Free summands of syzygies of modules over local rings

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Abstract

We give a new criterion for a commutative, noetherian, local ring to be Cohen-Macaulay.

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1. Introduction

In this paper we look to see whether syzygies of modules over local rings have free summands, where by *local* we will always mean a commutative, noetherian ring, with a unique maximal ideal. It has been shown that answers to these sorts of questions can shed light on properties of local rings. In particular, Dutta proves the following in [2, Corollary 1.3]:

Theorem 1.1 (Dutta). *A local ring* $(A, \mathfrak{m}, k)$ *is regular if and only if* $\operatorname{Syz}_i(k)$ *has a free summand for some* $i \geqslant 0$.

Later in [4, Proposition 7], Martsinkovsky and in [3, Proposition 2.2], Koh and Lee give additional proofs of this theorem. Additionally, Takahashi proves variations of Theorem 1.1 in [6, Theorem 4.3 and Theorem 6.5] when he shows that A is regular if and only if some syzygy of k has a semidualizing summand and when he shows that A is Gorenstein if and only if some syzygy of k has a G-projective summand in degree less than or equal to $\operatorname{depth}(A) + 2$.

In this paper we consider a different sort of variation on Theorem 1.1. Here we seek to understand what can be said if $\operatorname{Syz}_i(M)$ has a free summand when M is some A-module of finite length. Of particular interest to us is the case when $M = A/\mathbf{x}$, where $\mathbf{x}$ is a system of parameters. In Section 2 of this paper, we prove the following theorem which characterizes Cohen-Macaulay rings:

Theorem 2.4. *A local ring* $(A, \mathfrak{m}, k)$ *is Cohen-Macaulay if and only if for some* $i > 0$, $\operatorname{Syz}_i(A/\mathbf{x})$ *has a free summand for some system of parameters* $\mathbf{x}$ *that form part of a minimal set of generators for* $\mathfrak{m}$.

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We note that Craig Huneke, Daniel Katz, and Janet Striuli read through an earlier version of this paper. From this they discovered a somewhat simpler proof of Theorem 2.4 than the one given in this paper. While our proof is based on ideas from [1] and [3], their proof is based on Propositions 2 and 3 from [5]; Appendix I, Section 2.

2. A characterization of Cohen-Macaulay rings

Inspired by a theorem of Dutta, we give a new characterization of Cohen-Macaulay rings. This criterion is based on whether a certain syzygy has a free summand. We will need some background before we can prove our theorem. Dutta proves the following in [2, Corollary 1.2]:

Theorem 2.1 (Dutta)**.** *Let A be a local ring of dimension n and M be a finitely generated A-module of finite length. The following hold:*

1. *If A is Cohen-Macaulay, only $\operatorname{Syz}_n(M)$ can have a free summand.*
2. *If $\operatorname{depth}(A) < n-1$, no syzygy of M can have a free summand.*
3. *If $\operatorname{depth}(A) = n-1$, only $\operatorname{Syz}_{n-1}(M)$ can have a free summand. Moreover, if the Improved New Intersection Conjecture is true for A, then $\operatorname{Syz}_{n-1}(M)$ cannot have a free summand.*

We find this theorem suggestive, as it tells us that if $\operatorname{Syz}_i(M)$ has a free summand for some A-module M of finite length, then A is Cohen-Macaulay whenever the Improved New Intersection Conjecture holds. With this theorem in mind, one might try to prove a result similar to the result given in Theorem 1.1, with the residue field replaced by some other module of finite length, using a proof that mirrors one of the proofs given in [2], [4], or [3]. While each of these proofs seems to use the fact that k is a field quite heavily, and hence cannot be directly applied, we have found them illuminating nonetheless. In particular, we will use the following definition of Koh and Lee which was presented in [3]:

Definition. Given a local ring $(A, \mathfrak{m})$, a minimal free complex $(F_\bullet, \partial_\bullet)$ bounded on the right at degree z is said to satisfy condition (#) if

$$\operatorname{Ker}(\partial_i \otimes_A A/\mathfrak{m}^2) = \mathfrak{m}(F_i \otimes_A A/\mathfrak{m}^2)$$

for all $i > z$.

With this definition, Koh and Lee prove the following in [3, Lemma 2.3]:

Lemma 2.2 (Koh and Lee)**.** *Let $(A, \mathfrak{m})$ be a local ring, the following hold:*

1. *If $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of elements that form part of a minimal set of generators for $\mathfrak{m}$, then the Koszul complex $K_\bullet(\mathbf{x})$ satisfies condition (#).*
2. *Let M be a finitely generated A-module with submodule $N \subset \mathfrak{m}M$. Further suppose that a minimal resolution of M satisfies condition (#). If $\operatorname{Syz}_i(N)$ has a free summand, then $\operatorname{Syz}_{i+1}(M/N)$ has a free summand.*

Finally we will need the following theorem, which appears implicitly in [1, Section 2]. Here we explicitly state and prove the theorem, giving Dutta's proof.

Theorem 2.3 (Dutta)**.** *Let $(A, \mathfrak{m}, k)$ be a local ring of dimension n and depth $n-1$, $K_\bullet$ be the Koszul complex with respect to a system of parameters $\mathbf{x}$, $F_\bullet$ be a minimal free resolution of k, and $\varphi_\bullet$ be a lift of the canonical surjection $A/\mathbf{x} \to k$. Consider the following commutative diagram:*

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_n & \xrightarrow{d_n} & K_{n-1} & \longrightarrow & \cdots & \longrightarrow & K_0 & \longrightarrow & A/\mathbf{x} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & & & \downarrow{\scriptstyle \varphi_0} & & \downarrow & & \\
F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & k & \longrightarrow & 0
\end{array}
$$

If $\mathbf{x}$ is a system of parameters such that φ_n is zero, then $\mathrm{Syz}_{n-1}(H_1)$ has a free summand where H_1 is the first Koszul homology with respect to $\mathbf{x}$.

Proof. Since $\operatorname{depth}(A) = n - 1$, the following complex is exact:

$$0 \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_1/\operatorname{Im}(d_2) \longrightarrow 0$$

Letting $L_\bullet$ be a minimal free resolution of H_1, we may lift the canonical map $H_1 \to K_1/\operatorname{Im}(d_2)$ to obtain the following commutative diagram:

$$
\begin{array}{ccccccccccc}
L_n & \xrightarrow{\lambda_n} & L_{n-1} & \longrightarrow & \cdots & \longrightarrow & L_0 & \longrightarrow & H_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_n} & & & & \downarrow{\scriptstyle \gamma_1} & & \downarrow & & \\
0 & \longrightarrow & K_n & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_1/\operatorname{Im}(d_2) & \longrightarrow & 0
\end{array}
$$

By [1, Theorem 1.3], since $\varphi_n = 0$, it follows that $\operatorname{Im}(\gamma_n) = K_n = A$. Hence there is a basis $e_1, \ldots, e_b$ for L_{n-1} such that $\gamma_n(e_1) = 1$ and $\gamma_n(e_i) = 0$ for $i > 1$. By the commutativity of the above diagram, we see

$$\operatorname{Im}(\lambda_n) \subset \operatorname{Ker}(\lambda_{n-1}) \subset M$$

where M is the module generated by $e_2, \ldots, e_b$. Since

$$\mathrm{Syz}_{n-1}(H_1) \simeq L_{n-1}/\operatorname{Im}(\lambda_n),$$

we see that $\mathrm{Syz}_{n-1}(H_1)$ has a free summand, specifically the summand generated by the image of e_1. □

We now give a criterion for a local ring to be Cohen-Macaulay; compare with Dutta's result, Theorem 2.1.

Theorem 2.4. *A local ring $(A, \mathfrak{m}, k)$ is Cohen-Macaulay if and only if for some $i > 0$, $\mathrm{Syz}_i(A/\mathbf{x})$ has a free summand for some system of parameters $\mathbf{x}$ that form part of a minimal set of generators for $\mathfrak{m}$.*

Proof. ($\Rightarrow$) If A is Cohen-Macaulay, then a Koszul complex on a system of parameters is a minimal free resolution of $A/\mathbf{x}$. Hence $\mathrm{Syz}_n(A/\mathbf{x})$ has a free summand.

($\Leftarrow$) Seeking a contradiction, suppose that A is not Cohen-Macaulay and that for some $i > 0$, $\mathrm{Syz}_i(A/\mathbf{x})$ has a free summand. By Theorem 2.1, A must be of dimension n and depth $n-1$ and the only possible value for i is $n-1$. Working as in the proof of [2, Theorem 1.1], let $(L_\bullet, \lambda_\bullet)$ be a minimal free resolution of $A/\mathbf{x}$. Applying $(-)^* = \mathrm{Hom}_A(-, A)$ we obtain:

$$0 \longrightarrow L_0^* \longrightarrow L_1^* \longrightarrow \cdots \longrightarrow L_{n-2}^* \xrightarrow{\lambda_{n-1}^*} L_{n-1}^*$$

In this case, $H_0(L_\bullet^*) = \mathrm{Coker}(\lambda_{n-1}^*)$ and

$$H_i(L_\bullet^*) = \mathrm{Ext}_A^{n-1-i}(A/\mathbf{x}, A) = 0 \qquad \text{for } 0 < i \leqslant n-1.$$

Since $\mathrm{Syz}_{n-1}(A/\mathbf{x})$ has a free summand, we may write $\mathrm{Syz}_{n-1}(A/\mathbf{x}) = A \oplus B$, where B is some A-module and A is a summand of L_{n-1}. Applying $(-)^*$ to

$$L_n \xrightarrow{\lambda_n} L_{n-1} \longrightarrow A \oplus B \longrightarrow 0$$

we obtain

$$0 \longrightarrow A \oplus B^* \longrightarrow L_{n-1}^* \xrightarrow{\lambda_n^*} L_n^*$$

where A is a summand of L_{n-1}^*, hence we see that a minimal generator of L_{n-1}^* is contained in $\mathrm{Ker}(\lambda_n^*)$. Since

$$\mathrm{Ext}_A^{n-1}(A/\mathbf{x}, A) = \frac{\mathrm{Ker}(\lambda_n^*)}{\mathrm{Im}(\lambda_{n-1}^*)} \subset \frac{L_{n-1}^*}{\mathrm{Im}(\lambda_{n-1}^*)} = \mathrm{Coker}(\lambda_{n-1}^*)$$

we see that $\mathrm{Coker}(\lambda_{n-1}^*)$ has a minimal generator, call it e^*, killed by $\mathbf{x}$. Setting $C = \mathrm{Coker}(\lambda_{n-1}^*)$, consider the composition

$$A/\mathbf{x} \xrightarrow{\mu} C \xrightarrow{\nu} k$$

where μ maps the image of 1 to e^*, and ν maps e^* to the image of 1 and any other generators of C to 0. Let $(K_\bullet, d_\bullet)$ be the Koszul complex on $\mathbf{x}$ and let $F_\bullet$ be a minimal free resolution of k. Lifting μ and ν above, we obtain the following diagram with commutative squares:

$$\begin{array}{ccccccccccc}
K_n & \longrightarrow & K_{n-1} & \longrightarrow & \cdots & \longrightarrow & K_0 & \longrightarrow & A/\mathbf{x} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} & & & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & L_0^* & \longrightarrow & \cdots & \longrightarrow & L_{n-1}^* & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle \nu_n} & & \downarrow{\scriptstyle \nu_{n-1}} & & & & \downarrow{\scriptstyle \nu_0} & & \downarrow{\scriptstyle \nu} & & \\
F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

If we set $\varphi = \nu \circ \mu$, and $\varphi_\bullet = \nu_\bullet \circ \mu_\bullet$, we have a lift of the canonical map from $A/\mathbf{x}$ to k such that, $\varphi_n : K_n \to F_n$ is the zero map. Hence by Theorem 2.3, writing the homology of $K_\bullet$ as $H_\bullet$, we see that $\operatorname{Syz}_{n-1}(H_1)$ has a free summand.

Consider the following short exact sequence:

$$0 \to H_1 \to K_1/\operatorname{Im}(d_2) \to \mathbf{x}A \to 0$$

By our choice of $\mathbf{x}$, the following complex

$$0 \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1$$

satisfies condition (#) by part (1) of Lemma 2.2. Moreover, this complex is actually a resolution of $K_1/\operatorname{Im}(d_2)$ since $\operatorname{depth}(A) = n-1$. Finally, since $H_1 \subset \mathfrak{m}(K_1/\operatorname{Im}(d_2))$ and $\operatorname{Syz}_{n-1}(H_1)$ has a free summand, Lemma 2.2 implies that $\operatorname{Syz}_n(\mathbf{x}A)$ has a free summand and so $\operatorname{Syz}_{n+1}(A/\mathbf{x})$ also has a free summand. This contradicts Theorem 2.1. □

As we stated before, Theorem 2.4 is analogous to Theorem 1.1. This analogy parallels another analogy. It is a well known theorem that a local ring $(A, \mathfrak{m}, k)$ is a regular local ring if and only if k has finite projective dimension. Moreover, it is a consequence of the New Intersection Theorem, that if there exists any nonzero module of finite length and finite projective dimension, then the ring is Cohen-Macaulay. Hence, it is easy to see that a ring is Cohen-Macaulay if and only if $A/\mathbf{x}$ has finite projective dimension for some system of parameters $\mathbf{x}$.

We find this parallel between implications given by modules having finite projective dimension and implications given by syzygies of modules having free summands to be interesting. In particular, the validity of the Improved New Intersection Conjecture shows that if some syzygy of a module of finite length has a free summand, then A is Cohen-Macaulay. We would like to know if more results of this kind can be proved independently of the Improved New Intersection Conjecture.

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