BOUNDED COHOMOLOGY AND AMENABLE GROUPS

BOBBY RAMSEY

Abstract. We recall the definition of bounded cohomology for a discrete group and give an elementary proof that for amenable groups this bounded cohomology is trivial.

1. Bounded Cohomology

Let G be a discrete group, and let $C^n(G) = \{\phi : G^n \to \mathbb{R}\}$ with $\delta_n : C^n(G) \to C^{n+1}(G)$ defined by

$$\begin{aligned}(\delta_n\phi)(x_1, ..., x_{n+1}) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1x_2, x_3, ..., x_{n+1}) \\ &+... + (-1)^i\phi(x_1, ..., x_ix_{i+1}, ..., x_{n+1}) + ... \\ &+(-1)^{n+1}\phi(x_1, ..., x_n)\end{aligned}$$

This yields a cochain complex:

$$C^0(G) \xrightarrow{\delta_0} C^1(G) \xrightarrow{\delta_1} C^2(G) \xrightarrow{\delta_2} ...$$

The cohomology of this complex, denoted by $H^*(G)$, is the standard cohomology of the group G (with $\mathbb{R}$ coefficients).

Let us now restrict our attention to bounded functions [Ger]. Denote by $C_b^n(G)$ those $\phi \in C^n(G)$ with

$$\|\phi\|_\infty = \sup_{(x_1,...,x_n)\in G^n} |\phi(x_1, ..., x_n)| < \infty$$

If $\phi \in C_b^n(G)$, then consider $\delta_n\phi \in C^{n+1}(G)$

$$\begin{aligned}(\delta_n\phi)(x_1, ..., x_{n+1}) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1x_2, x_3, ..., x_{n+1}) \\ &+... + (-1)^i\phi(x_1, ..., x_ix_{i+1}, ..., x_{n+1}) + ... \\ &+(-1)^{n+1}\phi(x_1, ..., x_n)\end{aligned}$$

Date: June 2006.

So that when considering boundedness:

$$\begin{aligned}\|\delta_n\phi\|_\infty &\leq \|\phi\|_\infty + \|\phi\|_\infty \\ &\quad +... + \|\phi\|_\infty + ... \\ &\quad + \|\phi\|_\infty \\ &\leq (n+1)\|\phi\|_\infty\end{aligned}$$

Thus the differentials preserve boundedness. That is

$$C_b^0(G) \xrightarrow{\delta_0} C_b^1(G) \xrightarrow{\delta_1} C_b^2(G) \xrightarrow{\delta_2} ...$$

is a sub-cochain complex of the above complex. The cohomology of this complex, denoted by $H_b^*(G)$, is the *bounded cohomology* of the group G. As an example we will start by showing the following:

Proposition 1.1. *For any discrete group G, $H_b^1(G) = 0$.*

Proof. Let $\phi \in C_b^1(G)$ with $\delta_1\phi = 0$. Then for all a, $b \in G$ we have $(\delta_1\phi)(a,b) = \phi(b) - \phi(ab) + \phi(a) = 0$. That is, $\phi(ab) = \phi(a) + \phi(b)$ so that ϕ is a homomorphism from G to $\mathbb{R}$.

Assume that there is a $g \in G$ with $\phi(g) \neq 0$. Then $\phi(g^n) = n\phi(g)$ so that $\lim_{n\to\infty} |\phi(g^n)| = \infty$, contradicting that $f \in C_b^1(G)$.

Thus $\ker \delta_1 = 0$, so $H_b^1(G) = 0$. □

Now $C_b^0(G) = \{\phi : G^0 \to \mathbb{R} | \|\phi\|_\infty < \infty\}$, which we can identify with $\mathbb{R}$, with $\delta_0 = 0$. Thus we also have $H_b^0(G) = \mathbb{R}$, but this is not surprising as we are dealing with $\mathbb{R}$-coefficients.

2. Amenable Groups

Definition 2.1. [Pat] *A discrete group G is amenable if there is a left-invariant mean $m : \ell^\infty G \to \mathbb{C}$.*

This means that m is a positive linear functional on $\ell^\infty G$, with $\|m\| = m(1) = 1$. Moreover, for each $\phi \in \ell^\infty G$ and $g \in G$, $m(\phi) = m(\phi_g)$ where $\phi_g(x) = \phi(g \cdot x)$.

The following result is attributed in [Gro] to Hirsch and Thurston [HT]. We give an elementary proof.

Theorem 2.1. *Let G be discrete amenable group. Then for $n \geq 1$ we have $H_b^n(G) = 0$.*

Proof. Denote a left-invariant mean by m. We will first note that, as m is positive, $m(\psi) \in \mathbb{R}$ for any real-valued $\psi \in \ell^\infty G$. We have already seen that $H_b^1(G) = 0$. Let us start by examining $H_b^2(G)$.

Let $\phi \in C_b^2(G)$ with $\delta_2\phi = 0$. That is, for all a, b, and $c \in G$ we have

$$(\delta_2\phi)(a,b,c) = \phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b) = 0$$

So that $\phi(a,b) = \phi(b,c) - \phi(ab,c) + \phi(a,bc)$. Let $f : G \to \mathbb{R}$ be defined by $f(a) = m(\phi(a,x))$, where by $\phi(a,x)$ we mean the bounded function from G to $\mathbb{R}$ obtained by fixing the first argument of ϕ. As $f(a) = m(\phi(a,x)) \leq \|m\|\|\phi\|_\infty < \infty$, we have $f \in C_b^1(G)$.

$$
\begin{aligned}
(\delta_1 f)(a,b) &= f(b) - f(ab) + f(a) \\
&= m(\phi(b,x)) - m(\phi(ab,x)) + m(\phi(a,x)) \\
&= m(\phi(b,x)) - m(\phi(ab,x)) + m(\phi(a,bx)) \\
&= m(\phi(b,x) - \phi(ab,x) + \phi(a,bx)) \\
&= m(\phi(a,b)) \\
&= \phi(a,b) \cdot m(1) \\
&= \phi(a,b)
\end{aligned}
$$

Thus ϕ is in the image of δ_1, so $H_b^2(G) = 0$.

The case for general $n > 1$ is similar in technique to this case. Let $\phi \in C_b^n(G)$ with $\delta_n \phi = 0$. For all $x_1, ..., x_{n+1} \in G$

$$
\begin{aligned}
(\delta_n \phi)(x_1, ..., x_{n+1}) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1 x_2, x_3, ..., x_{n+1}) \\
&\quad + ... + (-1)^i \phi(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) + ... \\
&\quad + (-1)^{n+1} \phi(x_1, ..., x_n) \\
&= 0
\end{aligned}
$$

So that

$$
\begin{aligned}
(-1)^n \phi(x_1, ..., x_n) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1 x_2, x_3, ..., x_{n+1}) \\
&\quad + ... + (-1)^i \phi(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) + ... \\
&\quad + (-1)^n \phi(x_1, ..., x_n x_{n+1})
\end{aligned}
$$

Let $f : G^{n-1} \to \mathbb{R}$ be defined by

$$f(x_1, ..., x_{n-1}) = m(\phi(x_1, ..., x_{n-1}, x))$$

where, as before, $\phi(x_1, ..., x_{n-1}, x)$ is treated as a bounded function of one variable obtained by fixing the first $n-1$ arguments of ϕ.

Applying δ_{n-1} to f, we obtain the following:

$$
\begin{aligned}
(\delta_{n-1}f)(x_1,...,x_n) &= f(x_2,...,x_n) - f(x_1x_2,x_3,...,x_n) \\
&\quad +... + (-1)^i f(x_1,...,x_ix_{i+1},...,x_n) + ... \\
&\quad +(-1)^n f(x_1,...,x_{n-1}) \\
&= m(\phi(x_2,...,x_n,x)) - m(\phi(x_1x_2,x_3,...x_n,x)) \\
&\quad +... + (-1)^i m(\phi(x_1,...,x_ix_{i+1},...,x_n,x)) + ... \\
&\quad +(-1)^n m(\phi(x_1,...,x_{n-1},x)) \\
&= m(\phi(x_2,...,x_n,x)) - m(\phi(x_1x_2,x_3,...x_n,x)) \\
&\quad +... + (-1)^i m(\phi(x_1,...,x_ix_{i+1},...,x_n,x)) + ... \\
&\quad +(-1)^n m(\phi(x_1,...,x_{n-1},x_nx)) \\
&= m(\phi(x_2,...,x_n,x) - \phi(x_1x_2,x_3,...x_n,x) \\
&\quad +... + (-1)^i \phi(x_1,...,x_ix_{i+1},...,x_n,x) + ... \\
&\quad +(-1)^n \phi(x_1,...,x_{n-1},x_nx)) \\
&= m((-1)^n\phi(x_1,...,x_n)) \\
&= (-1)^n\phi(x_1,...,x_n)\cdot m(1) \\
&= (-1)^n\phi(x_1,...,x_n)
\end{aligned}
$$

Thus $\phi = \delta_{n-1}(-1)^n f$, so the kernel of δ_n lies in the image of δ_{n-1}, resulting in $H_b^n(G) = 0$. □

References

[Ger] S. M. Gersten. Bounded Cohomology and Combings of Groups. Preprint, University of Utah (1991). http://www.math.utah.edu/~sg/old-eprints.htm

[Gro] M. Gromov. Volume and Bounded Cohomology. IHES Publ. Math. **52** (1982), 5 – 99.

[HT] J. Hirsch, W. Thurston. Foliated Bundles, Invariant Measures and Flat Manifolds. Ann. of Math., **101** (1975), 369 – 390.

[Pat] A. Patterson. *Amenability*. Mathematical Surveys and Monographs, 29. American Mathematical Society, Providence, RI, 1988.