

# Unofficial Math 501 Problems

The following document consists of problems collected from previous classes and comprehensive exams given at the University of Illinois at Urbana-Champaign. In the exercises below, all rings contain identity and all modules are unitary unless otherwise stated. Please report typos, suggestions, gripes, complaints, and criticisms to: [snapp@math.uiuc.edu](mailto:snapp@math.uiuc.edu)

1) Let  $R$  be a commutative ring with identity and let  $I$  and  $J$  be ideals of  $R$ . If the  $R$ -modules  $R/I$  and  $R/J$  are isomorphic, prove  $I = J$ . Note, we really do mean *equals*!

2) Let  $R$  and  $S$  be rings where we assume that  $R$  has an identity element. If  $M$  is any  $(R, S)$ -bimodule, prove that

$$\mathrm{Hom}_R({}_R R, M) \stackrel{S}{\simeq} M.$$

3) Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence of  $R$ -modules where  $R$  is a ring with identity. If  $A$  and  $C$  are projective, show that  $B$  is projective.

4) Let  $R$  be a ring with identity.

(a) Prove that every finitely generated right  $R$ -module is noetherian if and only if  $R$  is a right noetherian ring.

(b) Give an example of a finitely generated module which is not noetherian.

5) Let  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  be a short exact sequence of  $R$ -modules where  $R$  is any ring. Assume there exists an  $R$ -module homomorphism  $\delta : B \rightarrow A$  such that  $\delta\alpha = \mathrm{id}_A$ . Prove that there exists an  $R$ -module homomorphism  $\gamma : C \rightarrow B$  such that  $\beta\gamma = \mathrm{id}_C$ .

6) If  $R$  is any ring and  ${}_R M$  is an  $R$ -module, prove that the functor  $- \otimes_R M$  is right exact.

7) Let  $S$  and  $R$  be rings, with  $S$  a subring of  $R$  containing  $1_R$ .

(a) Use tensor products to construct a covariant functor

$$\mathcal{F} : \mathbf{Mod}_S \rightarrow \mathbf{Mod}_R.$$

- (b) If  $M_S$  is a flat module, prove that  $M_S$  is  $S$ -isomorphic with a submodule of  $\mathcal{F}(M_S)$ .
- 8) Let  $R$  be any ring and let  $M_R, N_R$  be flat  $R$ -modules. Prove that  $M \oplus N$  is also flat.
- 9) Let  $R$  be a domain and let  $M_R$  be a flat  $R$ -module. Prove that  $M$  is  $R$ -torsion-free.
- 10) Give an example of a commutative ring  $R$  with identity and an  $R$ -module  $M$  such that the  $R$ -torsion elements of  $M$  do not form a submodule of  $M$ .
- 11) Let  $(C, \mu_\lambda)_{\lambda \in \Lambda}$  and  $(C', \mu'_\lambda)_{\lambda \in \Lambda}$  be coproducts of set of objects  $\{A_\lambda : \lambda \in \Lambda\}$  in some category  $\mathbf{C}$ . Prove that there is an equivalence  $\beta \in \mathbf{C}(C, C')$  such that  $\mu'_\lambda = \beta \mu_\lambda$  for all  $\lambda$  in  $\Lambda$ .
- 12) Prove that the direct sum  $(\bigoplus_{\lambda \in \Lambda} M_\lambda, \mu_\lambda)$  is the coproduct in the category  ${}_R \mathbf{Mod}$  where  $\mu_\lambda : M_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} M_\lambda$  via

$$(\mu_\lambda(a))_\nu = \begin{cases} a & \text{if } \nu = \lambda \\ 0_{M_\nu} & \text{if } \nu \neq \lambda. \end{cases}$$

- 13) Prove that  $\text{Hom}_R(-, M)$  is a contravariant functor from  ${}_R \mathbf{Mod}$  to  $\mathbf{Ab}$ .
- 14) Let  $R$  and  $S$  be rings and  $\alpha : R \rightarrow S$  a ring homomorphism. Show that there is a corresponding covariant functor  $\mathcal{F}_\alpha : {}_S \mathbf{Mod} \rightarrow {}_R \mathbf{Mod}$ .
- 15) Let  $L, M, N$  be submodules of an  $R$ -module and assume that  $N \subseteq M$ . Prove that  $L + M/L + N$  is  $R$ -isomorphic with a quotient of  $M/N$  and  $L \cap M/L \cap N$  is  $R$ -isomorphic with a submodule of  $M/N$ .
- 16) Let  $(F, \mu)$  and  $(F', \mu')$  be free on a set  $X$  in  ${}_R \mathbf{Mod}$ . Prove that  $F \stackrel{R}{\cong} F'$ .

17) In the following diagram the squares are commutative, the rows are exact, and  $\lambda$  and  $\nu$  are isomorphisms of modules. All maps are module homomorphisms.

$$\begin{array}{ccccccc} & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & & \end{array}$$

Prove:

- (a)  $\alpha$  is injective.
- (b)  $\beta'$  is surjective.

(c)  $\mu$  is an isomorphism.

**18)** Let  $R$  be a ring with identity. Prove that every  $R$ -module is projective if and only if  $R$  is a semisimple ring.

**19)** Let  $R$  be an arbitrary ring and let  $M$  be an  $R$ -module with submodules  $M_1, M_2, \dots, M_k$  such that each  $M/M_i$  is a simple module and

$$M_1 \cap M_2 \cap \dots \cap M_k = \{0\}.$$

Prove that  $M$  is a semisimple module.

**20)** Let  $G$  be a group and  $F$  a field. Define a map  $\theta : FG \rightarrow F$  by

$$\theta \left( \sum_{g \in G} f_g g \right) = \sum_{g \in G} f_g.$$

(a) Prove that  $\theta$  is a ring homomorphism.

(b) Prove that  $\text{Ker}(\theta) = I_G$  where  $I_G$  is generated as an  $F$ -vector space by the set  $\{g - 1 : g \in G\}$ .

(c) Assume that  $G$  has finite order not divisible by  $\text{char}(F)$ . Prove

$$FG = J_0 \oplus J_1$$

where  $J_0, J_1$  are ideals of  $FG$  and  $I_G J_0 = 0, I_G J_1 = J_1$ .

(d) Show that  $\dim_F(J_0) = 1$  and find a generator for  $J_0$ .

**21)** Let  $G$  be a nontrivial finite  $p$ -group and let  $F$  be a field of characteristic  $p$ , a prime. Let  $I_G$  be as in the above problem. Prove  $I_G$  is not a direct summand of the module  ${}_{FG}(FG)$ . What does this tell you about the ring  $FG$ ?

**22)** Let  $R$  be a PID and denote by  $F$  the field of fractions of  $R$ . Regard  $F$  as an  $R$ -module in the obvious way.

(a) If  $X$  is a finitely generated  $R$ -submodule of  $F$ , show that  $X = Rf$  for some  $f \in F$ .

(b) If  $F$  is a finitely generated  $R$ -module, prove  $R = F$ .

**23)** Let  $M$  be a finitely generated module over a PID  $R$ .

(a) If  $M$  is a torsion module prove that  $M$  is artinian.

(b) Assuming that  $R$  is not a field, show that  $M$  has a composition series if and only if it is a torsion module.

24) Describe all isomorphism types of abelian groups of order 2250, and give the invariant factors for each type.

25) Let  $M$  and  $N$  be finite abelian  $p$ -groups and let their invariant factors be

$$\underbrace{p, \dots, p}_{m_1}, \underbrace{p^2, \dots, p^2}_{m_2}, \dots, \underbrace{p^r, \dots, p^r}_{m_r},$$

and

$$\underbrace{p, \dots, p}_{n_1}, \underbrace{p^2, \dots, p^2}_{n_2}, \dots, \underbrace{p^r, \dots, p^r}_{n_r},$$

respectively, where  $m_i, n_i \geq 0$ . What are the invariant factors of  $M \otimes_{\mathbb{Z}} N$ ? Justify your answer.

26) Prove that an artinian integral domain is field. [Use the DCC directly, no major theorems necessary.]

27) Let  $R$  be a ring and  $M$  a left  $R$ -module. If  $r \in R$  and  $a \in M$ , define  $a \cdot r$  to be  $r \cdot a$ .

(a) Prove that the assignment  $(a, r) \mapsto a \cdot r$  makes  $M$  into a right module over  $R^{\text{Opp}}$ .

(b) Define a covariant functor  $\text{Opp} : {}_R \mathbf{Mod} \rightarrow \mathbf{Mod}_{R^{\text{Opp}}}$ .

28) Give an example of a nonzero abelian group  $A$  such that  $A \otimes_{\mathbb{Z}} A = 0$ .

29) Let  $M_\lambda$  be a right  $R$ -module for  $\lambda \in \Lambda$  and let  $N$  be a left  $R$ -module, where  $R$  is any ring. Prove that

$$\left( \bigoplus_{\lambda \in \Lambda} M_\lambda \right) \otimes_R N \simeq \bigoplus_{\lambda \in \Lambda} (M_\lambda \otimes_R N).$$

30) Let  $A = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ . Let  $A^{\otimes n}$  denote the  $n$ th tensor power,  $A \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A$ , ( $n$  factors). Prove that

$$A^{\otimes n} \simeq \mathbb{Z} \oplus \mathbb{Z}_2^{2^n - 1} \oplus \mathbb{Z}_3^{3^n - 1}, \quad (n > 0),$$

where  $\mathbb{Z}_k^m = \mathbb{Z}_k \oplus \cdots \oplus \mathbb{Z}_k$ , ( $m$  factors).

31) Let  $R$  be a commutative ring and let  $M$  be an  $R$ -module. The **exterior square**  $M \wedge_R M$  is an abelian group generated by elements  $a \wedge b$ , ( $a, b \in M$ ), with all the properties of the tensor product *plus*  $a \wedge a = 0$ , for all  $a \in M$ .

(a) Explain how to define  $M \wedge_R M$ .

- (b) Prove that  $a \wedge b = -(b \wedge a)$ , where  $a, b \in M$ .
- (c) A mapping  $\alpha : M \times M \rightarrow A$ , where  $A$  is an abelian group, is called **alternating middle linear** if it is middle linear and also  $\alpha((a, a)) = 0$ , for all  $a \in M$ . Construct a “canonical” alternating middle linear mapping

$$\theta : M \times M \rightarrow M \wedge_R M.$$

- (d) Use alternating middle linear maps to characterize  $M \wedge_R M$  up to isomorphism by a mapping property.

**32)** Let  $R$  be an arbitrary ring with identity.

- (a) Prove that a left  $R$ -module  $M$  is simple if and only if  $M \simeq R/I$  for some maximal left ideal  $I$ .
- (b) Use Zorn’s Lemma to prove that simple left  $R$ -modules exist.

**33)** Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

be a matrix with entries in a field  $k$  of characteristic  $p \geq 5$ .

- (a) Prove that  $A$  is similar to a Jordan block  $J$ .
- (b) Find the order of  $A$  in the group  $GL_4(k)$ .

**34)** Let  $R$  be a commutative ring and let  ${}_R \mathbf{Mod}$  be the category of all  $R$ -modules.

- (a) If  $M$  is an  $R$ -module, prove that  $R \otimes_R M$  is also an  $R$ -module.
- (b) Prove that  $M \simeq R \otimes_R M$ .
- (c) Prove that  $\mathcal{S}$ , the identity functor on  ${}_R \mathbf{Mod}$ , is naturally equivalent to the functor  $R \otimes_R -$ .

**35)** Define semisimple ring, and prove that a commutative ring  $R$  is semisimple if and only if  $R$  is isomorphic to a finite direct product of fields.

**36)** Let  $S^2 = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$  be the unit sphere in  $\mathbb{R}^3$ . Prove that

$$I = \{f(x, y, z) \in \mathbb{R}[x, y, z] : f(a, b, c) = 0 \text{ for all } (a, b, c) \in S^2\}$$

is a finitely generated ideal in  $\mathbb{R}[x, y, z]$ .

- 37)** Prove that a UFD satisfies the ascending chain condition for principal ideals, but that the ascending chain condition on all ideals need not hold.
- 38)** Let  $R$  be a finitely generated commutative ring with identity. Prove that  $R$  is isomorphic with some quotient ring of a polynomial ring  $\mathbb{Z}[t_1, \dots, t_n]$ . Explain why  $R$  is noetherian.
- 39)** Compute  $(\mathbb{Z}/14\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/12\mathbb{Z})$ .
- 40)** A module is called **semisimple** if it is a direct sum of simple modules.
- (a) If a module is generated by simple submodules, prove that it is semisimple.
  - (b) Let  $N$  be a submodule of a semisimple module  $M$ . Prove that  $N$  is a direct summand of  $M$ .
  - (c) Prove that a submodule of a semisimple module is semisimple.
- 41)** Prove that a noetherian module is a direct sum of finitely many (directly) indecomposable modules.
- 42)** Let  $M$  be a module with a submodule  $N$ . If  $N$  and  $M/N$  are artinian modules, show that  $M$  is artinian.
- 43)** Let  $R$  be a commutative ring with identity.
- (a) Define what it means to be a projective  $R$ -module.
  - (b) Give three more characterizations of a projective  $R$ -modules.
  - (c) Let  $P$  and  $Q$  be projective modules over a commutative ring  $R$  with identity. Prove that  $P \otimes_R Q$  is a projective  $R$ -module.
- 44)** Let  $M$  be a module and let  $\theta$  be a module endomorphism of  $M$ . Assume that  $r$  and  $s$  are positive integers which are minimal subject to  $\text{Im}(\theta^r) = \text{Im}(\theta^{r+1})$  and  $\text{Ker}(\theta^s) = \text{Ker}(\theta^{s+1})$ . Prove that  $r = s$  and that  $M = \text{Im}(\theta^r) \oplus \text{Ker}(\theta^r)$ .
- 45)** Let  $G$  be a finite abelian group and let  $F$  be a finite field of order  $q$  where  $q$  is relatively prime to  $|G|$ .
- (a) Prove that the group algebra  $FG$  is isomorphic as an  $F$ -algebra to a direct sum  $F_1 \oplus F_2 \oplus \dots \oplus F_k$  where  $F_i = GF(q^{e_i})$  and  $e_1 + e_2 + \dots + e_k = |G|$ .
  - (b) If  $|G| = 3$  and  $|F| = 5$ , find the correct values of the  $e_i$ 's.
- 46)** Prove that a finitely generated torsion module over a PID  $R$  is artinian. [Hint: Show that  $R/p^i R$  is artinian where  $p \in R$  is irreducible.]

47) If  $P$  is an  $R$ -module, define

$$\text{tr}(P) = \sum_{\alpha} \alpha(P)$$

where  $\alpha \in \text{Hom}_R(P, R)$ .

(a) Show that if  $P$  is free, then  $\text{tr}(P) \simeq R$ .

(b) Show that if  $P$  is projective, then  $\text{tr}(P)$  is a 2-sided ideal of  $R$ .

48) Let  $A \leq B$  be  $R$ -modules. Call  $A$  **large** in  $B$  if the only submodule  $S$  of  $B$  such that  $A \cap S = 0$  is  $S = 0$ .

(a) Show that given  $A \leq B$  there is an intermediate submodule  $S$  that is maximal among the submodules that contain  $A$  as a large submodule.

(b) In the case where  $R$  is an integral domain,  $A = R$ , and  $B$  is the field of fractions of  $R$ , what is this intermediate  $S$ ?

49) Show that if  $R$  is a commutative noetherian ring, then so are  $R[x]$  and  $S^{-1}R$ , where  $S$  is a multiplicative subset of  $R$ . Do not use Hilbert's Basis Theorem.

50) Let  $A$  be a finitely generated module over a PID  $R$ . Show if  $A \otimes_R A = 0$ , then  $A = 0$ . Show that  $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$ .

51) Let  $A$  be an  $R$ -module.  $A$  is called **divisible** if given  $a \in A$  and  $r \in R$  with  $r$  not a zero divisor in  $R$ , there exists  $a' \in A$  such that  $ra' = a$ . Show:

(a) Every injective module is divisible.

(b) Every divisible module over a PID is injective.

52) Let  $A$  be an  $n \times n$  matrix over an algebraically closed field  $k$ . Prove there exist matrices  $D$  and  $N$  such that:

(a)  $A = D + N$ .

(b)  $DN = ND$ .

(c)  $D$  is similar to a diagonal matrix.

(d)  $N^n = 0$ .

53) Let  $G$  be a finite group and let  $F$  be an algebraically closed field whose characteristic does not divide  $|G|$ . Prove:

(a)  $FG \simeq \text{Mat}_{n_1}(F) \times \cdots \times \text{Mat}_{n_t}(F)$  where  $\text{Mat}_n(F)$  is the ring of all  $n \times n$  matrices over  $F$ .

- (b)  $\dim_{\mathbb{C}}(Z(\mathbb{C}G)) = t$ , where  $t$  is the number of matrix summands in the decomposition of  $\mathbb{C}G$ .  $Z(R)$  denotes the center of the ring  $R$ .

**54)** Let  $M$  be a module over a ring with identity and assume that  $M$  satisfies both the ascending and descending chain condition on submodules.

- (a) If  $\alpha$  is a module endomorphism of  $M$ , prove that  $M = \text{Ker}(\alpha^i) \oplus \text{Im}(\alpha^i)$ .
- (b) Show that there are submodules  $L$  and  $N$  such that  $M = L \oplus N$  where  $\alpha(L) = L$  and  $\alpha|_L$  is an automorphism, while  $\alpha(N) \subseteq N$  and  $\alpha|_N$  is nilpotent.

**55)** Let  $M$  and  $N$  be nonzero  $R$ -modules, where  $R$  is a commutative ring with identity.

- (a) A mapping  $\varphi : \text{Hom}_R(M, R) \otimes_R N \rightarrow \text{Hom}_R(M, N)$  is defined by

$$\varphi(f \otimes y) : x \mapsto f(x)y,$$

where  $f \in \text{Hom}_R(M, R)$ ,  $y \in N$ , and  $x \in M$ . Show that  $\varphi$  is a well-defined  $R$ -module homomorphism.

- (b) Given that  $M$  is a finitely generated free  $R$ -module, prove that  $\varphi$  is an isomorphism.
- (c) Deduce from (b) that  $\varphi$  will also be an isomorphism if  $M$  is a finitely generated projective module.

**56)** Let  $M$  be a module over  $R$ , a commutative ring with identity. Then  $M$  is called  **$R$ -flat** if, for every ideal  $I$  of  $R$ , the natural map  $I \otimes_R M \rightarrow R \otimes_R M = M$  is injective. Also  $M$  is **faithfully flat** if it is flat and if  $N \otimes_R M = 0$  always implies that  $N = 0$ .

- (a) Show that  $\mathbb{Q}$  is  $\mathbb{Z}$ -flat.
- (b) Is  $\mathbb{Q}$  faithfully flat?
- (c) Prove that a finitely generated free  $R$ -module is faithfully  $R$ -flat.

**57)** Let  $A$  be an abelian group.

- (a) Suppose  $A$  is generated by  $n$  elements subject to  $r$  defining relations, where  $n > r$ . Prove that there are at least  $n - r$  infinite summands in a direct sum decomposition of  $A$  with cyclic summands.

- (b) Let  $A$  be generated by  $x_1, x_2, x_3, x_4$  subject to the following defining relations:

$$\begin{cases} 3x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ x_1 + 2x_3 + x_4 = 0 \\ x_1 + 2x_2 + x_3 + 3x_4 = 0 \end{cases}$$

Determine the structure of the group  $A$ .

- 58)** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{Q}$ . Describe in detail, but without proofs, the rational canonical form of  $A$ .

- 59)** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . Describe in detail, but without proofs, the Jordan canonical form of  $A$ .

- 60)** Let  $A$  be an  $n \times n$  matrix over  $\mathbb{Q}[x]$ . Describe in detail, but without proofs, the Smith normal form of  $A$ .

- 61)** Suppose that  $A$  is a  $6 \times 6$  matrix with rational entries satisfying

$$(A^3 + A + 1)^2 = 0.$$

Find all possible rational canonical forms of  $A$  and indicate the invariant factors.

- 62)** In the following diagram of  $R$ -modules and  $R$ -module homomorphisms the rows are exact and the squares are commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \end{array}$$

- (a) If  $\lambda$  and  $\nu$  are surjective, prove that  $\mu$  is also surjective.
- (b) Let  $M$  be an  $R$ -module. Form a corresponding diagram for  $\text{Hom}_R(A, M)$ ,  $\text{Hom}_R(B, M)$  etc., and prove that its squares commute.
- 63)** You are given an abelian group  $A$  with three generators subject to the relations:

$$\begin{aligned} 7a + 5b + 2c &= 0 \\ 3a + 3b &= 0 \\ 13a + 11b + 2c &= 0 \end{aligned}$$

By the structure theorem,  $A$  is a direct sum of cyclic groups. Find such a decomposition.

**64)** Prove the following statements, using only the definitions of projectivity and injectivity:

- (a)  $\mathbb{Q}$  is not a projective  $\mathbb{Z}$ -module.
- (b)  $\mathbb{Z}$  is not an injective  $\mathbb{Z}$ -module.

**65)** Let  $I$  be a principal ideal of a domain  $R$ . Prove that  $I \otimes_R I \stackrel{R}{\simeq} I^2$  and deduce that  $I \otimes_R I$  is  $R$ -torsion free.

**66)** Let  $R = \mathbb{Z}[t]$  and  $I = (2, t)$ .

- (a) Use the mapping property to construct an  $R$ -module homomorphism  $\theta : I \otimes_R I \rightarrow \mathbb{Z}/2\mathbb{Z}$  such that  $\theta(p \otimes q) = \frac{1}{2}p(0)q'(0) + 2\mathbb{Z}$ . Note that  $p(0)$  must be even and  $q'$  is the usual derivative. Also  $\mathbb{Z}/2\mathbb{Z}$  is an  $R$ -module via  $R \rightarrow \mathbb{Z}$ .
- (b) Prove that  $u = t \otimes 2 - 2 \otimes t$  in  $I \otimes_R I$  is nonzero by using (a).
- (c) Prove that  $u$  in (b) is an  $R$ -torsion element.

**67)** Let  $R$  be a commutative noetherian ring with identity,  $M$  a finitely generated  $R$ -module, and let  $f : M \rightarrow M$  be a surjective  $R$ -linear map. Show that  $f$  is an isomorphism.

**68)** Let  $T : V \rightarrow V$  be a linear transformation of a vector space  $V$  over a the field  $F$ .

- (a) Given  $T$ , explain how  $V$  becomes a  $F[X]$ -module, where  $X$  is an indeterminate.
- (b) Suppose the dimension of  $V$  is finite. Define the order of  $v \in V$  by

$$O(v) = \{g \in F[X] : g(T)v = 0\}.$$

Prove that  $O(v)$  is a nonzero ideal of  $F[X]$ .

- (c) Define  $T : V \rightarrow V$  on a basis  $e_1, \dots, e_n$ , ( $n \geq 2$ ) of  $V$  by  $T(e_1) = ae_1$  and  $T(e_{i+1} + e_i) = ae_{i+1} + e_i$  for  $a \in F$  and  $i = 1, \dots, n-1$ . Determine explicitly with proof the ideals  $O(e_i)$ ,  $i = 1, \dots, n$ .

**69)** Let  $R$  be a commutative ring with identity. Consider an exact sequence of finitely generated free  $R$ -modules

$$0 \rightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \rightarrow 0.$$

- (a) Show that there exists an  $R$ -linear map  $\gamma : H \rightarrow G$  such that  $\beta\gamma = \text{id}_H$ .

(b) Show that  $F \oplus H \simeq \alpha(F) \oplus \gamma(H) = G$ .

**70)** Let  $\alpha : M \rightarrow M$  be a surjective homomorphism of modules. Show that  $\alpha$  need not be injective if  $M$  is artinian.

**71)** Let  $R$  be a commutative ring with identity. For  $R$ -modules  $A, B, C, F, P, Q$ , prove that the following are equivalent:

(a) Given any right-exact sequence  $B \rightarrow C \rightarrow 0$  and a homomorphism  $f : P \rightarrow C$ , there exists  $\varphi : P \rightarrow B$  such that the diagram below commutes.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \varphi & \downarrow f & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

(b) Every short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$  is split exact.

(c) There is a free module  $F$  such that  $F \simeq P \oplus Q$  for some  $R$ -module  $Q$ .

**72)** Let  $F$  be the free abelian group with basis  $x_1, x_2, x_3, x_4$ . Define elements

$$\begin{aligned} y_1 &= 2x_1 + 2x_2 + 4x_3 + 6x_4, \\ y_2 &= 2x_1 + x_2 - x_4, \\ y_3 &= 4x_1 + 2x_2 + 4x_3, \end{aligned}$$

each in  $F$ , and let  $S$  be the subgroup generated by  $y_1, y_2, y_3$ . Determine the structure of the finitely generated abelian group  $F/S$ .

**73)** Compute the following tensor product:  $(\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \otimes_{\mathbb{Z}} (\mathbb{Z}_3 \oplus \mathbb{Z}_6)$ .

**74)** Let  $G$  be a group of prime order  $p$ . Prove that  $\mathbb{Q}G = I_1 \oplus I_2$  where  $I_1$  and  $I_2$  are ideals such that  $I_1 \stackrel{\mathbb{Q}G}{\simeq} \mathbb{Q}$  and  $I_2 \stackrel{\mathbb{Q}G}{\simeq} \mathbb{Q}(e^{\frac{2\pi i}{p}})$ .

**75)** Let  $M$  be a finitely generated module over a PID  $R$ . Prove that there exist irreducible elements  $p_1, \dots, p_k$  of  $R$  and a positive integer  $\ell$  such that

$$\bigcap_{i=1}^k p_i^{\ell} M$$

is a torsion-free module.

**76)** Use rational canonical form to find all similarity types of  $4 \times 4$  rational matrices  $A$  such that  $A^2 + A^4 = 0$ .

**77)** Let  $A = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus \mathbb{Z}_6$  and  $B = \mathbb{Q} \oplus \mathbb{Z}_4$ . Compute the tensor product  $A \otimes_{\mathbb{Z}} B$  in simplest terms.

**78)** Let  $M$  be a finitely generated module over a commutative noetherian ring  $R$ .

(a) Prove that  $M$  is a noetherian module.

(b) If  $N$  is a submodule such that  $M \stackrel{R}{\simeq} M/N$ , show that  $N = 0$ .

**79)** Let  $A$  be the  $n \times n$  ( $n > 1$ ) matrix over  $\mathbb{Q}$  all of whose entries equal 1

(a) Prove that the minimum polynomial of  $A$  is  $x(x - n)$ .

(b) Find the characteristic polynomial of  $A$ .

(c) Show that  $A$  is diagonalizable and find its diagonal form.

**80)** Let  $G$  be a cyclic group of order 6. Explain why the group algebra  $\mathbb{Q}G$  is a direct sum of fields and identify the fields.

**81)** Let  $M$  be a finitely generated abelian group. State the Structure Theorem for  $M$  in terms of invariant factors.

**82)** Prove that every nonzero vector space has a basis.

**83)** Let  $V$  be an  $n$ -dimensional vector space over the field of  $p$  elements. If  $v_1, \dots, v_r$  in  $V$  are linearly independent, prove that there are  $p^n - p^r$  vectors  $v \in V$  such that  $v_1, \dots, v_r, v$  are linearly independent.

Let  $GL_n(p)$  be the group of all nonsingular  $n \times n$  matrices over the field of  $p$  elements. Deduce from above that  $GL_n(p)$  has order  $(p^n - 1)(p^n - p) \dots (p^n - p^{n-1})$ .

**84)** Let  $R$  and  $S$  be rings and let  $A, B, C$  be a right  $R$ -module, an  $(R, S)$ -bimodule, and a right  $S$ -module respectively. Prove that there is an isomorphism of abelian groups

$$\text{Hom}_S(A \otimes_R B, C) \simeq \text{Hom}_R(A, \text{Hom}_S(B, C)).$$

**85)** Let  $R$  be a ring with identity.

(a) Let  $M_R$  and  ${}_R N$  be a right and a left  $R$ -module and let  $M_0, N_0$  be submodules of  $M, N$  respectively. Prove that

$$(M/M_0) \otimes_R (N/N_0) \simeq (M \otimes_R N)/S$$

where  $S$  is the subgroup generated by all  $a \otimes b$  where  $a \in M_0$  or  $b \in N_0$ .

- (b) If  $I$  is a left ideal of  $R$ , prove that  $M \otimes_R (R/I) \simeq M/MI$ .
- 86)** Define what it means to be an injective module and show the following
- (a)  $\mathbb{Q}$  is an injective  $\mathbb{Z}$ -module.
- (b) The module  $\mathbb{Z}/n\mathbb{Z}$  is injective over itself as a ring, where  $n > 1$ .
- 87)** Suppose  $V$  is a finite dimensional vector space over  $\mathbb{C}$ . Show that the ring  $\text{Hom}_{\mathbb{C}}(V, V)$  is simple.
- 88)** Prove that the following are equivalent for a commutative ring  $R$ :
- (a)  $R$  satisfies the ascending chain condition on ideals.
- (b) Each ideal of  $R$  is finitely generated.
- (c) Every set of ideals in  $R$  contains a maximal element with respect to set theoretic inclusion.
- 89)** Suppose that  $R$  is a PID and that  $F$  is a finitely generated free  $R$ -module. Show that any  $R$ -submodule  $M$  of  $F$  is a finitely generated free  $R$ -module.
- 90)** Let  $R$  be a PID and  $Q$  its field of quotients. Assume that  $R$  is not equal to  $Q$ , so  $R \subsetneq Q$  and  $Q$  may be viewed as an  $R$ -module. Prove that  $Q$  is not a finitely generated  $R$ -module.
- 91)** Let  $R$  be a ring with identity and  $e \in R$  be an idempotent element—that is  $e = e^2$ . Prove that the left ideal  $Re$  is a projective module.
- 92)** Let  $V$  be the  $\mathbb{Q}$ -vector space of 3-tuples of column vectors. Let  $T$  be the linear transformation defined on the natural basis by

$$T : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Then  $V$  becomes a module over the polynomial ring  $\mathbb{Q}[x]$  when (left) multiplication by  $x$  has the same effect as  $T$ :  $xv = T(v)$ .

- (a) Describe the decomposition of  $V$  into a direct sum of indecomposable  $\mathbb{Q}[x]$ -submodules.
- (b) Find all the  $\mathbb{Q}[x]$  submodules of  $V$ . [Hint: There are 6 of them.]
- 93)** Let  $R$  be a PID and  $F = R^n$ , the free module consisting of all  $n$ -tuples over  $R$ . Let  $y = (a_1, \dots, a_n)$  be an element of  $F$ . Show that there is some basis  $x_1, \dots, x_n$  of  $F$  over  $R$  with  $x_1 = y$  if and only if the ideal  $Ra_1 + \dots + Ra_n$  equals  $R$ .

**94)** Let  $y = (6, 5, 2)$  be an element of  $\mathbb{Z}^3$ . Find a basis of  $\mathbb{Z}^3$  over  $\mathbb{Z}$  that includes  $y$ .

**95)** Let  $A$  be a finite abelian group. State the Structure Theorem for  $M$  in terms of elementary divisors.

(a) State a condition on the elementary divisors of  $A$  that hold if and only if  $A$  is a cyclic group.

(b) Prove that the multiplicative group of nonzero elements of a finite field is a cyclic group.

**96)** Let  $R$  be a ring with identity. Prove that every left  $R$ -module is projective if and only if every left  $R$ -module is injective.

**97)** Let  $R$  be a ring with identity and  $M$  a left  $R$ -module. For any left  $R$ -modules  $A$  and  $B$  and  $R$ -homomorphism  $f : A \rightarrow B$ , there is a homomorphism of abelian groups

$$f^* : \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M)$$

defined by  $f^*(h) = h \circ f$ . If

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

is an exact sequence of  $R$ -modules and  $R$ -module homomorphisms, prove

$$0 \rightarrow \text{Hom}_R(C, M) \xrightarrow{g^*} \text{Hom}_R(B, M) \xrightarrow{f^*} \text{Hom}_R(A, M)$$

is an exact sequence of abelian groups.

**98)** Let  $F = \mathbb{Z} \oplus \mathbb{Z}$  be the free abelian (additive) group on two generators and let  $R$  be the subgroup of  $F$  generated by

$$r_1 = (4, 6), \quad r_2 = (8, -9), \quad r_3 = (4, -36).$$

Express the quotient  $F/R$  as a direct sum of cyclic groups each of which is either infinite or finite of prime power order.

**99)** Suppose  $A$  is a principal ideal domain and that  $F$  is a finitely generated free  $A$ -module. Show that any  $A$ -submodule  $M$  of  $F$  is a finitely generated free  $A$ -module.

**100)** Let  $G$  be a cyclic group of order 5. Explain why the group algebra  $\mathbb{Q}G$  is a direct sum of fields and identify the fields.

**101)** Prove that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$  and deduce that  $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \neq 0$ .

**102)** Prove that  $\text{Hom}_R(M, \prod_{\lambda \in \Lambda} A_\lambda) \simeq \prod_{\lambda \in \Lambda} \text{Hom}_R(M, A_\lambda)$  where  $A_\lambda, M$  are left  $R$ -modules.

**103)** Given rings  $R, S, T$  and bimodules  ${}_S A_{R,T} B_R$ , say what the bimodule structure of  $\text{Hom}(A, B)$  is and prove your assertion.

**104)** Let  $R$  be a commutative ring with identity. Prove that if every  $R$ -module is free, then  $R$  must be a field. [Hint: it's enough to prove that  $R$  has no proper non-zero ideals.]

**105)** Let  $V$  be an  $n$ -dimensional vector space over a field  $K$ , and let  $R$  be the ring of all linear transformations on  $V$ . Prove that  $R$  has the invariant basis property. [Hint:  $\dim_K(R) = n^2$ .]

**106)** Let  $R$  be a ring with identity and let  $M$  be a direct summand of an injective  $R$ -module  $J$ . Prove that  $M$  is an injective  $R$ -module.

**107)** Let  $F$  be a free  $R$ -module where  $R$  is a ring with identity. Suppose that  $S$  is a finitely generated submodule of  $F$ . Prove that there exists a surjective  $R$ -module homomorphism  $f : F \rightarrow F_1$  where  $F_1$  is a finitely generated  $R$ -module and  $\text{Ker}(f) \cap S = 0$ .

**108)** Suppose  $P_1 \rightarrow Q_1 \rightarrow M_1 \rightarrow 0$  and  $P_2 \rightarrow Q_2 \rightarrow M_2 \rightarrow 0$  are exact sequences of  $R$ -modules, where  $P_1$  and  $Q_1$  are projective. Suppose also that there is a  $R$ -module homomorphism  $\mu : M_1 \rightarrow M_2$ . Prove that there are homomorphisms  $\alpha : P_1 \rightarrow P_2$  and  $\beta : Q_1 \rightarrow Q_2$  making a commutative diagram. [Hint: Use the definition of projective modules.]

**109)** If  $F$  is a field of prime characteristic  $p$  and  $G$  is a non-trivial finite  $p$ -group, prove that the group algebra  $FG$  is not semisimple. [Hint: the center of  $G$  is non-trivial.]

**110)** Let  $F$  be the free abelian group with basis  $\{x_1, x_2, \dots, x_n\}$ , and let  $A = [a_{ij}]$  be any  $m \times n$  integer matrix. Define  $m$  elements  $y_i$  of  $F$  by  $y_i = \sum_{j=1}^n a_{ij}x_j$ , for  $i = 1, 2, \dots, m$ . Let  $K = (y_1, y_2, \dots, y_m)$  and put  $G = F/K$ , a finitely generated abelian group. The following procedure is an algorithm to determine the structure of  $G$ .

- (a) Show that applying a ROW operation to  $A$  does not change  $K$ .
- (b) Show that applying a COLUMN operation to  $A$  changes the basis but not  $K$ . [Note: Row and column operations for integer matrices are defined as in linear algebra, but rows and columns can only be multiplied by  $\pm 1$ .]
- (c) Show that by a suitable sequence of row and column operations  $A$  can be transformed into a diagonal matrix  $N = \text{diag}(d_1, d_2, \dots, d_k, 0, \dots, 0)$  where the  $d_i$  are positive integers.

- (d) Prove that  $G \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  where the number of  $\mathbb{Z}$ 's is  $n - k$ .
- (e) Show that  $G$  is finite if and only if  $k = n$ .
- (f) Let  $n = 3$  and define  $K = (2x_1 - 4x_2 + 6x_3, 3x_1 + 9x_2 - 9x_3, 7x_1 + x_2 + 3x_3)$ . Use the above method to determine the structure of the group  $G = F/K$ .

[Comment: The procedure just described is an algorithm to determine the structure of an abelian group with a set of  $n$  generators subject to a given set of  $m$  relations. For a background see Hungerford, p. 335-345.]

- 111)** Define “torsion-submodule”.
- 112)** Define “torsion-free module”.
- 113)** Let  $A = \prod_p \mathbb{Z}_p$ . Identify the torsion submodule.
- 114)** Prove that if  $M$  is finitely generated torsion module over a PID, then  $\text{Ann}_R(M) \neq 0$ .
- 115)** Give an example of a torsion module  $M$  over a domain  $R$  such that  $\text{Ann}_R(M) = 0$ .
- 116)** Show that  $\mathbb{Q}$  is NOT a free abelian group.
- 117)** TRUE or FALSE: If  $M$  is a torsion-free quotient of a finitely generated module over a PID, then  $M$  is free.
- 118)** List all isomorphism types of abelian groups of order 600.
- 119)** What are the invariant factors of  $\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$ .
- 120)** Let  $R = F[x, y]$  where  $F$  is a field. Prove that  $Rx + Ry$  is NOT a cyclic submodule of  $R$  (viewed as a left  $R$ -module).
- 121)** List all similarity types of nilpotent  $4 \times 4$  matrices.
- 122)** Let  $R$  be a domain and  $F$  its field of fractions, regarded as an  $R$ -module in the natural way. If  $S$  is a finitely generated  $R$ -submodule of  $F$ , prove that  $S \simeq I$  for some ideal  $I$  of  $R$ .
- 123)** Let  $R$  and  $F$  be as in the previous problem. If  $F$  is  $R$ -isomorphic with an ideal of  $R$ , prove that  $R = F$ . [Hint:  $F = rF$  if  $0 \neq r \in R$ .]
- 124)** Let  $F$  be a field and let  $f_1, f_2, \dots, f_k$  be distinct monic irreducible polynomials in  $F[t]$ . Use the Structure Theorem for finitely generated modules over PID's to prove the following statement: there is an  $n$ -dimensional  $F$ -vector space  $V$  and an  $\alpha \in \text{End}_F(V)$  such that  $f_1 f_2 \cdots f_k$  is the minimum polynomial of  $\alpha$  if and only if  $n = \ell_1 n_1 + \cdots + \ell_k n_k$  where  $n_i = \deg(f_i)$  and the  $\ell_i$  are positive integers.

**125)** Let  $A$  and  $B$  be torsion-free abelian groups with subgroups  $S$  and  $T$  respectively. Prove that the natural map  $S \otimes_{\mathbb{Z}} T \rightarrow A \otimes_{\mathbb{Z}} B$  is injective. Deduce that  $A \otimes B \neq 0$  if  $A \neq 0$  and  $B \neq 0$ . [Hint: Flat modules.]

**126)** Suppose  $R$  is a commutative ring with identity where each prime ideal is finitely generated. Prove that  $R$  is noetherian.