THE HOPF ALGEBRA STRUCTURE OF THE COMPLEX BORDISM OF THE LOOP SPACES OF THE SPECIAL ORTHOGONAL GROUPS

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ABSTRACT. The Hopf algebra structure of $MU_*(\Omega SO(n))$ is computed, generalizing the results of Bott for ordinary homology and of Clarke for K-theory. In fact, rephrased in terms of formal group structure, their results hold for any MU-algebra theory.

1. Introduction.

There has been a steady interest in calculating the BP-homology of well known spaces. Most of the effort has been directed towards spaces with torsion. However, determining *all* the relevant structure in 'easily' expressible terms is not straight forward even for torsion-free spaces, as the cases of loop spaces of orthogonal and symplectic groups shows. The latter has been studied in [Cl2]. Here we will study the former. Actually, I have not been able to determine the module of coalgebra primitives, but the rest of the structure is determined in terms of the formal group structure associated to complex cobordism. These results then transfer to any MU-algebra theory, including ordinary homology and K-theory. These two cases have been settled by [Bt] and [Cl1] respectively, and their results are special cases of ours. Actually, we do use Bott's calculations; but with a little more effort we can do with only the general results of Bott applicable to any Lie group and the rational calculations of Leray, Borel *et. al.*

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The plan of this paper is as follows: We state the major results in the next section. Section 3 contains a summary of facts about forward homomorphisms and Thom classes that we need. These results are surely well known; but I have not been able to find a suitable reference. I have stated the results about forward homomorphisms in slightly greater generality than we need. The fourth section contains some calculations of intersection numbers. The results concerning the generating varieties are proved in Section 5. The last section contains the proof of the main theorem and the obligatory list of the first few explicit formulas for BP-theory. The main feature of our result is that the generators are explicitly defined. This is crucial for the study of the bar spectral sequence converging to the Morava K-theories of $SO(2n+1)$ in [Ra].

We refer the reader to [Ad] and [Wi] for background material on, and unexplained notation involving complex bordism, BP-theory and their formal groups.

I would like to thank Steve Wilson for his comments improving the exposition of this paper.

After completing this paper, I learnt of [Bk], where the structure of $MU_*(\Omega Spin)$, including the module of primitives, is computed. However, it is not clear if the methods used there can be adapted to deal with the case of $\Omega SO(n)$ for finite n. Note further that, in our approach, the case $n = \infty$ is an easy consequence of the well-known structure of $MU_*(\mathbb{C}P^\infty)$ and Bott's results.

2. Statement of results.

Let $Q_n = SO(2+n)/(SO(2) \times SO(n))$ be the Grassmanian of oriented 2-planes in $\mathbb{R}^{n+2}$. This is a generating variety for the homology of $\Omega SO(n+2)$: There is a map from Q_n to $\Omega SO(n+2)$ that takes an oriented 2-plane P to the loop formed by rotations in P, followed by rotations in a fixed oriented 2-plane. Note that this maps into the component of the trivial loop. The induced map on homology is monomorphic and the image of $H_*(Q_n)$ generates $H_*(\Omega_0 SO(n+2))$ (see [Bt]). As we will see, this is true for any MU-algebra theory. We give Q_n an almost complex structure which is the conjugate of the usual complex structure [see Section 4 for details]. Note that we have an embedding $\overline{\mathbb{C}P}^n \hookrightarrow Q_{2n}$ of almost complex manifolds. We also have the obvious embeddings $f_n\colon Q_n \hookrightarrow Q_{n+1}$ with direct limit $Q_\infty = BSO(2) = \mathbb{C}P^\infty$; this gives a complex line bundle ξ over Q_n. As a real bundle, this is the tautological oriented 2-plane bundle over Q_n. Let $x \in MU^2(Q_n)$ be the Conner-Floyd Chern class of ξ. We denote by z_n and y_{n+1} the 'Atiyah-Poincaré' duals of $\overline{\mathbb{C}P}^n$ in Q_{2n} and Q_{2n+1}. [See Section 4 for an alternate definition]. Let $\theta(t) = ([2]t)/t$ in $MU^*[[t]]$; here $[n]t$ is the n-series ([Wi, *p.* 14]).

Remarks: Q_n is the quadratic hypersurface in $\mathbb{C}P^{n+1}$ given by $\sum_{j=0}^{n+1} z_j^2 = 0$ (with the conjugate complex structure). As the referee pointed out, the embeddings $\mathbb{S}^n \hookrightarrow Q_n$ and $\overline{\mathbb{C}P}^n \hookrightarrow Q_{2n}$ used in Section 4, are given by

$$(x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n, \imath] \text{ and } [z_0, \ldots, z_n] \mapsto [z_0, \ldots, z_n, \imath z_0, \ldots, \imath z_n]$$

respectively. This can be used to give alternate proofs of the statements in Section 4.

The first part of the following proposition, as well as the first part of its corollary are of course classical. They are included for the sake of completeness.

Proposition 1. (1) *$MU^*(Q_\infty) = MU^*[[x]]$.*
(2) *$MU^*(Q_{2n-1})$ is the free MU^*-module on $\{x^i, x^i y_n \mid 0 \le i < n\}$. The product is given by $x^n = \theta(x) y_n$; $x^i y_n = 0$ if $i \ge n$.*
(3) *$MU^*(Q_{2n})$ is the free MU^*-module on $\{x^i, x^i z_n \mid 0 \le i \le n\}$. The product is given by $x^{n+1} = ([2]x) z_n$; $x^i z_n = 0$ if $i > n$; $z_n^2 = 0$ if n is odd and $x^n z_n$ if n is even.*

It follows that $\{x^i \mid 0 \le i < 2n\}$ and $\{x^i \mid 0 \le i < 2n\} \cup \{\theta(x) z_n - x^n\}$ are bases for $MU\mathbb{Q}^*(Q_{2n-1})$ and $MU\mathbb{Q}^*(Q_{2n})$ respectively. Let the dual bases of $MU\mathbb{Q}_*(Q_{2n-1})$ and $MU\mathbb{Q}_*(Q_{2n})$ be $\{\beta_i \mid 0 \le i < 2n\}$ and $\{\beta_i \mid 0 \le i \le 2n\} \cup \{\epsilon\}$ respectively. Define α_i to be $\theta(x) \frown \beta_i$.

Corollary 2. (1) *$MU_*(Q_\infty)$ is free on $\{\beta_i \mid 0 \le i < \infty\}$.*
(2) *$MU_*(Q_{2n-1})$ is free on $\{\beta_i, \alpha_{n+i} \mid 0 \le i < n\}$.*
(3) *$MU_*(Q_{2n})$ is free on $\{\beta_i, \alpha_{n+i} \mid 0 \le i < n\} \cup \{\alpha_{2n}, \beta_n - \epsilon\}$; 2ϵ is spherical.*

The generating map $Q_n \hookrightarrow \Omega_0 SO(2+n)$ induces a monomorphism in complex bordism since it does so in ordinary homology and the spaces are torsion free. So we can identify the elements of $MU_*(Q_n)$ with their images in $MU_*(\Omega_0 SO(2+n))$ under the generating map.

If M is a graded MU_*-module M, we denote $MU_*[[t]] \otimes_{MU_*} M$ by $M[[t]]$ and grade it by giving t the grading -2. The MU-exp series will be denoted by $b(t)$: $b(t) = \sum_0^\infty t^i \otimes b_i \in MU_*MU[[t]]$. Define $B_n(t) \in MU_*(\Omega_0 SO(n+2))[[t]]$ to be $\sum_{i=0}^n t^i \otimes \beta_i$. Note that $\beta_0 = 1$.

Recall that $\pi_0(\Omega SO(n)) = \mathbb{Z}/2$ if $n \ge 3$. So, as Hopf algebras,

$$MU_*(\Omega SO(n)) = MU_*(\Omega_0 SO(n)) \otimes_{\mathbb{Z}} (\mathbb{Z}[\gamma]/(\gamma^2 = 1)).$$

Henceforth, we will work with $MU_*(\Omega_0 SO(n))$.

Let $B \subset A$ be two commutative algebras with unity and let S be a subset of A. We say that A is simply generated over B by S if monomials of the form $s_1 s_2 \cdots s_m$, where $s_i \in S$ are all distinct, form a basis for A over B. Note that A is free over B as a module.

Theorem 3. (1) *A simple system of generators for $MU_*(\Omega_0 SO)$ over MU_* is given by $\{\beta_i \mid 0 < i < \infty\}$.*

(2) *The MU_*-algebra $MU_*(\Omega_0 SO(2n+1))$ is simply generated over $MU_*[\alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n-1}]$ by $\beta_1, \ldots, \beta_{n-1}$.*

(3) *As an MU_*-algebra, $MU_*(\Omega_0 SO(2n+2))$ is simply generated by $\beta_1, \ldots, \beta_{n-1}, \beta_n - \epsilon$ over $MU_*[\alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}]$.*

(4) *The comodule structure of $MU_*(\Omega_0 SO(n+2))$ is given implicitly by the rational relation $\psi(B_n(t)) = B_n(b(t)) \pmod{t^{n+1}}$.*

(5) *The coalgebra structure of $MU_*(\Omega_0 SO(2n+r))$, $r = 1$ or 2, is given by*

$$\Delta(\beta_k) = \sum_{i=0}^{k} \beta_i \otimes \beta_{k-i} \qquad \text{if } k < n;$$

$$\Delta(\epsilon) = 1 \otimes \epsilon + \epsilon \otimes 1 \qquad \text{if } r = 2;$$

$$\Delta(\alpha_{n+k}) = \sum_{i=0}^{k} (\alpha_{n+i} \otimes \beta_{k-i} + \beta_{k-i} \otimes \alpha_{n+i}) \qquad \text{if } k < n$$

$$\Delta(\alpha_{2n}) = (-1)^n 2\epsilon \otimes 2\epsilon + \sum_{i=0}^{n} (\alpha_{n+i} \otimes \beta_{n-i} + \beta_{n-i} \otimes \alpha_{n+i}) \qquad \text{if } r = 2.$$

(6) *The algebra structure of $MU_*(\Omega_0 SO(n+2))$ is given implicitly by the rational relation*

$$B_n(t) B_n([-1]t) = \begin{cases} 1 & \text{if } n \text{ is odd or } \infty \\ 1 + (-1)^{n/2} t^n \epsilon^2 & \text{if } n \text{ is even} \end{cases} \pmod{t^{n+1}}.$$

Remarks: The last statement of the theorem is to be interpreted as follows: Expand the left-hand side in powers of t. In the result the coefficient of t^{2i} for $i \le n/2$ will be of the form $(-1)^i \beta_i^2 +$ other terms. One can then equate coefficients and solve for β_i^2 in terms of β_j^2, $j < i$ and simple monomials in β's and α's. Now we can inductively calculate β_i^2 in terms of the basis given by the first three statements of Theorem 3.

This gives the multiplication table for that basis. Part 4 of the theorem is interpreted in a similar but simpler fashion; there ψ refers to

$$MU\mathbb{Q}_*(\Omega_0 SO(n+2)) \to MU_*MU \otimes_{MU_*} MU\mathbb{Q}_*(\Omega_0 SO(n+2))\,.$$

Parts 4 and 6 of the theorem are not as explicit as one would like. The fact that we use both the [2]-series and the [−1]-series complicates matters, even if we specialize to Morava K-theories. I do not know if there are other sets of generators that do not suffer from this defect.

The tangent bundle of a manifold X will be denoted by τX. The total space and the Thom space of a vector bundle ξ are denoted by $E\xi$ and $M\xi$ respectively. The image, under $MU \to H$ and $MU \to H \to MU \wedge H\mathbb{Q} = MU\mathbb{Q}$, of an element in the (co) bordism of a space is indicated by attaching a (sub) superscript H. The normal bundle of an embedding f is denoted by νf. We also repeatedly use the following well known corollary of the Atiyah-Hizebruch spectral sequence (see, for example [CS]):

4. *If the reduction to ordinary homology of the elements of a subset B of $MU_*(X)$ is a basis for $H_*(X)$, then $MU_*(X)$ is free on B.*

3. Preliminaries I.

Let $f\colon X \to Y$ be a C^r-map of C^r-manifolds, $r \geq 1$. A complex orientation for f is a factorization of f as $X \xrightarrow{i} E \xrightarrow{p} Y$ where p is a complex bundle and i is an embedding with a stably complex normal bundle. Let f be proper and complex oriented. Then we can find a tubular neighborhood U of $i(X)$ in E and a neighborhood V of U such that $V \cap p^{-1}(y)$ is a closed disk for all $y \in Y$. If h is a complex oriented cohomology theory (this phrase has a different meaning than above when applied to spectra!), we define $f_!\colon h^i(X) \to h^{i+s}(Y)$, the forward homomorphism induced by f, to be

$$\begin{aligned} h^i(X) \xrightarrow{Th} h^{i+s+t}(U, U\setminus X) \xleftarrow{\cong} h^{i+s+t}&(E, E\setminus X) \\ &\to h^{i+s+t}(E, E\setminus V) \xrightarrow{Th} h^{i+s}(Y). \end{aligned}$$

Here Th stands for Thom isomorphisms, $s = dim\,Y - dim\,X$ and $t = dim\,E - dim\,Y$. Note that complex oriented theories have canonical Thom classes for stably complex bundles. If $f\colon X \to Y$ is an embedding with a stably complex normal bundle νf we can dispense with E:

$$f_! \quad = \quad \Big(h^i(X) \xrightarrow{Th} h^{i+s}(M(\nu f)) \xrightarrow[\text{construction}]{\text{Pontryagin-Thom}} h^{i+s}(Y) \Big)\,.$$

Note that $f_!(1)$ is just the Poincaré dual of X in Y.

This definition of the forward homomorphism is (more or less) due to [AH]. This is closely related to the transfer as defined in [MMM]. The following properties are easily proved and essentially restate the fact that the definition of complex cobordism as in [Q] agrees with the usual definition in terms of spectra. Below, $f: X \to Y$ is proper and complex-oriented.

1. *For $y \in h^*(Y)$ and $x \in h^*(X)$, $f_!(f^*(y)\, x) = y\, f_!(x)$.*

2. *If ζ is a complex bundle over Y and id_Y is oriented as $Y \xrightarrow{0} E(\zeta) \longrightarrow Y$, then $(id_Y)_! = (id_Y)_*$.*

3. *Let $f' : X' \to Y'$ be another proper and complex oriented map. Give $f \times f'$ the obvious complex orientation. Then $(f \times f')_!(x \times x') = (f_! x) \times (f'_! x')$.*

4. *Let $g : Z \hookrightarrow Y$ be an embedding transversal to f and $\tilde{g}: f^{-1}(Z) \hookrightarrow Z$ the pull-back of g via f. Then $(f|_Z)_! \tilde{g}^* = g^* f_!$.*

Combining 1–4, we see that we can replace p by $p \oplus \zeta$ in the definition of $f_!$, where ζ is a complex bundle, without changing $f_!$.

5. *Let f be an embedding. Then $f^* f_!$ is multiplication by the Euler class of νf.*

6. *Let $f: X \to Y$ and $g: Y \to Z$ be embeddings with stably complex normal bundles. Then $(gf)_! = \pm g_! f_!$. If f and g are embeddings of almost complex manifolds, then $(gf)_! = g_! f_!$.*

7. *Let f, g be as in (6) and be transversal to each other and let h be $X \cap Z = W \hookrightarrow Y$. Complex orient h by identifying νh with $\nu f|_W \oplus \nu g|_W$. Then $f_!(x) g_!(z) = h_!((x|_W)(z|_W))$.*

Next we turn our attention to $MU\mathbb{Q}$, the rational bordism theory. This has two complex orientations, x_{MU} and x_H, coming from MU and H respectively. Let $\lambda(t) = \log_{MU}(t)/t$; thus $x_H = x_{MU} \lambda(x_{MU})$. A stably complex bundle ξ over X has two Thom classes : $t_H(\xi)$ and $t_{MU}(\xi)$. There is a unique unit $u(\xi) \in MU\mathbb{Q}^0(X)$ such that $t_H(\xi) = u(\xi) t_{MU}(\xi)$. The following properties of u are easily verified:

8. *$u(\xi \times \eta) = u(\xi) \times u(\eta)$; $u(f^* \xi) = f^* u(\xi)$; $u(\xi \oplus \eta) = u(\xi) u(\eta)$.*

9. *$u(\xi)$ can be expressed as a power-series in the Chern classes of ξ, or in the Conner-Floyd Chern classes of ξ.*

10. *If ξ is a complex line bundle, then $u(\xi) = \lambda(cf_1(\xi))$, where $cf_1(\xi)$ is the first Conner-Floyd Chern class of ξ.*

11. *Let $f\colon X \to Y$ be an embedding with a stably complex normal bundle ν. Then $(f_H)_!(x) = u(\nu)(f_{MU})_!(x)$ for all $x \in MU\mathbb{Q}^* X$.*

Finally we need u of a certain bundle over $\mathbb{C}P^{n-1}$:

12. *Let ξ be the canonical complex line bundle over $\mathbb{C}P^{n-1}$ and $\xi_{n-1}^{\perp}$ be the $(n-1)$-plane bundle complementary to ξ. Then $u(\xi \oplus (\xi \otimes \xi_{n-1}^{\perp})) = (\lambda(x_{MU}))^n \theta(x_{MU})/2$.*

Proof. Note that by 10 and formal group calculations,

$$u(\xi \otimes \xi) = \lambda([2]x_{MU}) = 2\lambda(x_{MU})/\theta(x_{MU})$$

in $MU\mathbb{Q}^*(\mathbb{C}P^\infty)$. Also

$$u(\xi \otimes \xi)\, u(\xi \otimes \xi_{n-1}^{\perp}) = u(\xi)^n = (\lambda(x_{MU}))^n \ ,$$

because of 8, 10 and the fact that $(\xi \otimes \xi) \oplus (\xi \otimes \xi_{n-1}^{\perp}) = \xi^{\oplus n}$. Another appeal to 8 and 10 completes the proof.

4. Preliminaries II.

In this section we find various intersection numbers we will need later. These are straightforward calculations; we will give only enough detail to reproduce the full calculations. In particular, we will not prove transversality or check local intersection numbers.

Let G be a Lie group and H a closed subgroup of G. Then there is a well known *exact* functor from finite dimensional linear representations of H to vector bundles over G/H: To a representation V, associate the bundle $G \times_H V \to G/H$. For example, $\tau G/H$ is associated to the adjoint representation of H on g/h where g and h are the Lie algebras of G and H respectively. We use this to identify tangent and normal bundles.

Recall that $Q_n = SO(2+n)/(SO(2) \times SO(n))$ is the Grassmanian of oriented 2-planes in $\mathbb{R}^{2+n}$ and that $Q_\infty = BSO(2) \cong \mathbb{C}P^\infty$. We identify $so(2{+}n)/(so(2){+}so(n))$ with $Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{C})$ by sending an $(n{+}2)\times(n{+}2)$ matrix $[a_{jk}]$ to the row matrix $[a_{1j}{+}\imath a_{2j}]_{j=3}^{n+2}$. Thus the tangent bundle of Q_n is associated to the representation of $SO(2){\times}SO(n)$ on $Hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{C})$ where $SO(n)$ acts on $\mathbb{R}^n$ and $SO(2) = U(1)$ on $\mathbb{C}$. We use this to give Q_n an almost complex structure. In the same way the tangent bundle of $\overline{\mathbb{C}P}^n = U(1+n)/(U(1) \times U(n))$ is associated to the obvious representation of $U(1) \times U(n)$ on $Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$.

We have the obvious embedding $f_n\colon Q_n \hookrightarrow Q_{n+1}$ and the embedding $\overline{\mathbb{C}P}^n \hookrightarrow Q_{2n}$ induced by $U(m) \hookrightarrow SO(2m)$. These are embeddings of almost complex manifolds. The elements $y_n \in MU^*(Q_{2n-1})$ and $z_n \in MU^*(Q_{2n})$ can be defined as the image of 1 under the forward homomorphisms induced by $\mathbb{C}P^{n-1} \hookrightarrow Q_{2n-2} \hookrightarrow Q_{2n-1}$ and $\mathbb{C}P^n \hookrightarrow Q_{2n}$ respectively. Clearly the normal bundle of f_n is associated to the representation $pr\colon SO(2) \times SO(n) \to SO(2) = U(1)$. It is easy to see that the normal bundle of $\overline{\mathbb{C}P}^n$ in Q_{2n} is associated the obvious representation of $U(1) \times U(n)$ on $\mathbb{C} \otimes \mathbb{C}^n$. Thus we have the next statement, where ξ is the canonical complex line bundle over $Q_\infty = \mathbb{C}P^\infty$:

1. *$\nu(\mathbb{C}P^n \hookrightarrow Q_{2n}) = \xi \otimes \xi_n^\perp$ and $\nu(\mathbb{C}P^{n-1} \hookrightarrow Q_{2n-2} \hookrightarrow Q_{2n-1}) = \xi \oplus (\xi \otimes \xi_{n-1}^\perp)$. Here $\xi_m^\perp$ is the m-plane bundle complementary to $\xi|\mathbb{C}P^m$. Also $\nu f_n = \xi|Q_n$.*

Let $D(\tau\mathbb{S}^n) = \{(u,v) \mid u \perp v, \|u\| \leq 1\} \subset \mathbb{S}^n \times \mathbb{R}^{1+n}$ be the tangent disk bundle of $\mathbb{S}^n$. We define a map $D(\tau\mathbb{S}^n) \to Q_n$ by $(u,v) \mapsto \mathbb{R}u + \mathbb{R}(v + \sqrt{1-\|v\|^2} \cdot e_{n+2})$ where $\{e_1, \ldots, e_{n+2}\}$ is the standard basis of $\mathbb{R}^{2+n}$. It is easy to verify that this map carries $S(\tau\mathbb{S}^n)$, the tangent sphere bundle of $\mathbb{S}^n$, into Q_{n-1} and that

$$Q_n = Q_{n-1} \coprod_{S(\tau\mathbb{S}^n)} D(\tau\mathbb{S}^n). \tag{2}$$

We identify Q_n/Q_{n-1} with $M(\tau\mathbb{S}^n)$. Note that $\mathbb{S}^n \hookrightarrow D(\tau\mathbb{S}^n) \hookrightarrow Q_n$ gives an embedding i of $\mathbb{S}^n$ into Q_n. Embedding $\mathbb{R}^{n+1}$ in $\mathbb{R}^{n+2}$ as $\mathbb{R}e_2 + \cdots + \mathbb{R}e_{n+2}$ gives another embedding j of $\mathbb{S}^n = SO(1+n)/SO(n)$ into $SO(2+n)/(SO(2) \times SO(n)) = Q_n$ that is seen to be isotopic to i if n is even.

We will need the following facts later:

3. *In Q_{2n}, Q_{2n-1} and $\overline{\mathbb{C}P}^n$ intersect transversely at $\overline{\mathbb{C}P}^{n-1}$.*

Proof. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we have

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n-1}, \mathbb{C}) \cap Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) = Hom_{\mathbb{C}}(\mathbb{C}^{n-1}, \mathbb{C}).$$

Thus it is enough to prove the second claim. Any $L \in \overline{\mathbb{C}P}^n \cap Q_{2n-1}$ is a complex line of $\mathbb{C}^{1+n}$ contained in $\mathbb{R}^{2n+1}$. Hence the image of L under the projection of $\mathbb{C}^{1+n}$ onto the last complex coordinate is contained in $\mathbb{R}$. So this image is trivial and $L \subset \mathbb{C}^n$.

4. *$j(\mathbb{S}^{2n})$ and $\overline{\mathbb{C}P}^n$ intersect transversely at a single point with intersection number 1*

Proof. Since the tangent bundles of $j(\mathbb{S}^{2n})$ and $\mathbb{C}P^n$ are associated to $Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \imath\mathbb{R})$ and $Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ considered in the obvious manner as subspaces of $Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{C})$, it is enough to show that $j(\mathbb{S}^{2n}) \cap \mathbb{C}P^n$ is a singleton. Note that $j(u_1, \dots u_{2n+1}) = \mathbb{R}e_1 + \mathbb{R}(\sum_{k=1}^{2n+1} u_k e_{k+1})$. So the only complex line contained in $j(\mathbb{S}^{2n})$ is $\mathbb{C}e_1$.

5. *The self-intersection number of $i(\mathbb{S}^{2n})$ in Q_{2n} is $(-1)^n 2$*

Proof. Define yet another embedding j' of $\mathbb{S}^{2n}$ into Q_{2n} by

$$j'(u_1, \dots, u_{2n+1}) = \mathbb{R}(u_1 e_1 + \sum_{k=2}^{2n+1} u_k e_{k+1}) + \mathbb{R}e_2.$$

One easily verifies that the intersection of $j'(\mathbb{S}^{2n})$ and $j(\mathbb{S}^{2n})$ consists of exactly two points, namely $\mathbb{R}e_1 + \mathbb{R}e_2$ and $\mathbb{R}e_2 + \mathbb{R}e_1$. Now the tangent spaces to $j(\mathbb{S}^{2n})$ and $j'(\mathbb{S}^{2n})$ at $\mathbb{R}e_1 + \mathbb{R}e_2$ are identified with $Hom_{\mathbb{R}}(\mathbb{R}^{2n}, i\mathbb{R})$ and $Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R})$ respectively; we consider these two as subspaces of $Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{C})$. Thus the intersection at $\mathbb{R}e_1 + \mathbb{R}e_2$ is transversal with local intersection number $(-1)^n$. Define $\sigma\colon \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ by

$$\sigma(e_i) = \begin{cases} -e_i & \text{if } i = 2 \text{ or } 3; \\ e_i & \text{otherwise.} \end{cases}$$

Then $\sigma j = j'$ and $\sigma(\mathbb{R}e_1 + \mathbb{R}e_2) = \mathbb{R}e_2 + \mathbb{R}e_1$. Hence j' is isotopic to j and so to i, and the local intersection numbers at the two points are the same.

6. *The intersection number of Q_n and $\overline{\mathbb{C}P}^n$ in Q_{2n} is 1*

Proof. Identify $\mathbb{R}^{2+2n}$ with $\mathbb{C}^{1+n}$ with basis $e_0, e_1, \dots, e_n$. Then $Q_n \subset Q_{2n}$ is isotopic to $\widetilde{Q}_n$, the Grassmannian of oriented 2-planes in $\mathbb{C}e_0 + \mathbb{R}e_1 + \dots + \mathbb{R}e_n$. Clearly $\widetilde{Q}_n \cap \overline{\mathbb{C}P}^n = \{\mathbb{C}e_0\}$. The tangent spaces at this point are $\{f \in Hom_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}) \mid f(\imath e_j) = 0 \,\forall\, j > 0\}$ and $Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$. From this it follows that the intersection number is 1.

7. *The two composites $Q_{n+1} \to M(\xi|Q_n) \to MU(1)$ and $Q_{n+1} \to Q_\infty = \mathbb{C}P^\infty \to MU(1)$ are homotopic. Thus $(f_n)_!(1) = x$.*

Proof. Identify $MU(1) \simeq \mathbb{C}P^\infty$ with $K(\mathbb{Z}, 2)$. Then $Q_n \to Q_{n+1}$ composed with either of the two maps above represents the Euler class of ξ. If $n > 1$ then $H^2(Q_{n+1}) \to H^2(Q_n)$ is monomorphic, by 2. This proves the claim in that case. Now $\mathbb{S}^2 \longrightarrow Q_2 \longrightarrow M(\xi|Q_1)$ is null-homotopic for $i(\mathbb{S}^2)$ and Q_1 are disjoint. As $i^*\xi \cong j^*\xi$ is trivial, $\mathbb{S}^2 \longrightarrow Q_2 \longrightarrow Q_\infty$ is null homotopic. By 2, $H^2(Q_2) \to H^2(Q_1) \oplus H^2(\mathbb{S}^2)$ is monomorphic. These three facts complete the proof if $n = 1$.

5. Proof of Proposition 1.

Let M_n be the Thom space of the tangent bundle of $\mathbb{S}^{2n}$. Since the latter is stably trivial, M_n is stably equivalent to $(\mathbb{S}^{2n})^+ \wedge \mathbb{S}^{2n}$. Let $a_n \in \widetilde{MU}^{2n}(M_n)$ be the Thom class and c the canonical generator of $\widetilde{MU}^*(\mathbb{S}^{2n})$.

1. *$\widetilde{MU}^*(M_n)$ is freely generated by a_n and $(a_n \smile a_n)/2$. $\mathbb{S}^{2n} \to M_n$ maps a_n to $2c$. The element $(a_n \smile a_n)/2$ is stably the suspension of a generator of $\widetilde{MU}^*(\mathbb{S}^{2n})$.*

Proof. The second statement is well known. Let D and S be the tangent disk and sphere bundles of $\mathbb{S}^{2n}$. Then $MU^*((D,S) \times (D,S)) \to MU^*(D \times (D,S))$ maps $a_n \smile a_n$ to $2c \smile a_n$. The proof is completed by appealing to the Thom isomorphism theorem and by noting that, in this case, the Thom isomorphism coincides with suspension upto sign.

As we will see below, we have a split short exact sequence

$$0 \longrightarrow \widetilde{MU}^*(M_n) \longrightarrow MU^*(Q_{2n}) \longrightarrow MU^*(Q_{2n-1}) \longrightarrow 0 \tag{2}$$

where we identify M_n with Q_{2n}/Q_{2n-1} as in the last section. Thus we can and will abuse notation by identifying the elements of $\widetilde{MU}^*(M_n)$ with their images in $MU^*(Q_{2n})$.

3. *$(f_{2n})_!(z_n) = y_{n+1}$; $f^*_{2n-1}(z_n) = y_n$; $i_!(1) = a_n$; $a_n x = 0$; $f^*_{2n-1}(a_n) = 0$; $f^*_n(x) = x$.*

Proof. We use the definition of y_n and z_n given in Section 4. The first equality follows from 3.6; the second from 3.4 and 4.3; and the third by definition. The fourth and fifth follow from 4.6 and the fact that Q_{2n-1} and $i(\mathbb{S}^{2n})$ are disjoint. The last statement is obvious.

Next we prove by induction on n that

4. *$MU^*(Q_{2n-1})$ is free over MU^* with basis $\{x^i, x^i y_n \mid 0 \le i < n\}$. Furthermore $2y_{n,H} = x^n_H$.*

5. *$MU^*(Q_{2n})$ is free over MU^* with basis $\{x^i, x^i z_n \mid 0 \le i \le n\}$. In addition $a_{n,H} = 2z_{n,H} - x_H^n$; $a_{n,H}^2 = (-1)^n 2x_H^n z_{n,H}$.*

4 is true for $n = 1$. Now $\xi | Q_1$ is the tangent bundle of $Q_1 = \mathbb{S}^2$ and x_H its Euler class. Since $\overline{\mathbb{C}P}^0$ is a point, $y_{1,H} = x_H/2$. So $H^*(Q_1)$ is free on $\{1, y_{1,H}\}$. This is enough by 2.4.

4 for n implies 5 for n. The long exact sequence of (Q_{2n}, Q_{2n-1}) becomes the split short exact sequence

$$0 \longrightarrow \widetilde{H}^*(M_n) \longrightarrow H^*(Q_{2n}) \longrightarrow H^*(Q_{2n-1}) \longrightarrow 0,$$

for by the induction assumption and 1, $H^*(Q_{2n-1})$ and $H^*(M_n)$ are free and concentrated in even dimensions. [Note that this argument proves 2 too, inductively.] The induction assumption and 3 imply that $f_{2n-1}^*(2z_{n,H} - x_H^n) = 2y_{n,H} - x_H^n = 0$. Hence $2z_{n,H} - x_H^n = k a_{n,H}$ for some $k \in \mathbb{Z}$. But k has to be 1:

$$\begin{aligned} 2kc_H &= i^*(ka_{n,H}) && \text{by 1} \\ &= j^*(ka_{n,H}) && \text{since } i \simeq j \\ &= j^*(2z_{n,H} - x_H^n) && \text{by hypothesis} \\ &= 2c_H && \text{by 4.4 and the fact that } j^*(x) = 0. \end{aligned}$$

The last statement of 5 follows from 4.5 and 4.6. Now 1, 3, the induction hypothesis and the split short exact sequence above imply that $H^*(Q_{2n}) \cong H^*(Q_{2n-1}) \oplus \widetilde{H}^*(M_n)$ is free on

$$\{x_H^i, x_H^i z_{n,H} \mid 0 \le i < n\} \cup \{2z_{n,H} - x_H^n,\ (-1)^n x_H^n z_{n,H}\}.$$

So $\{x^i, x^i z_n \mid 0 \le i \le n\}$ reduces to a basis of $H^*(Q_{2n})$. By 2.4, we are done.

5 for n implies 4 for $n+1$. First, I claim that $(f_{2n})_!(a_n) = 0$: Note that, if the g is the standard embedding of $\mathbb{S}^{2n}$ into $\mathbb{S}^{2n+1}$, then

$$\left(\mathbb{S}^{2n} \xrightarrow{i} Q_{2n} \xrightarrow{f_{2n}} Q_{2n+1}\right) \quad \simeq \quad \left(\mathbb{S}^{2n} \xrightarrow{g} \mathbb{S}^{2n+1} \xrightarrow{i} Q_{2n+1}\right).$$

Together with 3.5, this implies that $(f_{2n})_!(a_n) = (f_{2n})_! i_!(1) = \pm i_! g_!(1)$. But clearly $g_!(1) = 0$. Note that by the same argument, $i^*(x) = 0$. Next we show that $2y_H^{n+1} = x_H^{n+1}$:

$$\begin{aligned} 2y_{n+1,H} - x_H^{n+1} &= 2(f_{2n})_!(z_{n,H}) - x_H^n (f_{2n})_!(1) && \text{by 3 and 3.1} \\ &= (f_{2n})_!(a_{n,H}) \\ &= 0 && \text{as above.} \end{aligned}$$

By [Bt, 9.1] $H^*(Q_{2n+1})$ is freely generated by $\{x_H^k, x_H^{n+k}/2 \mid 0 \le k < n+1\}$. Now appeal to 2.4.

Now we return to the proof of Proposition 2.1. Note that $x^i y_n = 0$ if $i \ge n$ and $x^i z_n = 0$ if $i > n$ because of dimension considerations. Next we show that $x^n = \theta(x) y_n$ in $MU\mathbb{Q}^*(Q_{2n-1}) \supset MU^*(Q_{2n-1})$:

$$\begin{aligned} \lambda(x_{MU})^n x_{MU}^n &= \big(\log_{MU}(x_{MU})\big)^n = x_H^n \\ &= 2y_{n,H} && \text{by 4} \\ &= \big(\lambda(x_{MU})\big)^n \theta(x_{MU}) y_{n,MU} && \text{by 4.1, 3.11 and 3.12.} \end{aligned}$$

We need to make one more digression:

$$a_n = \theta(x) z_n - x^n \quad \text{in} \quad MU^*(Q_{2n}). \tag{6}$$

Proof. Using 2 and the fact that $f_{2n-1}^*(a_n) = 0$, we see that there exist $k \in MU^0$ and $u \in MU\mathbb{Q}^*$ such that $a_n = k(\theta(x) z_n - x^n) + u x^{2n}$. Reduction to ordinary homology and 5 show that $k = 1$. But u must be 0, because in $MU\mathbb{Q}^*(Q_{2n+1})$,

$$\begin{aligned} \theta(x) y_{n+1} - x^{n+1} &= 0 && \text{proved above} \\ &= (f_{2n})_!(a_n) && \text{by 3} \\ &= (f_{2n})_!(\theta(x) z_n - x^n + u x^{2n}) && \text{by assumption} \\ &= \theta(x) y_{n+1} - x^{n+1} + u x^{2n+1} && \text{by 3, 3.1 and 4.7.} \end{aligned}$$

Since $a_n x = 0$, we see that $x^{n+1} = z_n([2]x)$ in $MU^*(Q_{2n})$. By 5, $a_n^2 = k x^n z_n$ for some $k \in MU^0$. Reduction to ordinary homology and 5 show that $k = (-1)^n 2$. An easy calculation using 6 determines z_n^2 and completes the proof of Proposition 2.1.

Proof of Corollary 2.2. : The proofs of (1), (2) and (3) are similar. We will prove (3): We have the rational basis $1, x, \ldots, x^{2n}, \theta(x) z_n - x^n$ and the integral basis $1, x, \ldots, x^n, z_n, x z_n, \ldots, x^n z_n$. Let c_i be the coefficients of the 2-series: $[2]x = \sum_0^\infty c_i x^{i+1}$. As $x^{n+j} = \sum_{i=0}^{n-j} c_i x^{i+1} z_n$, the dual of the latter basis is

$$\{\beta_i \mid 0 \le i < n\} \cup \{\beta_n - \epsilon\} \cup \{c_j \epsilon + \sum_{i=1}^{j} c_{j-i} \beta_{n+i} \mid 0 \le j \le n\}.$$

Clearly we can replace $c_j \epsilon + \sum_{i=1}^{j} c_{j-i} \beta_{n+i}$ by

$$\left(c_j \epsilon + \sum_{i=1}^{j} c_{j-i} \beta_{n+i}\right) + \left(c_j(\beta_n - \epsilon) + \sum_{i=1}^{n} c_{j+i} \beta_{n-i}\right) = \alpha_{n+j}.$$

The last sentence of Corollary 2.2.(3) is proved by using the fact that $i^*(a_n)/2$ is a generator of $\widetilde{MU}^*(\mathbb{S}^{2n})$ and that $i^*(x) = 0$.

6. Proof of Theorem 2.3 and Miscellaneous Remarks.

Proofs of (1), (2) and (3) are similar to each other. We prove (2): Note that the reduction of α_j to ordinary homology is $2\beta_j^H$. By [Bt, Section 9], $H_*(\Omega_0 SO(2n+1))$ is free on

$$\{(\beta_1^H)^{j_1}\cdots(\beta_{n-1}^H)^{j_{n-1}}\cdot(2\beta_n^H)^{k_0}\cdots(2\beta_{2n-1}^H)^{k_n}\mid j_i=0 \text{ or } 1\}.$$

Now use 2.4.

Note that parts 4 and 5 of Theorem 2.3 are really statements about $MU_*(Q_n)$. The latter follows by dualizing the multiplication tables for the bases $\{1,x,\dots,x^{2n},\theta(x)z_n - x^n\}$ and $\{1,x,\dots,x^{2n-1}\}$ of $MU^*(Q_{2n})$ and $MU_*(Q_{2n-1})$ respectively. The only nontrivial facts needed, namely that $(\theta(x)z_n-x^n)x=0$ and that $(\theta(x)z_n-x^n)^2=(-1)^n x^{2n}$, follow from the last part of the proof of Proposition 2.1.(3).

Proof of Theorem 2.3.(4). This is standard for $n=\infty$. As the homomorphism $MU\mathbb{Q}_*(Q_{2m-1})\to MU\mathbb{Q}_*(Q_\infty)$ is a monomorphism, the case of odd n follows. Now we turn to the case of even n: By Corollary 2.2.(3), $\psi(\epsilon)=1\otimes\epsilon$. The kernel of $MU\mathbb{Q}_*(Q_n)\to MU\mathbb{Q}_*(Q_\infty)$ is generated by ϵ, and β_i is in the image of $MU\mathbb{Q}_*(Q_{n-1})\to MU\mathbb{Q}_*(Q_n)$ for $i<n$. Thus the only ambiguity in ψ is in $\psi(\beta_n)$ and is of the form $\alpha\otimes\epsilon$: $\psi(B_n(t))=B_n(b(t))+t^n\alpha\otimes\epsilon \pmod{t^{n+1}}$. Consider the map $p\colon Q_n\to Q_n/Q_{n-1}\cong M(\tau\mathbb{S}^n)$. Recall that $M(\tau\mathbb{S}^n)$ is stably equivalent to $(\mathbb{S}^n)^+\wedge\mathbb{S}^n$. The results of the previous section imply that p^* maps the suspensions of the generators of $MU\mathbb{Q}^*(\mathbb{S}^n)$ to $a_{n/2}$ and $(-1)^{n/2}x^n/2$. Thus $p_*(\beta_i)=0$ if $i<n$, and $p_*(2\epsilon)$ and $(-1)^{n/2}p_*(\beta_n)$ are the suspensions of the canonical generators of $MU_*(\mathbb{S}^n)$. The following calculation then shows that $\alpha=0$:

$$\begin{aligned}1+t^n\otimes p_*(\beta_n)&=\psi(p_*(B_n(t))=p_*(B_n(b(t))+t^n\alpha\otimes\epsilon)\\&=1+t^n\otimes p_*(\beta_n)+t^n\alpha\otimes p_*(\epsilon).\end{aligned}$$

Proof of Theorem 2.3.(6). Define a self-homeomorphism σ of $Q_n=\widetilde{G}_{2+n,2}$ that reverses the orientation of oriented 2-planes. It is easy to see that $\sigma^*\xi=\overline{\xi}$. So $\sigma^*(x)=[-1]x$, $\sigma^*(x_H)=-x_H$ and $\sigma_*(B_n(t))=B_n([-1]t)$. Let n be even. Let D and S be the tangent disk and sphere bundles of $\mathbb{S}^n$, and $g\colon(D,S)\to(Q_n,Q_{n-1})$ the relative homeomorphism of Section 4. Then $\sigma g=g\sigma'$ where $\sigma':D\to D$ is induced by the antipodal map of $\mathbb{S}^n$. Hence $\sigma p=p\sigma'$ and so $\sigma^*(a_{n/2})=-a_{n/2}$. It follows that $\sigma_*(\epsilon)=-\epsilon$.

Let h be the composition

$$Q_n \xrightarrow{\text{diag}} Q_n \times Q_n \xrightarrow{1\times\sigma} Q_n \times Q_n$$
$$\to \Omega SO(n+2) \times \Omega SO(n+2) \xrightarrow{\text{mult}} \Omega SO(n+2).$$

Then h induces the trivial homomorphism in reduced rational homology: By Theorem 2.3.(4) and the last paragraph, $h_*(\epsilon) = \epsilon - \epsilon$. Also

$$h_*(B_n^H(t)) = \begin{cases} B_n^H(t)B_n^H(-t) & \text{if } n \text{ is odd,} \\ B_n^H(t)B_n^H(-t) - (-1)^{n/2}t^n\epsilon & \text{if } n \text{ is even.} \end{cases}$$

Both are 1 by [Bt, 9.6, 10.6 and 10.10].

Since Q_n and $\Omega SO(n+2)$ are torsion-free, h induces the trivial homomorphism in reduced bordism. Now a calculation similar to the above completes the proof.

I have not been able to determine the module of primitives in general. In the case of $MU_*(\Omega_0 SO)$, the following is true [Bk]: Define $P(t) \in MU_*(\Omega SO)[[t]]$ by $P(t)B_\infty(t) = B'_\infty(t)$. Let p_i be the coefficient of t^{i-1} in $P(t)$; (p_i is the i-th Newton polynomial in the β_j's). Then p_i's are primitive. Furthermore $P(t) + [-1]'(t)P([-1]t) = 0$; so p_{even}'s are linear combinations over MU_* of p_{odd}'s. The module of primitives of $MU_*(\Omega SO)$ is freely generated by $\{p_{2i+1} \mid 0 \le i < \infty\}$.

It follows from Theorem 2.3 that β_n^2 is a linear combination of simple monomials in $\beta_1, \ldots, \beta_n, \alpha_{n+1}, \ldots, \alpha_{2n}$.

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