Free summands of syzygies of modules over local rings

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Abstract

We give a new criterion for a commutative, noetherian, local ring to be Cohen-Macaulay. Additionally, we present a class of chain complexes of finitely generated free modules with finite length homology whose existence in the most general setting is crucial to the validity of the Improved New Intersection Conjecture.

Key words: Cohen-Macaulay, Improved New Intersection Conjecture, syzygy
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1. Introduction

In this paper we look to see whether certain syzygies and cokernels of modules over local rings have free summands, where by *local* we will always mean a commutative, noetherian ring, with a unique maximal ideal. It has been shown that answers to these sorts of questions can shed light on properties of local rings. In particular, Dutta proves the following in [2, Corollary 1.3]:

Theorem 1.1 (Dutta)**.** *A local ring* $(A, \mathfrak{m}, k)$ *is regular if and only if* $\operatorname{Syz}_i(k)$ *has a free summand for some* $i \geqslant 0$*.*

Later in [8, Proposition 7], Martsinkovsky and in [7, Proposition 2.2], Koh and Lee give additional proofs of this theorem. Additionally, Takahashi proves variations of Theorem 1.1 in [11, Theorem 4.3 and Theorem 6.5] when he shows that A is regular if and only if some syzygy of k has a semidualizing summand and when he shows that A is Gorenstein if and only if some syzygy of k has a G-projective summand in degree less than or equal to $\operatorname{depth}(A) + 2$.

In this paper we consider a different sort of variation on Theorem 1.1. Here we seek to understand what can be said if $\operatorname{Syz}_i(M)$ has a free summand when M is some A-module of finite length. Of particular interest to us is the case when $M = A/\mathbf{x}$, where $\mathbf{x}$ is a system of parameters. In Section 2 of this paper, we prove the following theorem which characterizes Cohen-Macaulay rings:

Theorem 2.4. *A local ring* $(A, \mathfrak{m}, k)$ *is Cohen-Macaulay if and only if for some* $i > 0$, $\operatorname{Syz}_i(A/\mathbf{x})$ *has a free summand for some system of parameters* $\mathbf{x}$ *that form part of a minimal set of generators for* $\mathfrak{m}$*.*

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We note that Craig Huneke, Daniel Katz, and Janet Striuli read through an earlier version of this paper. From this they discovered a somewhat simpler proof of Theorem 2.4 than the one given in this paper. Their proof is based on Propositions 2 and 3 from [9]; Appendix I, Section 2.

In Section 3 of this paper, our work is motivated by a theorem of Dutta, Theorem 3.1 of this paper, see [2, Theorem 1.1]. With this theorem, Dutta shows that studying whether certain cokernels have free summands can shed light on the

Improved New Intersection Conjecture. *Let* $(A, \mathfrak{m})$ *be a local ring and*

$$F_\bullet : \qquad 0 \to F_n \to \cdots \to F_0 \to 0$$

be a complex of finitely generated free modules. If $\ell(H_i(F_\bullet)) < \infty$ *for* $i > 0$ *and* $H_0(F_\bullet)$ *has a nonzero minimal generator killed by a power of* $\mathfrak{m}$*, then* $\dim(A) \leqslant n$.

In [2], Dutta shows that a non-Cohen-Macaulay local ring of dimension n satisfies the Improved New Intersection conjecture if and only if cokernels of the nth differentials of a certain class of chain complexes do not have free summands. In particular, resolutions of modules of finite length fall into this class of chain complexes. In Theorem 3.2, we construct classes of chain complexes that are not resolutions satisfying the conditions in Dutta's theorem such that the cokernels of the nth differentials do not have free summands.

We should note that since Theorem 2.4 and Theorem 3.2 both follow from the Improved New Intersection theorem, they hold for in the equicharacteristic case, see [3] and [6], when the dimension of the ring is 3, see [5], and some other special cases. Our proofs are independent of the validity of the Improved New Intersection Conjecture.

2. A characterization of Cohen-Macaulay rings

Inspired by a theorem of Dutta, we give a new characterization of Cohen-Macaulay rings. This criterion is based on whether a certain syzygy has a free summand. We will need some background before we can prove our theorem. Dutta proves the following in [2, Corollary 1.2]:

Theorem 2.1 (Dutta)**.** *Let* A *be a local ring of dimension* n *and* M *be a finitely generated* A*-module of finite length. The following hold:*

1. *If* A *is Cohen-Macaulay, only* $\mathrm{Syz}_n(M)$ *can have a free summand.*
2. *If* $\mathrm{depth}(A) < n-1$*, no syzygy of* M *can have a free summand.*
3. *If* $\mathrm{depth}(A) = n-1$*, only* $\mathrm{Syz}_{n-1}(M)$ *can have a free summand. Moreover, if the Improved New Intersection Conjecture is true for* A*, then* $\mathrm{Syz}_{n-1}(M)$ *cannot have a free summand.*

We find this theorem suggestive, as it tells us that if $\mathrm{Syz}_i(M)$ has a free summand for some A-module M of finite length, then A is Cohen-Macaulay whenever the Improved New Intersection Conjecture holds. With this theorem in mind, one might try to prove a result similar to the result given in Theorem 1.1, with the residue field replaced by some other module of finite length, using a proof that mirrors one of the proofs given in [2], [8], or [7]. While each of these proofs seems to use the fact that k is a field quite heavily, and hence cannot be directly applied, we have found them illuminating nonetheless. In particular, we will use the following definition of Koh and Lee which was presented in [7]:

Definition. Given a local ring $(A, \mathfrak{m})$, a minimal free complex $(F_\bullet, \partial_\bullet)$ bounded on the right at degree z is said to satisfy condition $(\#)$ if

$$\mathrm{Ker}(\partial_i \otimes_A A/\mathfrak{m}^2) = \mathfrak{m}(F_i \otimes_A A/\mathfrak{m}^2)$$

for all $i > z$.

With this definition, Koh and Lee prove the following in [7, Lemma 2.3]:

Lemma 2.2 (Koh and Lee)**.** *Let $(A, \mathfrak{m})$ be a local ring, the following hold:*

1. *If $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of elements that form part of a minimal set of generators for $\mathfrak{m}$, then the Koszul complex $K_\bullet(\mathbf{x})$ satisfies condition $(\#)$.*
2. *Let M be a finitely generated A-module with submodule $N \subset \mathfrak{m}M$. Further suppose that a minimal resolution of M satisfies condition $(\#)$. If $\mathrm{Syz}_i(N)$ has a free summand, then $\mathrm{Syz}_{i+1}(M/N)$ has a free summand.*

Finally we will need the following theorem, which appears implicitly in [1, Section 2]. Here we explicitly state and prove the theorem, giving Dutta's proof.

Theorem 2.3 (Dutta)**.** *Let $(A, \mathfrak{m}, k)$ be a local ring of dimension n and depth $n-1$, $K_\bullet$ be the Koszul complex with respect to a system of parameters $\mathbf{x}$, $F_\bullet$ be a minimal free resolution of k, and $\varphi_\bullet$ be a lift of the canonical surjection $A/\mathbf{x} \to k$. Consider the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"] & K_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r] & K_0 \arrow[r] \arrow[d, "\varphi_0"] & A/\mathbf{x} \arrow[r] \arrow[d] & 0 \\
F_{n+1} \arrow[r] & F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

If $\mathbf{x}$ is a system of parameters such that φ_n is zero, then $\mathrm{Syz}_{n-1}(H_1)$ has a free summand where H_1 is the first Koszul homology with respect to $\mathbf{x}$.

Proof. Since $\mathrm{depth}(A) = n-1$, the following complex is exact:

$$0 \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_1/\mathrm{Im}(d_2) \longrightarrow 0$$

Letting $L_\bullet$ be a minimal free resolution of H_1, we may lift the canonical map $H_1 \to K_1/\operatorname{Im}(d_2)$ to obtain the following commutative diagram:

$$\begin{tikzcd}
L_n \arrow[r, "\lambda_n"] & L_{n-1} \arrow[r] \arrow[d, "\gamma_n"] & \cdots \arrow[r] & L_0 \arrow[r] \arrow[d, "\gamma_1"] & H_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_n \arrow[r] & \cdots \arrow[r] & K_1 \arrow[r] & K_1/\operatorname{Im}(d_2) \arrow[r] & 0
\end{tikzcd}$$

By [1, Theorem 1.3], since $\varphi_n = 0$, it follows that $\operatorname{Im}(\gamma_n) = K_n = A$. Hence there is a basis $e_1, \dots, e_b$ for L_{n-1} such that $\gamma_n(e_1) = 1$ and $\gamma_n(e_i) = 0$ for $i > 1$. By the commutativity of the above diagram, we see

$$\operatorname{Im}(\lambda_n) \subset \operatorname{Ker}(\lambda_{n-1}) \subset M$$

where M is the module generated by $e_2, \dots, e_b$. Since

$$\operatorname{Syz}_{n-1}(H_1) \simeq L_{n-1}/\operatorname{Im}(\lambda_n),$$

we see that $\operatorname{Syz}_{n-1}(H_1)$ has a free summand, specifically the summand generated by the image of e_1. ☐

We now give a criterion for a local ring to be Cohen-Macaulay, compare with Dutta's result, Theorem 2.1.

Theorem 2.4. *A local ring $(A, \mathfrak{m}, k)$ is Cohen-Macaulay if and only if for some $i > 0$, $\operatorname{Syz}_i(A/\mathbf{x})$ has a free summand for some system of parameters $\mathbf{x}$ that form part of a minimal set of generators for $\mathfrak{m}$.*

Proof. ($\Rightarrow$) If A is Cohen-Macaulay, then a Koszul complex on a system of parameters is a minimal free resolution of $A/\mathbf{x}$. Hence $\operatorname{Syz}_n(A/\mathbf{x})$ has a free summand.

($\Leftarrow$) Seeking a contradiction, suppose that A is not Cohen-Macaulay and that for some $i > 0$, $\operatorname{Syz}_i(A/\mathbf{x})$ has a free summand. By Theorem 2.1, A must be of dimension n and depth $n-1$ and the only possible value for i is $n-1$. Working as in the proof of [2, Theorem 1.1], let $(L_\bullet, \lambda_\bullet)$ be a minimal free resolution of $A/\mathbf{x}$. Applying $(-)^* = \operatorname{Hom}_A(-, A)$ we obtain:

$$0 \longrightarrow L_0^* \longrightarrow L_1^* \longrightarrow \cdots \longrightarrow L_{n-2}^* \xrightarrow{\lambda_{n-1}^*} L_{n-1}^*$$

In this case, $H_0(L_\bullet^*) = \operatorname{Coker}(\lambda_{n-1}^*)$ and

$$H_i(L_\bullet^*) = \operatorname{Ext}_A^{n-1-i}(A/\mathbf{x}, A) = 0 \qquad \text{for } 0 < i \leqslant n-1.$$

Since $\operatorname{Syz}_{n-1}(A/\mathbf{x})$ has a free summand, we may write $\operatorname{Syz}_{n-1}(A/\mathbf{x}) = A \oplus B$, where B is some A-module and A is a summand of L_{n-1}. Applying $(-)^*$ to

$$L_n \xrightarrow{\lambda_n} L_{n-1} \longrightarrow A \oplus B \longrightarrow 0$$

we obtain

$$0 \longrightarrow A \oplus B^* \longrightarrow L_{n-1}^* \xrightarrow{\lambda_n^*} L_n^*$$

where A is a summand of L_{n-1}^*, hence we see that a minimal generator of L_{n-1}^* is contained in $\operatorname{Ker}(\lambda_n^*)$. Since

$$\operatorname{Ext}_A^{n-1}(A/\mathbf{x}, A) = \frac{\operatorname{Ker}(\lambda_n^*)}{\operatorname{Im}(\lambda_{n-1}^*)} \subset \frac{L_{n-1}^*}{\operatorname{Im}(\lambda_{n-1}^*)} = \operatorname{Coker}(\lambda_{n-1}^*)$$

we see that $\operatorname{Coker}(\lambda_{n-1}^*)$ has a minimal generator, call it e^*, killed by $\mathbf{x}$. Setting $C = \operatorname{Coker}(\lambda_{n-1}^*)$, consider the composition

$$A/\mathbf{x} \xrightarrow{\mu} C \xrightarrow{\nu} k$$

where μ maps the image of 1 to e^*, and ν maps e^* to the image of 1 and the other generators to 0. Let $(K_\bullet, d_\bullet)$ be the Koszul complex on $\mathbf{x}$ and let $F_\bullet$ be a minimal free resolution of k. Lifting μ and ν above, we obtain the following diagram with commutative squares:

$$\begin{tikzcd}
K_n \arrow[r] \arrow[d, "\mu_n"] & K_{n-1} \arrow[r] \arrow[d, "\mu_{n-1}"] & \cdots \arrow[r] & K_0 \arrow[r] \arrow[d, "\mu_0"] & A/\mathbf{x} \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] \arrow[d, "\nu_n"] & L_0^* \arrow[r] \arrow[d, "\nu_{n-1}"] & \cdots \arrow[r] & L_{n-1}^* \arrow[r] \arrow[d, "\nu_0"] & C \arrow[r] \arrow[d, "\nu"] & 0 \\
F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

If we set $\varphi = \nu \circ \mu$, and $\varphi_\bullet = \nu_\bullet \circ \mu_\bullet$, we have a lift of the canonical map from $A/\mathbf{x}$ to k such that, $\varphi_n : K_n \to F_n$ is the zero map. Hence by Theorem 2.3, writing the homology of $K_\bullet$ as $H_\bullet$, we see that $\operatorname{Syz}_{n-1}(H_1)$ has a free summand.

Consider the following short exact sequence:

$$0 \to H_1 \to K_1/\operatorname{Im}(d_2) \to \mathbf{x}A \to 0$$

By our choice of $\mathbf{x}$, the following complex

$$0 \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1$$

satisfies condition (#) by part (1) of Lemma 2.2. Moreover, this complex is actually a resolution of $K_1/\operatorname{Im}(d_2)$ since $\operatorname{depth}(A) = n-1$. Finally, since $H_1 \subset \mathfrak{m}(K_1/\operatorname{Im}(d_2))$ and $\operatorname{Syz}_{n-1}(H_1)$ has a free summand, Lemma 2.2 implies that $\operatorname{Syz}_n(\mathbf{x}A)$ has a free summand and so $\operatorname{Syz}_{n+1}(A/\mathbf{x})$ also has a free summand. This contradicts Theorem 2.1. □

As we stated before, Theorem 2.4 is analogous to Theorem 1.1. This analogy parallels another analogy. It is a well known theorem that a local ring $(A, \mathfrak{m}, k)$ is a regular local ring if and only if k has finite projective dimension. Moreover, it is a consequence of the New Intersection Theorem, that if there exists any nonzero module of finite length and finite projective dimension, then the ring is Cohen-Macaulay. Hence, it is easy to see that a ring is Cohen-Macaulay if and only if $A/\mathbf{x}$ has finite projective dimension for some system of parameters $\mathbf{x}$.

We find this parallel between implications given by modules having finite projective dimension and implications given by syzygies of modules having free summands to be interesting. In particular, the validity of the Improved New Intersection Conjecture shows that if some syzygy of a module of finite length has a free summand, then A is Cohen-Macaulay. We would like to know if more results of this kind can be proved independently of the Improved New Intersection Conjecture.

3. Cokernels without free summands

Motivating this work is the following theorem of Dutta [2, Theorem 1.1]:

Theorem 3.1 (Dutta)**.** *Let A be a local ring of dimension n and depth $d < n$. Let $(L_\bullet, \lambda_\bullet)$ be a minimal complex which is bounded on the right at degree zero such that:*

1. $H_0(L_\bullet) \neq 0$.
2. $\ell(H_i(L_\bullet)) < \infty$ *for all* i.
3. $H_i(L_\bullet) = 0$ *for* $i \geqslant n - d$.

Then $\operatorname{Coker}(\lambda_i)$ *cannot have a free summand if* $i > 1$ *and* $i \neq n$. *Moreover given a ring A,* $\operatorname{Coker}(\lambda_n)$ *cannot have a free summand for all such* $L_\bullet$ *if and only if the Improved New Intersection Conjecture holds for the ring A.*

From the work discussed above, there are several examples of complexes $(L_\bullet, \lambda_\bullet)$ satisfying conditions (1), (2), and (3) above where one can show that $\operatorname{Coker}(\lambda_n)$ does not have a free summand, even when the validity of the Improved New Intersection Conjecture is not known. For instance consider the following:

- If $(A, \mathfrak{m}, k)$ is not a regular local ring, a minimal resolution of k will satisfy the above conditions by Theorem 1.1.

- If $(A, \mathfrak{m})$ is not a Cohen-Macaulay ring, a minimal resolution of $A/\mathbf{x}$ where $\mathbf{x}$ is a system of parameters that form part of a minimal set of generators for $\mathfrak{m}$ will also satisfy the above conditions by Theorem 2.4.

In light of the above examples, one may inquire if we can produce examples of complexes that are not minimal resolutions satisfying conditions (1), (2), (3) above where it can be verified that the cokernel of the nth differential does not have a free summand. Finding this sort of complex does not seem easy, recall that this property is crucial for the validity of the Improved New Intersection Conjecture. Following a similar construction as given in given in [10, Theorem 3.2], we address this problem with our next theorem:

Theorem 3.2. *Let $(A, \mathfrak{m}, k)$ be a local ring of dimension n and depth d such that $n - d \geqslant 1$. Then there exists minimal complexes of finitely generated free modules*

$$L_\bullet : \qquad \cdots \longrightarrow L_i \xrightarrow{\lambda_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\lambda_1} L_0 \longrightarrow 0$$

such that:

1. $H_0(L_\bullet) \neq 0$.
2. $\ell(H_i(L_\bullet)) < \infty$ *for* $i \geqslant 0$.
3. $H_i(L_\bullet) = 0$ *for* $i \geqslant n - d$.
4. $\operatorname{Coker}(\lambda_n)$ *does not have a free summand.*

Proof. For $n \leqslant 3$, the result follows from Theorem 3.1 and the work of Heitmann [5]. Assume that $n > 3$ and let

$$P_\bullet : \qquad \cdots \longrightarrow P_i \xrightarrow{\rho_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\rho_1} P_0 \longrightarrow Q \longrightarrow 0$$

be a minimal free resolution of some finitely generated A-module Q of infinite projective dimension. Let $M = \operatorname{Coker}(\rho_n^*)$, where $(-)^* = \operatorname{Hom}_A(-, A)$ and let

$$F_\bullet : \qquad \cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

be a minimal free resolution. Considering the canonical surjection $\theta : M \to M/\mathfrak{m}M$, lift this map to a map of complexes $\theta_\bullet : P_\bullet^* \to F_\bullet$ to obtain the following commutative diagram:

$$\begin{tikzcd}
P_0^* \arrow[r] \arrow[d, "\theta_n"] & P_1^* \arrow[r] \arrow[d, "\theta_{n-1}"] & P_2^* \arrow[r] \arrow[d, "\theta_{n-2}"] & \cdots \arrow[r] & P_{n-1}^* \arrow[r] \arrow[d, "\theta_1"] & P_n^* \arrow[r] \arrow[d, equal, "\theta_0"] & M \arrow[r] \arrow[d, "\theta"] & 0 \\
F_n \arrow[r] & F_{n-1} \arrow[r] & F_{n-2} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

Letting $\psi : \operatorname{Im}(\partial_{n-1}^*) \to \operatorname{Ker}(\partial_n^*)$ be the canonical injection, set $U = \operatorname{Syz}_2(Q)$, $\gamma = \theta_{n-1}^*|_{\operatorname{Ker}(\partial_n^*)}$, and $\widetilde{\theta} = \gamma \circ \psi$. We have the following commutative triangle:

$$\begin{tikzcd}
\operatorname{Im}(\partial_{n-1}^*) \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\theta}"] & \\
& \operatorname{Ker}(\partial_n^*) \arrow[dl, "\gamma"] \\
U &
\end{tikzcd}$$

Let $(G_\bullet, \tau_\bullet)$ be a minimal free resolution of $\operatorname{Ker}(\partial_n^*)$. Let $\psi_\bullet$ be a lift of ψ, $\gamma_\bullet$ be a lift of γ, and note that $\theta_\bullet^*$ lifts $\widetilde{\theta}$. We may put these lifts and complexes together into a long diagram.

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dd] \arrow[dr, "\psi_{n-1}"] & & F_0^* \arrow[rr, "\partial_1^*"] \arrow[dd, equal, "\theta_0^*"'] \arrow[dr, "\psi_{n-2}"] & & F_1^* \arrow[rr] \arrow[dd, "\theta_1^*"'] \arrow[dr, "\psi_{n-3}"] & & \cdots \arrow[rr] & & F_{n-2}^* \arrow[rr] \arrow[dd, "\theta_{n-2}^*"'] \arrow[dr, "\psi_0"] & & \mathrm{Im}(\partial_{n-1}^*) \arrow[rr] \arrow[dd, "\widetilde{\theta}"'] \arrow[dr, "\psi"] & & 0 \\
& G_{n-1} \arrow[rr] \arrow[dl, "\gamma_{n-1}"] & & G_{n-2} \arrow[rr] \arrow[dl, "\gamma_{n-2}"] & & G_{n-3} \arrow[rr] \arrow[dl, "\gamma_{n-3}"] & & \cdots \arrow[rr] & & G_0 \arrow[rr] \arrow[dl, "\gamma_0"] & & \mathrm{Ker}(\partial_n^*) \arrow[r] \arrow[dl, "\gamma"] & 0 \\
P_{n+1} \arrow[rr, "\rho_{n+1}"] & & P_n \arrow[rr] & & P_{n-1} \arrow[rr] & & \cdots \arrow[rr] & & P_2 \arrow[rr] & & U \arrow[rr] & & 0
\end{tikzcd}$$

Since $\theta_\bullet^*$ and $\gamma_\bullet \circ \psi_\bullet$ are both lifts of $\widetilde{\theta} = \gamma \circ \psi$, there are homotopy maps, h and h' such that

$$\gamma_{n-2} \circ \psi_{n-2} - \theta_0^* = \rho_{n+1} \circ h + h' \circ \partial_1^*$$

However, $\mathrm{Im}(\rho_{n+1} \circ h + h' \circ \partial_1^*) \subset \mathfrak{m}P_n$, hence

$$(\gamma_{n-2} \circ \psi_{n-2} - \theta_0^*) \otimes_A k = 0.$$

Thus, $\gamma_{n-2} \circ \psi_{n-2}$ and θ_0^* agree modulo $\mathfrak{m}$. From this we see that no minimal generator of F_0^* is mapped into $\mathfrak{m}G_{n-2}$ by ψ_{n-2}. Now consider the mapping cone of $\psi_\bullet$:

$$\cdots \to G_n \to F_0^* \oplus G_{n-1} \to F_1^* \oplus G_{n-2} \to \cdots \to F_{n-2}^* \oplus G_1 \to G_0$$

where the differential of this complex is given by

$$\Lambda_i = \begin{bmatrix} -\partial_{n-i}^* & 0 \\ -\psi_{i-1} & \tau_i \end{bmatrix}.$$

Since A is local, the mapping cone of $\psi_\bullet$ is the direct sum of a totally split subcomplex and a minimal subcomplex. Let $(L_\bullet, \lambda_\bullet)$ be this minimal subcomplex of the mapping cone of $\psi_\bullet$. Since ψ_{n-2} does not map minimal generators of F_0^* to $\mathfrak{m}G_{n-2}$, it is not a block of the matrix λ_{n-1} and so $\lambda_{n-1} = \tau_{n-1}$. Hence $L_{n-1} = G_{n-1}$ and so $\mathrm{Coker}(\lambda_n)$ has a free summand if and only if $\mathrm{Coker}(\tau_n)$ has a free summand.

To show that $\mathrm{Coker}(\tau_n)$ cannot have a free summand, we will use use the same technique as used in the proof of [2, Theorem 1.1]. Suppose that $\mathrm{Coker}(\tau_n)$ has a free summand. Write $\mathrm{Coker}(\tau_n) = A \oplus B$ and consider the following exact sequence:

$$0 \longrightarrow \mathrm{Coker}(\tau_n) \xrightarrow{f} G_{n-2} \xrightarrow{g} \mathrm{Im}(\tau_{n-2}) \longrightarrow 0$$

Setting V and W to be the cokernels of the injections $\mu : A \to G_{n-2}$ and $\nu : B \to G_{n-2}$ respectively, we obtain the exact sequences

$$0 \longrightarrow A \xrightarrow{\mu} G_{n-2} \xrightarrow{\alpha} V \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow B \xrightarrow{\nu} G_{n-2} \xrightarrow{\beta} W \longrightarrow 0$$

Working as in [2, Theorem 1.1], with notation similar to that used in [11, Lemma 3.1], we obtain the exact sequences

$$0 \longrightarrow B \xrightarrow{\alpha\circ\nu} V \xrightarrow{\delta} \mathrm{Im}(\tau_{n-2}) \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\beta\circ\mu} W \xrightarrow{\gamma} \mathrm{Im}(\tau_{n-2}) \longrightarrow 0$$

where $\delta(\alpha(x)) := g(x)$ and $\gamma(\beta(y)) := g(y)$. We also obtain one last exact sequence

$$0 \longrightarrow G_{n-2} \xrightarrow{\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]} V \oplus W \xrightarrow{\zeta} \mathrm{Im}(\tau_{n-2}) \longrightarrow 0 \qquad (2)$$

where $\zeta\left(\begin{smallmatrix}\alpha(x)\\ \beta(y)\end{smallmatrix}\right) := g(x-y)$. By (1), the projective dimension of V is 1 and hence $\mathrm{depth}(V) = d-1$. However, it is not hard to see that $\mathrm{depth}(\mathrm{Im}(\tau_{n-2})) \geqslant d$ and hence by (2), $\mathrm{depth}(V \oplus W)$ must be at least d, implying that $\mathrm{depth}(V) \geqslant d$, a contradiction. Hence $\mathrm{Coker}(\tau_n)$ does not have a free summand and neither does $\mathrm{Coker}(\lambda_n)$.

Now we'll examine the homology of $L_\bullet$. Since $L_\bullet$ is a minimal subcomplex of the mapping cone of $\psi_\bullet$, these two complexes have identical homology. Hence, replacing the homology of the mapping cone of $\psi_\bullet$ with the homology of $L_\bullet$ in the long exact sequence associated to the mapping cone

$$\cdots \to \underbrace{H_i(G_\bullet)}_{0} \to H_i(L_\bullet) \to H_{i-1}(F_\bullet^*) \to \underbrace{H_{i-1}(G_\bullet)}_{0} \to \cdots$$

we find that

$$H_i(L_\bullet) = \mathrm{Ext}_A^{n-i-1}(M/\mathfrak{m}M, A)$$

when $i > 1$. For the cases when $i = 0$ or $i = 1$, consider the following exact sequence:

$$\cdots \longrightarrow \underbrace{H_1(G_\bullet)}_{0} \longrightarrow H_1(L_\bullet) \xrightarrow{0} \underbrace{H_0(F_\bullet^*)}_{\mathrm{Im}(\partial_{n-1}^*)} \longrightarrow \underbrace{H_0(G_\bullet)}_{\mathrm{Ker}(\partial_n^*)} \longrightarrow H_0(L_\bullet) \longrightarrow 0 \qquad (3)$$

However, $\mathrm{Im}(\partial_{n-1}^*)$ injects into $\mathrm{Ker}(\partial_n^*)$ implying that

$$H_0(L_\bullet) \simeq \frac{\mathrm{Ker}(\partial_n^*)}{\mathrm{Im}(\partial_{n-1}^*)} = \mathrm{Ext}_A^{n-1}(M/\mathfrak{m}M, A),$$

and by [4, Proposition 2.3], we see that $H_0(L_\bullet) \neq 0$. Moreover, by (3), we see that $H_1(L_\bullet) = 0$. Thus:

1. $H_0(L_\bullet) \neq 0$.
2. $\ell(H_i(L_\bullet)) < \infty$ for $i \geqslant 0$.
3. $H_i(L_\bullet) = 0$ for $i \geqslant n-d$.
4. $\mathrm{Coker}(\lambda_n)$ does not have a free summand.

We have now shown that $L_\bullet$ is a complex exhibiting the desired properties. □

In light of Theorem 3.1, the Improved New Intersection Conjecture is equivalent to proving that every complex that satisfies conditions (1), (2), and (3) of Theorem 3.1 also satisfies condition (4) above. While Theorem 3.2 shows that there are complexes with nontrivial homology satisfying these conditions, we would like to find more. Currently, we are working to expand the class of complexes for which conditions (1), (2), (3), and (4) are known to hold.

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