A NOTE ON THE VARIETY OF PAIRS OF MATRICES WHOSE PRODUCT IS SYMMETRIC

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This paper is dedicated to Wolmer Vasconcelos.

Abstract. We give different short proofs for a result proved by C. Mueller in [9]: Over an algebraically closed field pairs of $n \times n$ matrices whose product is symmetric form an irreducible, reduced, and complete intersection variety of dimension $(3n^2 + n)/2$. Our work is connected to the work of Brennan, Pinto, and Vasconcelos in [2].

1. Introduction and notation

A central research theme in classical algebraic geometry is to study the scheme defined by a given ideal in a polynomial ring in order to determine whether it has desirable properties such as being normal, nonsingular, complete intersection, smooth, Gorenstein, Cohen-Macaulay, irreducible, or reduced. Many interesting results have been obtained in connection to such questions. However, there are also long-standing open problems that continue to generate interest and provide motivation for further research in this area. One of these open problems is the following

Conjecture 1.1 (M. Artin, M. Hochster)**.** Let $X = (x_{ij})$ and $Y = (y_{ij})$ be square $n \times n$ matrices in $2n^2$ indeterminates and let I be the ideal generated by entries of $XY - YX$ in the polynomial ring $R = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$, where k is an algebraically closed field. Then $\operatorname{Spec}(R/I)$ is reduced, and Cohen-Macaulay.

It is easy to show $\operatorname{Spec}(R/I)$ is Cohen-Macaulay when $n = 1$ and $n = 2$. However, *Macaulay*, see [1], was used to completely establish this when $n = 3$ (though several partial results were obtained in [12]), and *Macaulay 2*, see [5], was used to establish this when $n = 4$.

Several authors, including T. Motzkin and O. Taussky [8], M. Gerstenhaber [4], and R. Guralnick [6] have shown that commuting pairs of $n \times n$ matrices with entries in k, regarded as points in the $2n^2$-dimensional affine space, form an irreducible variety of dimension $n^2 + n$. Since the coordinate ring of this variety is $R/\operatorname{rad}(I)$, the scheme $\operatorname{Spec}(R/\operatorname{rad}(I))$ is referred to as the *commuting variety* in the literature.

In [2] a special commuting variety is studied, the variety of commuting pairs of symmetric matrices. Partially inspired by this work, we consider the ideal J generated by entries of $XY - Y^tX^t$ in the polynomial ring $R = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$, where X^t is the transpose of a matrix X. This ideal was studied by C. Mueller in [9].

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As a special case of his results, he finds that the scheme Spec (R/J) is reduced, irreducible and a complete intersection of dimension $(3n^2+n)/2$. In this paper we use completely different methods to give proofs of these results. We should note that closed points of Spec (R/J) can be identified with pairs of $n \times n$ matrices over k, whose product is symmetric.

In what follows, unless otherwise stated k will be an arbitrary field. We let $M_n(k)$ be the affine space of all $n \times n$ matrices with entries in k and $S_n(k)$ be the affine space of all symmetric $n \times n$ matrices with entries in k. We will use the notation X^t to denote the transpose of a matrix X.

2. Pairs of matrices whose product is symmetric

We begin by showing that if k is algebraically closed, pairs of $n \times n$ matrices whose product is symmetric regarded as points in the $2n^2$-dimensional affine space form an irreducible variety. We will use an idea similar to the proof of [6, Theorem 2]. First we need a few elementary facts.

Lemma 2.1. *Let $A \in M_n(k)$. Then there is a nonsingular symmetric $n \times n$ matrix T such that AT is symmetric.*

Proof. This is essentially proved in [11, Theorem 1]. By that theorem there is a nonsingular symmetric matrix T, such that $AT = TA^t$. Since T is symmetric, the previous equality shows that AT is symmetric. □

Proposition 2.2. Let $B \in M_n(k)$, and T be a nonsingular $n \times n$ matrix. Let x be an indeterminate. Then $\det(B + xT)$ is not identically zero.

Proof. Suppose $\det(B + xT)$ is identically zero. Then columns of the matrix $B+xT$ are linearly dependent. Hence, there are scalars $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i V_{B,i} = -x \sum_{i=1}^{n} \alpha_i V_{T,i},$$

where $\{V_{B,i}\}$ and $\{V_{T,i}\}$ are columns of B and T, respectively. Since this identity holds for all $x \in k$, we must have

$$\sum_{i=1}^{n} \alpha_i V_{T,i} = 0,$$

which is a contradiction since T is nonsingular and its columns are linearly independent. □

Lemma 2.3. *Let $A, B \in M_n(k)$. If $\det(B) \neq 0$ and AB is symmetric, then there is a symmetric matrix $S \in S_n(k)$ such that $A = B^t S$.*

Proof. To say AB is symmetric means $AB = B^t A^t$. Multiplying both sides of this equality by B^{-1} from the right, we obtain

$$A = B^t A^t B^{-1}.$$

We end the proof by showing that $S := A^t B^{-1}$ is a symmetric matrix. To see this multiply both sides of $AB = B^t A^t$ by B^{-1} on the right, and $(B^{-1})^t$ on the left. This gives us $(B^{-1})^t A = A^t B^{-1}$, hence $S^t = S$. □

Theorem 2.4. *Let k be an algebraically closed field and let*

$$V^n = \{(A, B) \in M_n(k) \times M_n(k) : AB \text{ is symmetric}\}.$$

Then V^n, viewed as a subset of the $2n^2$-dimensional affine space over k, is an irreducible variety.

Proof. Consider a pair $(A, B) \in V^n$. By Lemma 2.1 there is a nonsingular matrix T such that AT is symmetric. Thus, $(A, B+xT)$ is contained in V^n for every $x \in k$. Moreover, except for finitely many values of x, $\det(B + xT)$ is nonzero (because by Proposition 2.2 the determinant of $B + xT$ is a *nonzero* polynomial in x and vanishes for only finitely many values of x). Thus (A, B) is in the closure of the set of elements in V^n where the second term is nonsingular.

Now consider the map $\varphi : S_n(k) \times M_n(k) \longrightarrow V^n$, defined by

$$\varphi(S, B) = (B^t S, B).$$

The image of φ is irreducible (since the domain is) and is dense (since by Lemma 2.3 it contains all elements of V^n where the second term is nonsingular). Thus its closure is equal to V^n and is irreducible. □

Working from the fact that V^n is irreducible, we will use a technique introduced in [2] to show that the variety of pairs of matrices whose product is symmetric is reduced and a complete intersection. This technique relies on the notion of Jacobian module and some other related results. Let $I = (f_1, \ldots, f_m)$ be the generators of an ideal I in the polynomial ring $R = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$. Assume further that each f_i is a sum of terms of the form $x_{ij}y_{kl}$. Then, the Jacobian matrix Φ of the f_i's with respect to the variables y_{ij} is a matrix with entries in the ring $A = k[x_{11}, \ldots, x_{nn}]$ and can be used to define an A-module

$$A^m \xrightarrow{\Phi} A^{n^2} \longrightarrow E \to 0.$$

The Jacobian module of R/I is the A-module $E = \operatorname{coker}(\Phi)$. The importance of the Jacobian module is the following, see [2, Proposition 1.2]:

Proposition 2.5. Let $R = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$. Let $X = (x_{ij})$ and $Y = (y_{ij})$ be square $n \times n$ matrices in $2n^2$ indeterminates and let J be the ideal generated by entries of $XY - Y^tX^t$ in R. Let E be the Jacobian module of R/J. Then $R/J \simeq S(E)$. Moreover, if k is algebraically closed and V^n is as defined in Theorem 2.4, then the affine coordinate ring of V^n is isomorphic to $\operatorname{Spec}(S(E))_{\text{red}}$.

The following theorem follows from [13, Proposition 1.4.1] and [10, Theorem 3.4]:

Theorem 2.6. *Let R be a polynomial ring over an arbitrary field, and let E be a finitely generated R-module of projective dimension one. Then the following conditions are equivalent:*

(a) *$S(E)$ is an integral domain.*
(b) *$\operatorname{Spec}(S(E))$ is irreducible.*

Moreover, under these conditions, $S(E)$ is a local complete intersection over R.

Theorem 2.7. *Let k be an algebraically closed field and R and J be as defined in Proposition 2.5. Then the scheme $\operatorname{Spec}(R/J)$ is reduced and is a complete intersection.*

Proof. This will follow from Proposition 2.5 and Theorems 2.4 and 2.6, once we show that the Jacobian module of R/J has projective dimension one. The following argument is similar to the proof of [2, Lemma 3.2]. By definition of E we have a presentation

$$A^{n(n-1)/2} \xrightarrow{\Phi} A^{n^2} \longrightarrow E \longrightarrow 0,$$

where $A = k[x_{11}, \ldots, x_{nn}]$. It suffices to show that the presentation matrix Φ of E is injective. This will be achieved by showing that Φ has rank $n(n-1)/2$. If we specialize the matrix X (with notation of Proposition 2.5) to a generic diagonal matrix (specializing the matrix X will not increase the rank of Φ), it is easy to see that the entries of $Z = XY - Y^tX^t$ specialize to

$$z^*_{ij} = X_{ii}Y_{ij} - X_{jj}Y_{ji}$$

so that the corresponding Jacobian matrix has full rank. □

Remark 2.8. Mueller has shown in [9] that the scheme $\operatorname{Spec}(R/J)$ as defined in Theorem 2.7 is also normal.

3. A connection with the variety of commuting pairs of symmetric matrices

In $\operatorname{char} k \neq 2$ we can give another proof for the fact that the variety of pairs of $n \times n$ matrices whose product is symmetric is a complete intersection. This proof is based on results from [2] which we summarize in the following theorem:

Theorem 3.1 ([2])**.** *Let k be an algebraically closed field of characteristic $\neq 2$, and let X_s and Y_s be generic symmetric $n \times n$ matrices in $n(n+1)$ indeterminates. Let I_s be the ideal generated by entries of $X_sY_s - Y_sX_s$ in the polynomial ring in $n(n+1)$ indeterminates over k, which we denote by S. Then* $\operatorname{Spec}(S/I_s)$ *is reduced, irreducible and a complete intersection.*

Theorem 3.1 was proved in [2, Theorem 3.1]. Even though it appears that this proof is only written for characteristic 0, it works in fact in any characteristic $\neq 2$. The only place in their proof where characteristic 0 seems to have been used is in showing that every symmetric matrix commutes with a symmetric *nonderogatory* matrix [2, Lemma 3.5], which they need for proving the irreducibility of $\operatorname{Spec}(S/I_s)$. We would like to rewrite the proof given in [2, Lemma 3.5] for all characteristics $\neq 2$. While all we need to do is to change some references, the result needed is rather technical and it is most convenient to simply reproduce the proof. We recall that a square matrix A is called nonderogatory if its minimal polynomial coincides with its characteristic polynomial. This condition is also equivalent to saying that every matrix that commutes with A can be expressed as a polynomial in A.

Lemma 3.2 ([2, Lemma 3.5])**.** *Let B be a an element of $S_n(k)$, where k is an algebraically closed field of characteristic $\neq 2$. Then there exists a nonderogatory element of $S_n(k)$ that commutes with B.*

Proof. By the Jordan decomposition Theorem, see [3, Theorem 24.9], there are matrices D and N, such that $B = D+N$, where D is diagonalizable, N is nilpotent, and both D and N can be expressed as polynomials in B, therefore both are symmetric. It follows that D is in fact *orthogonally* similar to a diagonal matrix, see [7, Theorem 70]. Let O be an orthogonal matrix that diagonalizes D. Then

$$O^t B O = \bigoplus_{i=1}^{s} (\lambda_i I_i + N_i'),$$

with $N_i' = O^t N_i O$ nilpotent and symmetric. Then the matrix

$$O \left(\bigoplus_{i=1}^{s} (\mu_i I_i + N_i') \right) O^t$$

with distinct μ_i's, is nonderogatory and commutes with B. □

We are now ready to present our second proof of the fact that the variety of pairs of $n \times n$ matrices whose product is symmetric is a complete intersection.

Theorem 3.3. *Let k be an algebraically closed field of characteristic $\neq 2$, and let R, X, Y and J be as defined in Proposition 2.5. Then the ring R/J is a complete intersection of dimension $(3n^2 + n)/2$.*

Proof. Let S be the polynomial ring over k whose indeterminates are exactly those found in the set

$$\{x_{ij} : i \leq j\} \cup \{y_{ij} : i \leq j\}.$$

We see that R has $2n^2$ indeterminates and that S has n^2+n indeterminates. Define a surjective homomorphism $\varphi : R \to S$ where

$$x_{ij} \mapsto \begin{cases} x_{ij} & \text{if } i \leq j, \\ x_{ji} & \text{if } i > j, \end{cases} \qquad \text{and} \qquad y_{i,j} \mapsto \begin{cases} y_{ij} & \text{if } i \leq j, \\ y_{ji} & \text{if } i > j. \end{cases}$$

Note that the images of X and Y under φ are both symmetric matrices. Hence

$$\varphi(XY - Y^t X^t) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X). \tag{3.1}$$

By [2, Theorem 3.1] the distinct nonzero entries of the matrix

$$\varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$$

form a regular sequence in S of length $(n^2 - n)/2$. Since given any regular sequence in S, the preimage of this sequence will form a regular sequence in $R/\ker(\varphi)$, it follows from (3.1) that the distinct (up to sign) nonzero entries of $XY - Y^t X^t$ form a regular sequence of length $(n^2 - n)/2$ in $R/\ker(\varphi)$.

Next we examine $\ker(\varphi)$. It is easy to see that that $\ker(\varphi)$ is generated by the elements of the set

$$\{x_{ij} - x_{ji} : i < j\} \cup \{y_{ij} - y_{ji} : i < j\}$$

and that these generators are algebraically independent. Hence they form a regular sequence in R of length $(n^2 - n)$. Since these generators and the distinct (up to sign) nonzero entries of $XY - Y^t X^t$ are homogeneous elements forming a regular sequence, we conclude that the distinct (up to sign) nonzero entries of $XY - Y^t X^t$ also form a regular sequence of length $(n^2 - n)/2$ in R. Thus, R/J is a complete intersection ring of dimension $(3n^2 + n)/2$. □

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