The polynomially bounded conjugacy problem for relatively hyperbolic groups

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- G - finitely generated group
- P - finitely generated $\mathbb{Z}G$-module
- $P \oplus Q \cong (\mathbb{Z}G)^n$
- $\mathrm{End}_{\mathbb{Z}G}(P \oplus Q) \cong M_{n\times n}(\mathbb{Z}G)$

- Trace $Tr : M_{n\times n}(\mathbb{Z}G) \to \frac{\mathbb{Z}G}{[\mathbb{Z}G,\mathbb{Z}G]}$
- $\frac{\mathbb{Z}G}{[\mathbb{Z}G,\mathbb{Z}G]} \cong \bigoplus_{x\in<G>} \mathbb{Z}$
- P-rank of $g \in G$, $r_P(g) = \pi_{<g>} Tr Id_P$

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Conjecture. Classical Bass Conjecture *For any finitely generated projective* $\mathbb{Z}G$*-module* P*,* $r_P(g) = 0$ *for* $g \neq 1_G$*.*

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Conjecture. Classical Bass Conjecture *For any finitely generated projective $\mathbb{Z}G$-module P, $r_P(g) = 0$ for $g \neq 1_G$.*

Linell - True for nontrivial torsion elements.

- $HH_0(\mathbb{Z}G) \cong \bigoplus_{x \in <G>} \mathbb{Z}$
- $Tr^{HS} : K_0(\mathbb{Z}G) \to HH_0(\mathbb{Z}G)$
- $Tr^{HS} : K_0(\mathbb{C}G) \to HH_0(\mathbb{C}G)$

$$
\begin{array}{ccccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{C}G^{\otimes 3} & \xleftarrow{1-t} & \mathbb{C}G^{\otimes 3} & \xleftarrow{N} & \mathbb{C}G^{\otimes 3} & \xleftarrow{1-t} & \mathbb{C}G^{\otimes 3} & \xleftarrow{N} & \mathbb{C}G^{\otimes 3} & \longleftarrow & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & b\downarrow & & \\
\mathbb{C}G^{\otimes 2} & \xleftarrow{1-t} & \mathbb{C}G^{\otimes 2} & \xleftarrow{N} & \mathbb{C}G^{\otimes 2} & \xleftarrow{1-t} & \mathbb{C}G^{\otimes 2} & \xleftarrow{N} & \mathbb{C}G^{\otimes 2} & \longleftarrow & \\
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\end{array}
$$

$$
\begin{aligned}
b'(g_0, g_1, \ldots, g_n) &= \sum_{i=0}^{n-1} (-1)^i \, (g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \\
b(g_0, g_1, \ldots, g_n) &= \sum_{i=0}^{n-1} (-1)^i \, (g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \\
&\quad + (-1)^n \, (g_n g_0, g_1, \ldots, g_{n-1}) \\
t(g_0, g_1, \ldots, g_n) &= (-1)^n \, (g_n, g_0, \ldots, g_{n-1}) \\
N &= 1 + t + t^2 + \ldots + t^n
\end{aligned}
$$

Chern-Connes characters $ch_n^m : K_n(\mathbb{C}G) \to HC_{n+2m}(\mathbb{C}G)$

$$S \circ ch_n^m = ch_n^{m-1}$$

Conjecture. Strong Bass Conjecture *For each non-elliptic class x, the image of the composition $\pi_x \circ ch_* : K_*(\mathbb{C}G) \longrightarrow HC_*(\mathbb{C}G)_x$ is zero.*

$x \in < G >$ satisfies ‘nilpotency condition’ if
$S_x : HC_*(\mathbb{C}G)_x \longrightarrow HC_{*-2}(\mathbb{C}G)_x$ is nilpotent.

Observation 1. Let x be a non-elliptic conjugacy class satisfying the nilpotency condition. The composition
$K_n(\mathbb{C}G) \longrightarrow HC_n(\mathbb{C}G) \xrightarrow{\pi_x} HC_n(\mathbb{C}G)_x$ is zero for all $n \geq 0$. In particular the strong Bass conjecture holds for G.

For $h \in x$, $N_h = G_h/(h)$.
Burghelea Decomposition: x non-elliptic
$HC_*(\mathbb{C}G)_x \cong H_*(N_h)$.
$S_x : HC_*(\mathbb{C}G)_x \longrightarrow HC_{*-2}(\mathbb{C}G)_x$

$$0 \longrightarrow (h) \longrightarrow G_h \longrightarrow N_h \longrightarrow 0$$

If each N_h has finite cohomological dimension, then G satisfies the nilpotency condition.

Most basic topological extension, $\ell^1 G$.

Conjecture. ℓ^1 **- Bass Conjecture** *For each non-elliptic conjugacy class* x*, the image of the composition* $K_0(\ell^1 G) \to HC_0(\ell^1 G)_x$ *is zero.*

$$HC^*(\ell^1 G)_x \cong H_b^*(G_h) \otimes HC^*(\mathbb{C})$$

$\mathcal{S}G$ the set of all $\phi : G \to \mathbb{C}$

$$\|\phi\|_k := \sum_{g \in G} |\phi(g)| \left(1 + \ell(g)\right)^k < \infty$$

Conjecture. Strong ℓ^1 - Bass Conjecture *For each non-elliptic conjugacy class, the image of the composition $\pi_x \circ ch_* : K_*(\mathcal{S}G) \to HC_*(\mathcal{S}G)_x$ is zero.*

Is true whenever, for each non-elliptic conjugacy class x, $S_x : HC_*(\mathcal{S}G)_x \to HC_{*-2}(\mathcal{S}G)_x$ is nilpotent.

$C_n(H)$ the set of all finitely supported chains for $H^*(H)$

$$0 \leftarrow C_0(H) \leftarrow C_1(H) \leftarrow C_2(H) \leftarrow \ldots$$

$PC_n(H)$ the superset of those ϕ for which $\sum_{h_1,\ldots,h_n} |\phi(h_1,\ldots,h_n)| \left(1+\ell(h_0)+\ldots+\ell(h_n)\right)^k < \infty$.

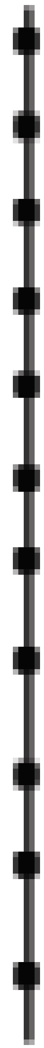
$$0 \leftarrow PC_0(H) \leftarrow PC_1(H) \leftarrow PC_2(H) \leftarrow \ldots$$

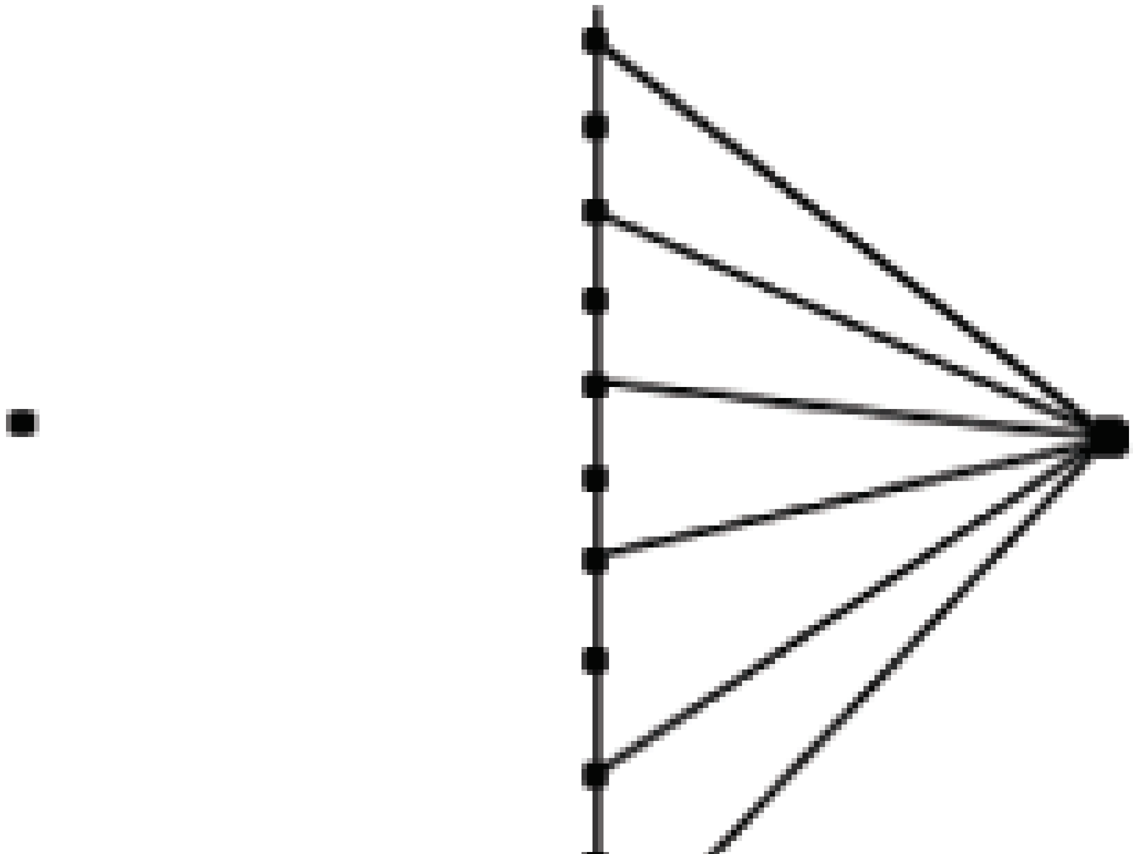
$HP_*^{\ell}(H)$
In some situations Burghelea decomposition extends to obtain $HC_*(\mathcal{S}G)_x \cong HP_*^{\ell_G}(N_h)$.

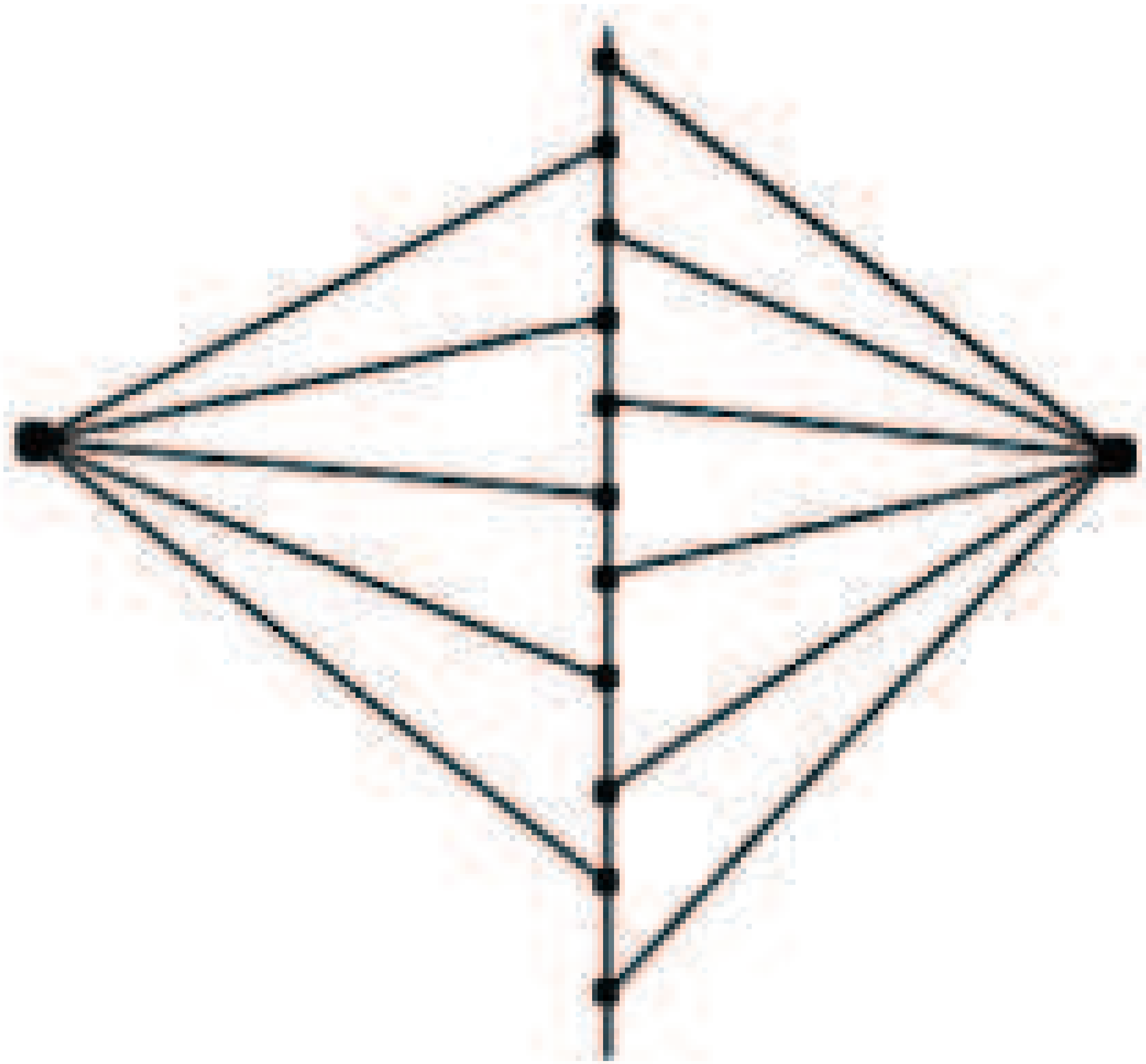
G satisfies a polynomial conjugacy problem if for each $x \in < G >$ there is P_x such that: $u, v \in x$ then there is $g \in G$ with $g^{-1}ug = v$ such that $\ell(g) \leq P_x(\ell(u) + \ell(v))$.

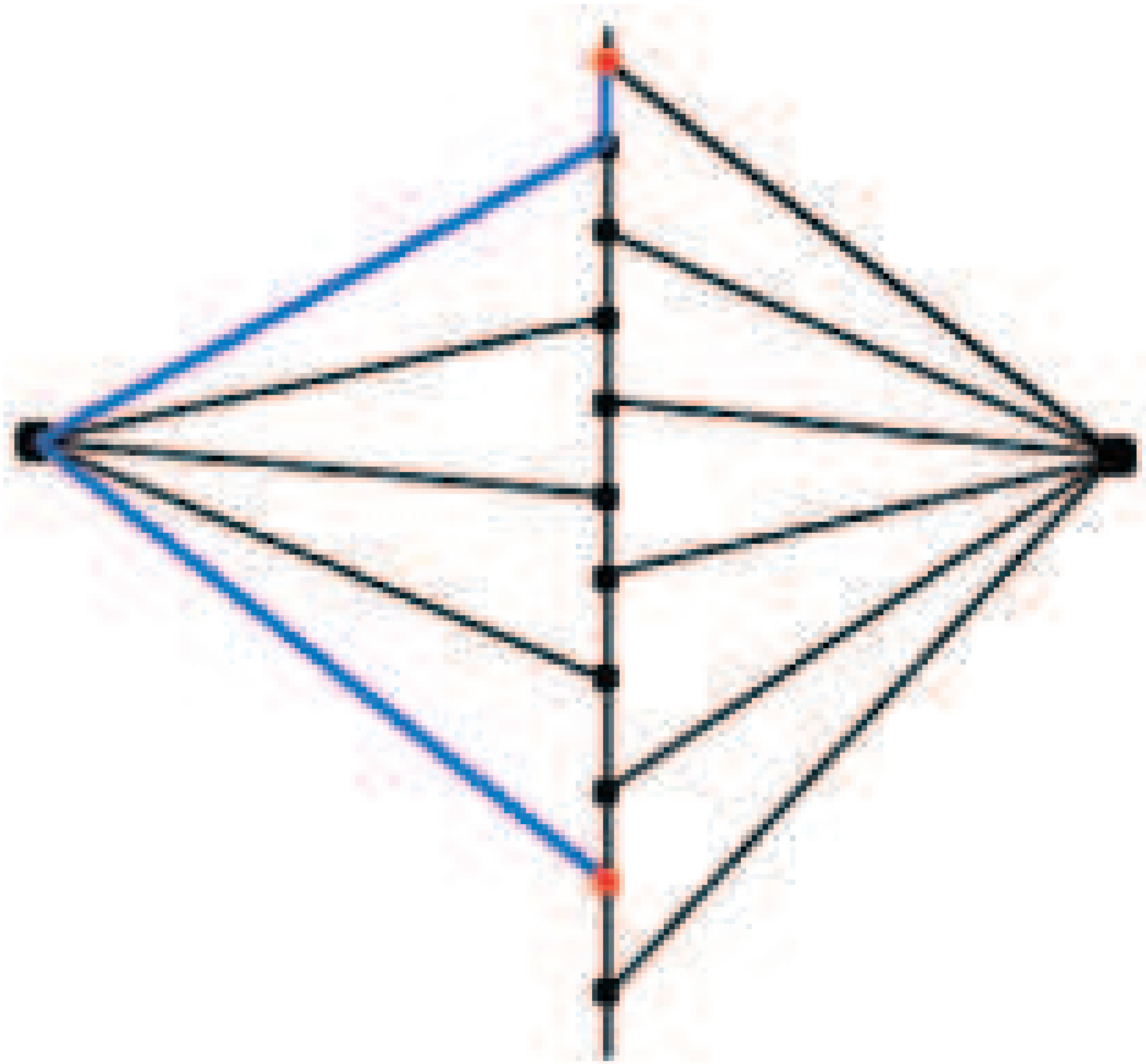
- Hyperbolic groups
- Pseudo-Anasov classes of Mapping class groups
- Mapping class groups
- 2-step f.g. Nilpotent groups

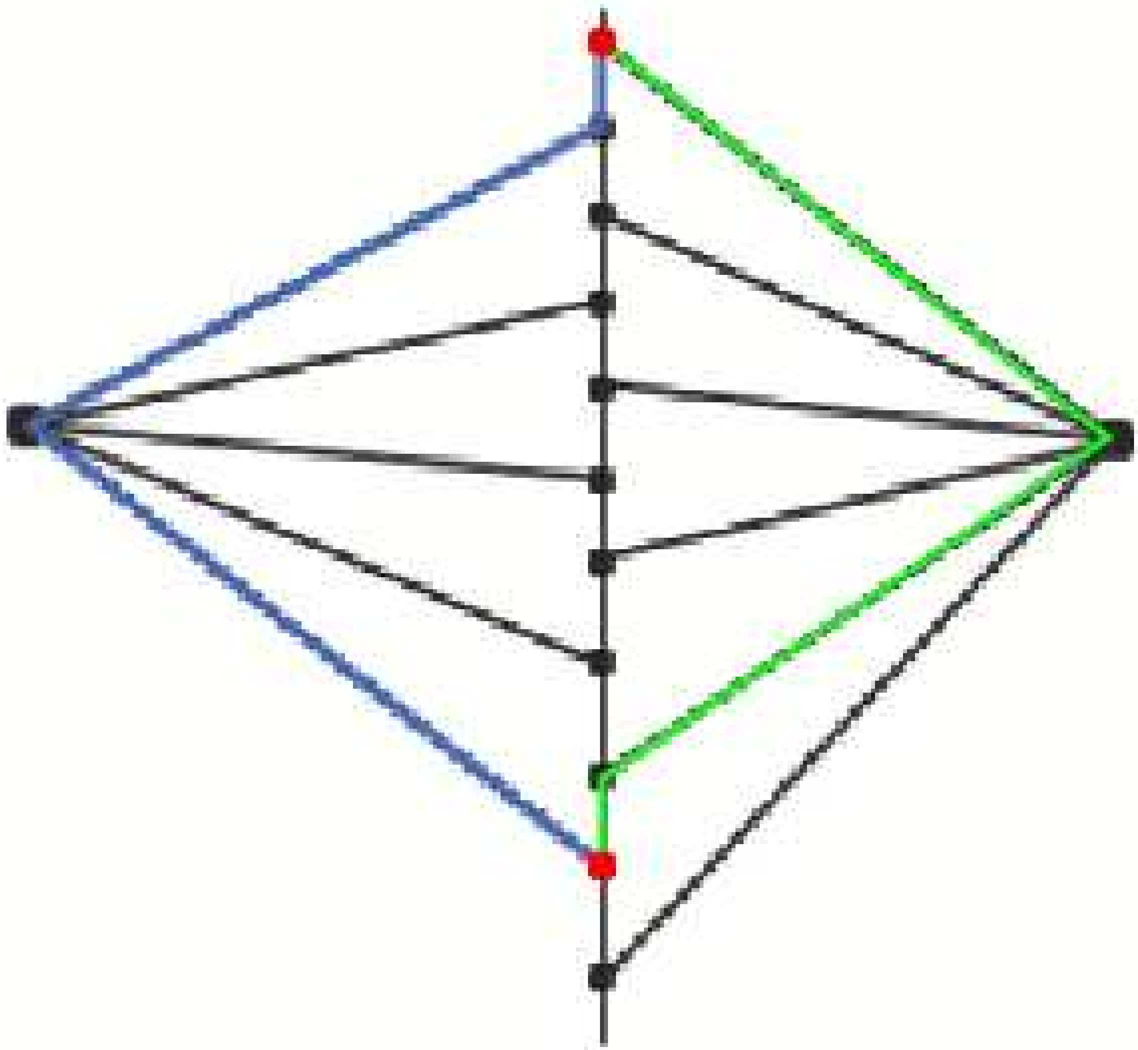
For these groups, if $HP_*^{\ell_G}(N_h) \cong H_*(N_h)$ and $S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x$ nilpotent then $S_x : HC_*(\mathcal{S}G)_x \to HC_{*-2}(\mathcal{S}G)_x$ is nilpotent.











G and $H_1, H_2, \ldots, H_n$ satisfy the bounded coset penetration property if for every λ there is a constant $c(\lambda)$ such that if p and q are two $(\lambda, 0)$-quasi-geodesics without backtracking, starting and ending at the same group element vertices then:

- If p and q both penetrate a coset gH_i, the points at which p and q enter (respectively exit) gH are at a distance no more than $c(\lambda)$ from one another.
- If p penetrates a coset gH_i which is not penetrated by q, then the points where p enters the coset and where p exits the coset are within $c(\lambda)$ from one another.

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G is relatively hyperbolic with respect to the H_i if the relative graph is hyperbolic and satisfies the bounded coset penetration property.

Lemma. *Let u and v be conjugate nontorsion hyperbolic elements of G. There is a constant K_h and a $g \in G$ with $u = g^{-1}vg$ and $\ell_{\hat{\Gamma}}(\hat{g}) \leq K_h(\ell(u) + \ell(v))$. Moreover K_h is independent of u and v.*

Maher's proof of the Pseudo-Anasov case for Mapping Class groups.

Lemma. *Drutu-Sapir Let* g *be an element in* $\gamma H\gamma^{-1}$*, and let* x *be a point in* $G \setminus \gamma H$*. Let* x_1 *be a nearest point projection of* x *onto* γH*. Then there exists a uniform constant* C *such that one of the following situations occur:*

1. $d_G(x_1, gx_1) \leq C$
2. $d_G(x, gx) \geq d_G(x, \gamma H) + \frac{1}{2} d_G(x_1, gx_1) - C$

This allows us to show:

Lemma. *Let* u *be a nontorsion parabolic element of* G*, lying in* $\gamma H\gamma^{-1}$*. There is a linear polynomial* K_p*, independent of* u*, such that* $\ell_{\hat{\Gamma}}(\hat{\gamma}) \leq K_p(\ell(u))$*.*

We'll use these two results, as well as some elementary estimates of Bumagin, to show

Theorem. *If each H_i has polynomially bounded conjugacy problem for each non-elliptic class, then so does G.*

Suppose that u and v are nontorsion conjugate elements in G, and denote $L = \ell(u) + \ell(v)$.

- u and v hyperbolic elements.
- g penetrates no more than $2L + 10c(8L)$
- Has relative length no more than $K_h L$.

$$K_h L \left(2L + 10c(8L)\right)$$

Otherwise u and v parabolic.

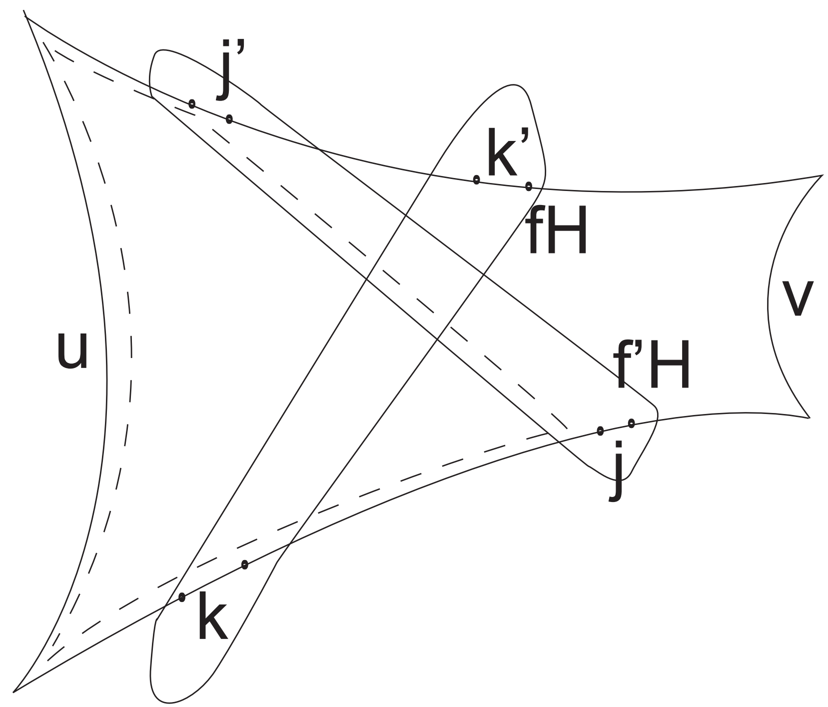
- $u, v \in H_i$
- Conjugate in H_i or not.

$g \in G \setminus H_i$ with $g^{-1}ug = v \in g^{-1}H_ig \cap H_i$

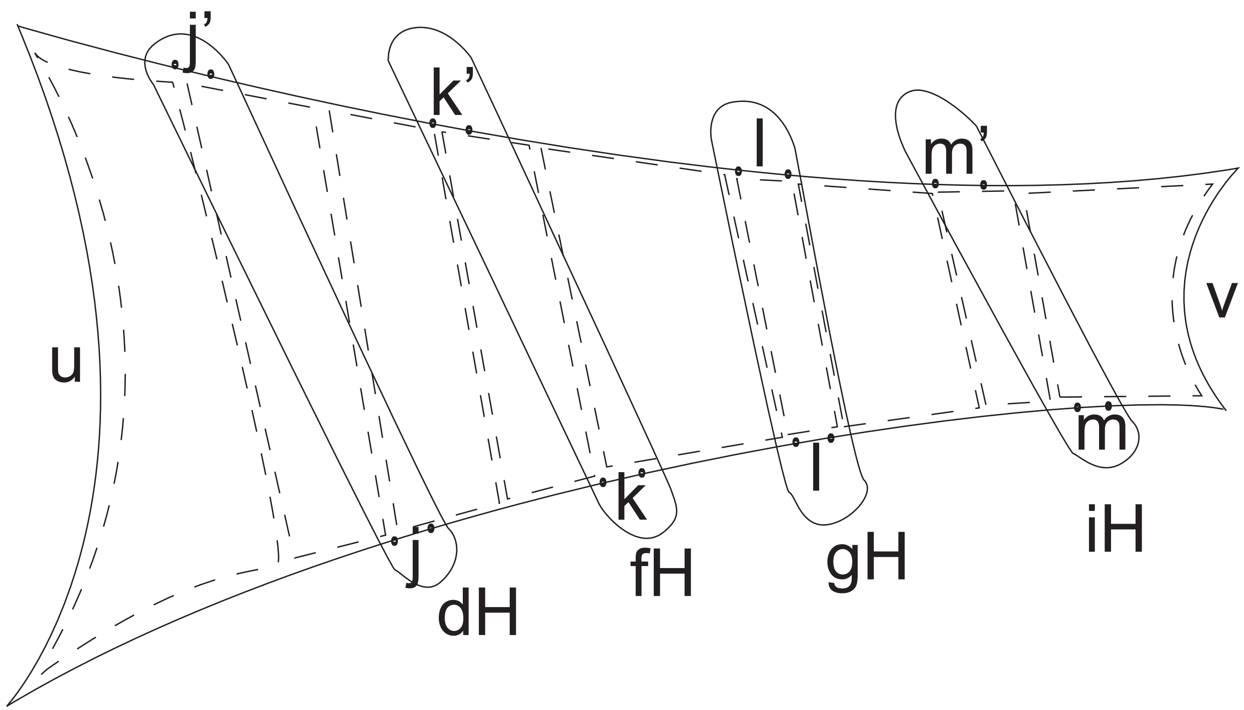
- $v \in H_i$, $u \in G \setminus H_i$.
- If g' an element of minimal relative length such that $g'^{-1}ug' = h \in H_i$ then $\ell(h) \leq c(7L)$.
- v and h in H_i
- Can assume g has minimal length among all elements conjugating u into H_i
- $\ell_{\hat{\Gamma}}(\hat{g}) \leq K_p(\ell(u))$

Geodesic quadrilateral in $\hat{\Gamma}$ with sides $[e, u]$, $\hat{p} = [u, ug]$, $[ug, g]$, and $\hat{q} = [e, g]$.
Suppose $\hat{p}$ penetrates fH_j along k. If $\hat{q}$ doesn't penetrate fH_j, $\ell(k) \leq 2L + c(2L + 1) + 2c(2)$.
Otherwise $\hat{q}$ penetrates fH_j along k'. If $k = k'$, then k conjugates γ_1 and γ_2 in H_j with $\ell(\gamma_i) \leq Q(L)$.

$k \neq k'$.



$$\ell(k) \leq c(L + \ell_{\hat{\Gamma}}(\hat{g}))$$


j'
k'
l
m'
v
u
m
l'
k
j
iH
gH
fH
dH

- u and v both conjugate into H_i, lie in $G \setminus H_i$.
- There is $h \in H_i$, $\ell(h) \leq c(7L)$
- Product of the two conjugators

Need to show c is polynomially bounded!

Lemma. *The BCP function* $c(\lambda)$ *can be chosen to be polynomially bounded.*

Lemma. *Let* (H, d) *be a* δ*-hyperbolic geodesic metric space. For every* k *and* R *there is a constant* $N = N(k, R)$ *such that if* p *and* q *are two* k*-quasigeodesics whose starting points are within* R *from one-another, and whose ending points are within* R *from one-another, then* p *and* q *lie within the* N*-neighborhood of one-another. Moreover,* $N(k, R)$ *can be chosen to be bounded by a polynomial in* k *and* R*.*