) \supset \mathcal{J}_0(\mathcal{B}_0)$.

For the case $I(\mathcal{B}_0) = \mathcal{J}_0(\mathcal{B}_0^{ref})$, by (2.11), $\mathcal{B}' = (\mathcal{B} - split(\mathcal{B})) \cup \{\mathcal{B}_{\mathcal{I}_i} : (\mathcal{I}_i; 0; \emptyset) \in split(\mathcal{B}), \mathcal{B}_{\mathcal{I}_i}$ a name of $\mathcal{I}_i\}$, is a name of $\mathcal{I}$. ∎

5. UNIQUENESS

We prove first an interesting lemma which shows that for any proper ideal $\mathcal{I}$ and for any finite set of bits $\mathcal{B}$ describing $\mathcal{I}$, if $\mathcal{B}$ contains an atom $B = (\mathcal{I}_1; 0; \emptyset)$, then $\mathcal{I}_1$ is contained in $\mathcal{J}_0(\mathcal{B}')$ for any finite set of bits $\mathcal{B}'$ describing $\mathcal{I}$, or B is splittable in $\mathcal{B}$. We also observe that splitting atoms tends to construct more trees, whereas shelling a bit tends to construct less. We show that the unique name of an ideal $\mathcal{I}$ is found when neither shelling nor splitting can be done on the bits.

Lemma 5.1 *Let $\mathcal{B}_1, \mathcal{B}_2$ be finite sets of bits such that $I(\mathcal{B}_1) = I(\mathcal{B}_2)$. Then:*

(i) if $B \in \mathcal{B}_1^{atm}$, then either $J_0(B) \subseteq J_0(B')$ for some $B' \in \mathcal{B}_2$, or $B \in split(\mathcal{B}_1)$, and

(ii) for $i = 1, 2$, if $I(\mathcal{B}_i) \supset \mathcal{J}_0(\mathcal{B}_i)$ and $\mathcal{B}_i$ satisfies (A2) and (A3), then $\mathcal{B}_1^{atm} = \mathcal{B}_2^{atm}$.

Proof. Let $J_0(B) = \mathcal{I}_1$. To get (i) we apply (1.7) to the following three inclusions: $I(\mathcal{B}_{\mathcal{I}_1}) \subseteq I(\mathcal{B}_2)$, $I(\mathcal{B}_1) \subseteq I(\mathcal{B}_2)$, and $I(\mathcal{B}_1 - \{B\}) \cup \mathcal{B}_{\mathcal{I}_1}) \subseteq I(\mathcal{B}_2)$. We are given that (1.7)(i) does not hold for all three pairs, because $\mathcal{I}_1$ is coherent and so $\mathcal{I}_1 \not\subseteq \mathcal{J}_0(\mathcal{B}_2)$. The first two inclusions are given and together they imply (1.7)(ii) holds for the third inclusion. Since $I(\mathcal{B}_1) = I(\mathcal{B}_2)$, we have $B \in split(\mathcal{B})$.

To obtain (ii), first suppose that $\mathcal{B}_1^+ = \emptyset$. Then, (ii) follows from (2.6)(i). If $\mathcal{B}_1^+ \neq \emptyset$, then $\mathcal{B}^{atm} = \mathcal{B}^{ref}$, or else by (2.10)(iii), we contradict (A3). Hence, by (i), if $B \in \mathcal{B}^{atm}$, we have $I(\{B\}) = J_0(B) \subseteq J_0(B')$. By (2.6)(iii), $B' \in \mathcal{B}'^{atm}$. We deduce, by symmetry and (2.4)$\neg(iv)$, that $B = B'$ and so $\mathcal{B}_1^{atm} \subseteq \mathcal{B}_2^{atm}$. Similarly, $\mathcal{B}_2^{atm} \subseteq \mathcal{B}_1^{atm}$ and so (ii) follows. ∎

Theorem 5.2 *Every proper ideal $\mathcal{I}$ has at most one name.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be names of $\mathcal{I}$. First, if $I(\mathcal{B}_i) = \mathcal{J}_0(\mathcal{B}_i)$ for $i = 1$ or 2, then, by (2.7), we have $\mathcal{B}_i = \mathcal{B}_i^{atm}$. Then, applying (5.1)(i) to each bit $B \in \mathcal{B}_i$ and to the set $\mathcal{B}_j, j \neq i$, we get $\mathcal{I} = \mathcal{J}_0(\mathcal{B}_j)$. By (A1), we have $I(\mathcal{B}_j) = \mathcal{J}_0(\mathcal{B}_j)$. Hence, by (2.7) and (1.5)(i), we deduce $\mathcal{B}_1 = \mathcal{B}_2$. Next, suppose $I(\mathcal{B}_i) \supset \mathcal{J}_0(\mathcal{B}_i)$ for both $i = 1, 2$. By (5.1)(ii), $\mathcal{B}_1^{atm} = \mathcal{B}_2^{atm}$. By (2.6)(iv), $\mathcal{B}_1 = \mathcal{B}_2$. ∎

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA E-mail: robertso@math.ohio-state.edu

DEPARTMENT OF APPLIED MATHEMATICS, ITI, CHARLES UNIVERSITY, PRAGUE, CZECH REPUBLIC E-mail: yared@kam.mff.cuni.cz